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FURTHER RESULTS ON THE E-K-R THEOREM FOR THE DISTANCE REGULAR GRAPHS $H_q(k, n)$

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Abstract. The distance regular graph $H_q(k, n)$ is defined on the set $M_{k\times n}(GF(q))$ of all $k\times n$ matrices over $GF(q)$ such that two matrices A and B are adjacent if and only if the rank of $A-B$ is 1. It has been shown that the maximum size of families $\mathscr{G}$ contained in $M_{k\times n}(GF(q))$ with the property that rank $(A-B)\leq k-r$ for all $A, B\in\mathscr{G}$ is $q^{n(k-r)}$ and those families $\mathscr{G}$ with size $q^{n(k-r)}$ are characterized whenever $n\geq k+1$ and $(n, q)\neq(k+1, 2)$, (Discrete Mathematics, 64 (1987), 191-198).

The remaining cases $(n,k,r,q)=(k+1,k,k-1,2)$, $(k,k,k-1,q)$ and $(4,3,1,2)$ are treated in this paper. Partial results for (k,k,r,q) are also derived.

1. Introduction. Let X be a set with n elements, and $r<k\leq n$. A family $\mathscr{G}\subseteq\binom{X}{k}$ is called a r-intersecting family if $|A\cap B|\geq r$ holds for all $A, B\in\mathscr{G}$. The first intersection theorem was proved by Erdős, Ko, and Rado in the late 1930, however it was not published until 1961.

THEOREM [5, 6, 14]. *Let n, k, r be integers with $n\geq k\geq r\geq 0$, and X be set of n elements. Suppose that $\mathscr{G}\subseteq\binom{X}{k}$ is a r-intersecting family. Then, for $n\geq n_0(k, r)=(r+1)(k-r+1)$,*

a) $|\mathscr{G}|\leq\binom{n-r}{k-r}$, *and*

b) $|\mathscr{G}|=\binom{n-r}{k-r}$ *if and only if* $\bigcap_{A\in\mathscr{G}}A$ *consists of r elements.*

A number of analogues of the E-K-R theorem have been obtained for structures other than subsets of a set. For example, among

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the others, Hsieh [9], Frankl and Wilson [8] obtained analogues for subspaces of a finite vector space, Frankl and Furedi [7], Moon [11] for integer sequences, Stanton [13] for Chevalley groups. Analogous results hold for matrices too. Let $M_{k\times n}(GF(q))$ be the set of all $k \times n$ matrices over $GF(q)$.

THEOREM 1 [10]: *Let* $\mathscr{G} \subseteq M_{k\times n}(q)$, *and* $\text{rank}(A-B) \leq k-r$ *for all* $A, B \in \mathscr{G}$ *where* $0 \leq r \leq k$. *Assume that* $n \geq k+1$, *and* $(n, q) \neq (k+1, 2)$, *then*

a) $|\mathscr{G}| \leq q^{n(k-r)}$, *and*

b) $|\mathscr{G}| = q^{n(k-r)}$ *if and only if, up to isomorphism,*
$\mathscr{G} = \{A \mid A \in M_{k\times n}(q)$ *with zero entries in the last* r *rows*$\}$.

REMARK. Using a different approach, Moon [12] proved Theorem 1 under the conditions that $n > k+1$ and $q \geq 3$. Moreover if $n > r+2$, assume $r \leq (q-1)\, q^{n-r-3}$.

In Section 2, we present some combinatorial structures on $M_{k\times n}(GF(q))$ which will be used later. The remaining cases $(n, k, r, q) = (k+1, k, k-1, 2)$, $(k, k, k-1, q)$ and $(4, 3, 1, 2)$ in the above theorem are treated in Section 3 and Section 4 respectively through different approaches. Some partial results for (k, k, r, q) are derived in Section 5.

2. **Some combinatorial structures on** $M_{k\times n}(GF(q))$. For the set $M_{k\times n}(GF(q))$ of all $k\times n$ matrices over $GF(q)$, let $\mathscr{R}_i = \{(A, B) \mid A, B \in M_{k\times n}(GF(q))$ with $\text{rank}(A-B) = i\}$, $0 \leq i \leq k$ $(\leq n)$. Then $\{\mathscr{R}_i \mid 0 \leq i \leq k\}$ forms a partition of $M_{k\times n}(GF(q))$, $\mathscr{R}_i^t = \mathscr{R}_i$ for all i, and furthermore, for $(A, B) \in \mathscr{R}_i$, the cardinality of the set $\{C \mid C \in M_{k\times n}(GF(q))$ with $(A, C) \in \mathscr{R}_j$ and $(C, B) \in \mathscr{R}_k\}$ is a function of i, j, and k only, independent of the choices of A and B. In other words, $(M_{k\times n}(GF(q)), \mathscr{R}_1)$ forms a distance regular graph of diameter k, denoted by $H_q(k, n)$. Refer to [1] for more details about distance regular graphs.

In addition to the structure of distance regular graphs, there is another interpretation for $M_{k\times n}(q)$, which we describe as following:

Let V be a $k+n$ dimensional vector space over a finite field $GF(q)$, $W \subseteq V$ be a fixed subspace of dimension n, and $\{w_1, w_2, \cdots, w_n\}$,

$\{e_1, e_2, \cdots, e_k, w_1, w_2, \cdots, w_n\}$ be bases of W and V respectively. Let and $U = \langle e_1, \cdots, e_k \rangle$, and

$$\mathcal{A}_k = \{A \mid A \subseteq V \text{ is a } k \text{ dimensional subspace with } A \cap W = 0\}.$$

It is known that each element A in $\mathcal{A}_k$ has a base $\{e_1 + w_1', \cdots, e_k + w_k'\}$ for uniquely chosen $w_1', \cdots, w_k'$ in W. Let $w_i' = \sum_{1 \le i \le k} a_{ij} w_j$ and $M(A)$ be the matrix

$$\begin{bmatrix} a_{11}\, a_{12} \cdots a_{1n} \\ a_{21}\, a_{22} \cdots a_{2n} \\ \cdots \\ a_{k1}\, a_{k2} \cdots a_{kn} \end{bmatrix}$$

then the correspondence $A \to M(A)$ defines a bijection from $\mathcal{A}_k$ onto $M_{k \times n}(q)$. Moreover, $\dim (A \cap B)$ is r if and only if the rank of $M(A) - M(B)$ is $k - r$. Hence, a subset $\mathcal{G} \subseteq M_{k \times n}(q)$ with the property that $\operatorname{rank}(A - B) \le k - r$ for all $A, B \in \mathcal{G}$ corresponds to an analongue of r-intersecting family in $\mathcal{A}_k$, i.e., $\mathcal{G} \subseteq \mathcal{A}_k$ such that $\dim (A \cap B) \ge r$ for all $A, B \in \mathcal{G}$. With the above correspondence, we shall make no difference between $M_{k \times n}(GF(q))$ and $\mathcal{A}_k$ in the rest of this note. For further details, refer to [10].

An analogue of the Erdös-Ko-Rado theorem for the distance-regular graphs of bilinear forms $H_q(k, n)$ is obtained by Delsarte (implicitly) and Huang. A subset $\mathcal{E} \subseteq M_{k \times n}(GF(q))$ with cardinality $|\mathcal{E}| = q^{nr}$ and the property that $\operatorname{rank}(A - B) \ge k - r + 1$ for all distinct $A, B \in \mathcal{E}$ was constructed by Delsarte [3, p. 237]. Another theorem of Delsarte [2, Theorem 3.9] shows that $|\mathcal{E}||\mathcal{G}| \le q^{nk}$ for each family $\mathcal{G} \subseteq M_{k \times n}(GF(q))$ with the property that $\operatorname{rank}(C - D) \le k - r$ for all $C, D \in \mathcal{G}$. The following theorem follows immediately.

THEOREM. *Let* $\mathcal{G} \subseteq M_{k \times n}(GF(q))$, *and* rank $(A - B) \le k - r$ *for all* $A, B \in \mathcal{G}$ *then* $|\mathcal{G}| \le q^{n(k-r)}$.

The extremal families are characterized by Huang.

THEOREM 2 [10]. *Let* $\mathcal{G} \subseteq M_{k \times n}(GF(q))$, *and* rank$(A - B) \le k - r$ *for all* $A, B \in \mathcal{G}$, *where* $0 \le r \le k$. *Assume that* $n \ge k + 1$, *and* $(n, q) \ne (k + 1, 2)$, *then*

$|\mathscr{G}| = q^{n(k-r)}$ *if and only if, up to isomorphism,*

$$\mathscr{G} = \{A \mid A \in M_{k\times n}(q) \text{ with zero entries on the last } r \text{ rows}\}.$$

In other words, $\dim(\bigcap_{F\in\mathscr{G}} F) = r$.

Under the given conditions, based on the interpretation of $M_{k\times n}(GF(q))$ as $\mathscr{A}_k$ together with the following inequality

$$\begin{bmatrix} r+p \\ r \end{bmatrix}\begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^p < q^{np}, \quad 1 \le p \le k-r,$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = (q^n-1)\cdots(q^n-q^{k-1})/((q^k-1)\cdots(q^k-q^{k-1}))$ is a Gaussian coefficient, the above theorem is proved by showing that $|\mathscr{G}| < q^{n(k-r)}$ whenever $\dim(\bigcap_{F\in\mathscr{G}} F) < r$, i.e., the above two theorems are proved simultaneously.

3. **The cases** $(n, k, k-1, 2)$ **where** $n = k,\ k+1$. In this section, all possible $(k-1)$-intersecting families of $\mathscr{A}_k$ are characterized, the conditions $n = k,\ k+1$ and $q = 2$ are not necessary. Suppose $\mathscr{G} \subseteq \mathscr{A}_k$ is a $(k-1)$-intersecting family, by the translation invariance of the rank function, we may assume that $U = \langle e_1, \cdots, e_k\rangle \in \mathscr{G}$, which corresponds to the zero matrix in $M_{k\times n}(GF(q))$. For $A \in \mathscr{G}$, the direct sum of A and U is denoted by $A + U$.

LEMMA 3.1. *If* $\dim(A\cap U) = \dim(A\cap B) = \dim(B\cap U) = k-1$, *then either* $B \subseteq A + U$, *or* $A \cap U \subseteq B$.

Proof. Suppose $A \cap U \nsubseteq B$. Since $\dim(A\cap B\cap U) \le k-2$ and $\dim(U\cap B) + \dim(A\cap B) \le \dim((A+U)\cap B) + \dim(A\cap B\cap U)$, we have $\dim((A+U)\cap B) \ge k$. Therefore, $B \subseteq A+U$ as required.

LEMMA 3.2. *Suppose that* A, B, *and* U *are in* $\mathscr{G}$.

(1) *If* $A\cap U \subseteq B$ *but* $B \nsubseteq A+U$, *then* $A\cap U \subseteq \bigcap_{F\in\mathscr{G}} F$.

(2) *If* $B \subseteq A+U$ *but* $A\cap U \nsubseteq B$, *then* $\bigcup_{F\in\mathscr{G}} F \subseteq A+U$.

Proof. For (1), suppose that $A\cap U \nsubseteq F$ for some $F\in\mathscr{G}$. Then $F \subseteq A+U$ by Lemma 3.1. Since $B\cap F \subseteq B\cap(A+U)$ and $B \nsubseteq A+U$, we have $B\cap F = B\cap(A+U)$ by compairing their dimensions. On the other hand, $A \cap U \subseteq B \cap (A+U)$, therefore we have $A \cap U \subseteq B \cap F \subseteq F$. This contradicts $A \cap U \nsubseteq F$ and (1) is proved. For (2), if there exists an element $F \in \mathscr{G}$ with $F \nsubseteq A+U$, then $A \cap U \subseteq F$ by Lemma 3.1, and $A \cap U \subseteq \bigcap_{F \in \mathscr{G}} F$ by (1). This contradicts $A \cap U \nsubseteq B$ and (2) is proved.

The previous two lemmas show that either

(i) $A \cap U \subseteq \bigcap_{F \in \mathscr{G}} F$, or

(ii) $\bigcup_{F \in \mathscr{G}} F \subseteq A + U$

for all $A \in \mathscr{G}$. Here, we note that $\dim(A \cap U) = k-1$, and $\dim(A+U) = k+1$. For a fixed A in $\mathscr{G}$, let

$$\mathscr{G}_1 = \{F \mid F \in \mathscr{A}_k \text{ with } A \cap U \subseteq F\},$$

and

$$\mathscr{G}_2 = \{F \mid F \in \mathscr{A}_k \text{ with } F \subseteq A + U\}.$$

Clearly, both $\mathscr{G}_1$ and $\mathscr{G}_2$ are maximal $(k-1)$-intersecting families of sizes q^n and q^k respectively. They are of equal size whenever $n = k$. The following theorem follows immediately.

THEOREM 3. *Let $\mathscr{G} \subseteq \mathscr{A}_k$ be an extremal $(k-1)$-intersecting family with $U, A \in \mathscr{G}$, then either $\mathscr{G} = \{F \mid F \in \mathscr{G}_k$ with $A \cap U \subseteq F\}$, with $|\mathscr{G}| = q^n$, or*

$$\mathscr{G} = \{F \mid F \in \mathscr{A}_k \text{ with } F \subseteq A + U\}, \text{ with } |\mathscr{G}| = q^k.$$

Both types are of equal size q^k whenever $n = k$.

The matrix representations of the above two types of maximal $(k-1)$-intersection families can be described as following:

$$\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ 0 & 0 \cdots 0 \\ & \cdots \\ 0 & 0 \cdots 0 \end{bmatrix}$$

for the first type, and

$$\begin{bmatrix} a_{11} & 0 \cdots 0 \\ a_{21} & 0 \cdots 0 \\ & \cdots \\ a_{k1} & 0 \cdots 0 \end{bmatrix}$$

for the second type.

The following corollary provides some information of $M_{k\times n}(GF(q))$ of $\mathscr{A}_k$ from the geometric view point if the extremal $(k-1)$-intersecting families are regarded as lines.

COROLLARY 3.4. (1) *Each* $A\in\mathscr{A}_k$ *is contained in exacitly* $(q^k-1)/(q-1)$ $(k-1)$*-intersecting families of size* q^n*, and is contained in exactly* $(q^n-1)/(q-1)$ $(k-1)$*-intersecting families of size* q^k.

(2) *In* $\mathscr{G}$ *and* $\mathscr{G}'\subseteq\mathscr{A}_k$ *are two* $(k-1)$*-intersecting families with different sizes, then* $|\mathscr{G}\cap\mathscr{G}'|=0$ *or* q.

Proof. Straightforward from Theorem 3.

4. **The case** $(n, k, r, q) = (4, 3, 1, 2)$. In the first half of this section, we provide some information for the more general case $(k+1, k, 1, 2)$. Then we assume that $k=3$ and Theorem 4 follows. Let $\mathscr{G}\subseteq M_{k\times(k+1)}(GF(2))$ be an 1-intersecting family. Without loss of generality, we may assume that the zero matrix belongs to $\mathscr{G}$. Thus $\operatorname{rank}(X)\le k-1$ for all $X\in\mathscr{G}$. Let $X\in\mathscr{G}$ with row vectors $X_1,\cdots,X_k$, then $X_1, X_2,\cdots, X_k$ are linear dependent, i.e., there exists a nonzero vector $(\alpha_1, \alpha_2,\cdots,\alpha_k)\in(GF(2))^k$ such that $\sum_{1\le i\le k}\alpha_i X_i=0$, the zero vector. Let $\vec{a}_i$ be the binary representation of i, $1\le i\le 2^k-1$, and

$$\mathscr{L}_i=\{X\in M_{k\times(k+1)}(GF(2)) \text{ with } \vec{a}_i X=0\},$$

$1\le i\le 2^k-1$, e.g.,

$$\mathscr{L}_1=\left\{\begin{bmatrix} x_{11} & \cdots & x_{1,k+1}\\ x_{21} & & x_{2,k+1}\\ & \cdots & \\ x_{n-1,1} & \cdots & x_{k-1,k+1}\\ 0 & \cdots & 0\end{bmatrix} \middle|\, x_{ij}\in GF(2)\right\}$$

for $1=[0, 0,\cdots, 0, 1]$.

Some observations about $\{\mathscr{L}_i\mid 1\le i\le 2^k-1\}$ are as following:

LEMMA 4.1. (1) $|\mathscr{L}_i|=2^{(k-1)(k+1)}$ *for all* $i\le 2^k-1$,

(2) $\mathscr{L}_i-\mathscr{L}_i\subseteq\mathscr{L}_i$, $1\le i\le 2^k-1$,

(3) $|\mathscr{L}_i\cap\mathscr{L}_j|=2^{(k-2)(k+1)}$ *if* $i\ne j$,

(4) rank $(X-Y)\leq k-1$ *for all* $X, Y\in\mathscr{L}_i$, $1\leq i\leq 2^k-1$,

(5) $\mathscr{G}\subseteq\bigcup_{1\leq i\leq 2^k-1}(\mathscr{G}\cap\mathscr{L}_i)$.

Proof. Straightforward.

Let $\mathscr{G}_i=\mathscr{G}\cap\mathscr{L}_i$, $1\leq i\leq 2^k-1$. If $\mathscr{G}_i=\mathscr{G}$ for some i, then $\mathscr{G}\subseteq\mathscr{L}_i$, and thus $\mathscr{G}$ can be enlarged to these intersecting families. Otherwise there are distinct i and j such that each of $\mathscr{G}_i$, $\mathscr{G}_j$, $\mathscr{G}_i-\mathscr{G}_j$, $\mathscr{G}_j-\mathscr{G}_i$ is nonempty. We assume that $i_0\leq 2^k-1$ is an index such that $|\mathscr{G}_{i_0}|\geq|\mathscr{G}_i|$ for all $i\leq 2^k-1$.

LEMMA 4.2. *If* $\mathscr{X}\subseteq\mathscr{G}_i$ *such that* $\mathscr{X}+A_l\subseteq\mathscr{L}_{t_l}$ *for* $1\leq l\leq r$, *where* $A_1,\cdots,A_l\in\mathscr{G}_j$ *and* $i, j, t_1,\cdots,t_l$ *are pairwise distinct. Then* $|\mathscr{X}|\leq 2^{(k-r-1)(k+1)}$.

Proof. This is clear since each column of each member in $\mathscr{X}$ contains at most $r+1$ independent entries.

We consider those values of j that each of $\mathscr{G}_j$, $\mathscr{G}_{i_0}-\mathscr{G}_j$ and $\mathscr{G}_j-\mathscr{G}_{i_0}$ is nonempty. For such j, let

$$\mathscr{D}_j=\mathscr{G}_{i_0}-\mathscr{G}_j,\quad\text{and}\quad\mathscr{E}_j=\mathscr{G}_j-\mathscr{G}_{i_0}.$$

Then both $\mathscr{D}_j$ and $\mathscr{E}_j$ are nonempty and $\mathscr{D}_j\cap\mathscr{E}_j$ is empty. Let

$$\{\mathscr{D}_{j_1},\cdots,\mathscr{D}_{j_r}\}\quad\text{and}\quad\{\mathscr{E}_{j_1},\cdots,\mathscr{E}_{j_s}\}$$

be partitions of $\mathscr{D}_j$ and $\mathscr{E}_j$ respectively such that $\mathscr{D}_{j_h}-\mathscr{E}_{j_l}$ is entirely contained in $\mathscr{L}_t$ for some $t\leq 2^k-1$.

A matrix $M_j=M(\mathscr{D}_j,\mathscr{E}_j)$ is introduced as a tool to estimate both $|\mathscr{G}_{i_0}|$ and $|\mathscr{G}_j|$. The matrix M_j is a $r\times s$ matrix in which the rows and columns are indexed by the above partitions of $\mathscr{D}_j$ and $\mathscr{E}_j$ respectively such that the (k,l)-entry of M_j is t whenever t is the smallest integer that $\mathscr{D}_{j_h}-\mathscr{E}_{j_l}\subseteq\mathscr{L}_t$. We assume those partitions of $\mathscr{D}_j$ and $\mathscr{E}_j$ are chosen such that no two rows (or columns) are identical.

The following lemmas provide some information about the matrix M_j, which enable us to approximate $|\mathscr{G}_{i_0}|$ and $|\mathscr{G}_j|$.

LEMMA 4.3. *For the matrix* $M_j=M(\mathscr{D}_j,\mathscr{E}_j)$,

(1) *both* i_0 *and* j *do not appear as entries in* M_j,

(2) *each line (either a row or a column) contains at most* $k-1$ *values.*

Proof. For (1), suppose, by contradiction, that j appears in M_j as an entry, then $\mathcal{D}_{j_h} - \mathcal{E}_{j_t} \subseteq \mathcal{L}_j$ for some pair (h, t), and hence $\mathcal{D}_{j_t} \subseteq \mathcal{L}_j - \mathcal{L}_j \subseteq \mathcal{L}_j$. This contradicts the fact that $\mathcal{D}_j \cap \mathcal{L}_j$ is empty. Similarly, i_0 does not appear as entry in M_j either. (2) is clear from Lemma 4.2.

LEMMA 4.4. *There do not exist any two rows (or two columns) which contain*

$$\begin{bmatrix} t_1\, t_2 \cdots t_{k-1} \\ t_1\, t_2 \cdots t_{k-1} \end{bmatrix}$$

(or its transpose) as its submatrix.

Proof. If there are two such rows which are indexed by $\mathcal{D}_{j_h}$ and $\mathcal{D}_{j_t}$, then $\mathcal{D}_{j_h} = \mathcal{D}_{j_t}$ is a singleton and and hence these two rows must be identical, a contradiction. Similar argument works for columns.

We assume $k=3$ in the following:

If M_j is a column vector for some j, then $|\mathcal{E}_j| \le 2^{(k-2)(k+1)} = 16$, and $|\mathcal{D}_j| \le 2^{(k-2)(k+1)+1} = 32$. Thus $|\mathcal{G}_{i_0} \cup \mathcal{G}_j| \le 16 + 32 + 2^{(k-2)(k+1)} = 64$. If M_j is a row vector, similarly, we have $|\mathcal{G}_{i_0} \cup \mathcal{G}_j| \le 64$.

On the other hand, we assume that M_j is neither a column vector nor a row vector. Let H be a submatrix of M_j which consists of two columns of M_j. For distinct $h, t \le 2^k - 1$,

1. the pair (h, t) occurs at most once in H and there are at most 4 such pairs by Lemma 4.3 and Lemma 4.4.
2. there is at most one form like (t, t) occurs in H.

Lemma 4.2 shows that

$$|\mathcal{D}_{j_k}| \le 2^{(k-3)(k+1)}$$
$$= 1 \quad \text{if the } \mathcal{D}_{j_k}\text{—row of this submatrix } H \text{ is } (h, t),$$
$$\le 2^{(k-2)(k+1)}$$
$$= 16 \quad \text{if the } \mathcal{D}_{j_k}\text{—row of this submatrix } H \text{ is } (t, t).$$

If follows that

$$|\mathcal{D}_j| \le 2^{(k-2)(k+1)} + 3 \cdot 2^{(k-3)(k+1)} = 2^4 + 3 = 19.$$

Similar arguments show that $|\mathcal{E}_j| \leq 19$ too. Hence $|\mathcal{G}_{i_0} \cup \mathcal{G}_j| \leq 19 + 19 + 2^{(k-2)(k+1)} = 54$. It follows, based on the above analysis, that

$$\begin{aligned}|\mathcal{G}| &\leq 64 + 32(2^k - 1 - 2)\\ &= 224\\ &< 256.\end{aligned}$$

THEOREM 4. *If $\mathcal{G} \subseteq M_{3\times 4}(GF(2))$ is an intersecting family with $|\mathcal{G}| = 256 (= 2^8)$, then $\mathcal{G} = \mathcal{L}_i$ for some $i \leq 7$, in other words,* $\dim(\bigcap_{F\in\mathcal{G}} F) = 1$.

Indeed, the argument above provides more information about the size $|\mathcal{G}|$ than what we need to conclude Theorem 4.

5. **The case** $(n, k, r, q) = (k, k, r, q)$. As suggested in Theorem 3, for the case $n = k$, there are at least two types of maximum r-intersecting families (with size $q^{k(k-r)}$). If $\mathcal{G} \subseteq \mathcal{A}_k$ is a r-intersecting family such that either $\dim(\bigcap_{F\in\mathcal{G}} F) = r$ or $\dim(\langle \bigcup_{F\in\mathcal{G}} F\rangle) = 2k - r$, where $\langle \bigcup_{F\in\mathcal{G}} F\rangle$ denotes the subspace spanned by $\bigcup_{F\in\mathcal{G}} F$, then $\mathcal{G}$ can be enlarged to be a r-intersecting family with maximum size $q^{k(k-r)}$. The following theorem provides some information for those r-intersecting families $\mathcal{G}$ with $|\mathcal{G}| = q^{k(k-1)}$ but $\dim(\bigcap_{F\in\mathcal{G}} F) \leq r - 1$.

Let V, W, U and $\mathcal{A}_k$ be as defined in Section 1. We assume that $k = n$ in the rest of this note, i.e., V is a vector space of dimension $2k$, U and W are subspaces of V of dimension k such that V is the direct sum of U and W.

THEOREM 5. *Let $\mathcal{G} \subseteq \mathcal{A}_k$ be a r-intersecting family, i.e., $\dim(A \cap B) \geq r$ for all A, $B \in \mathcal{G}$, and $|\mathcal{G}| = q^{k(k-r)}$, where $0 \leq r < k$ and $q \geq 3$. Then $\bigcap_{F\in\mathcal{G}} F \subseteq V$ is either a r-dimensional subspace or a trivial subspace.*

Proof. Suppose, by contradiction, that $X = \bigcap_{F\in\mathcal{G}} F \subseteq V$ is a subspace of dimension t, $1 \leq t \leq r - 1$. Consider the quotient spaces V/X, its subspaces U/X and W/X ($\simeq W$), and the family $\mathcal{G}/X = \{A/X \mid A \in \mathcal{G}\}$. Then

(1) V/X, a vector space of dimension $2k-t$ over $GF(q)$, is the direct sum of U/X and W/X,

(2) each member A/X of $\mathcal{G}/X$ is a $k-t$ dimensional subspace of V/X with $A/X \cap W/X = 0$,

(3) $A/X \cap B/X$ is a subspace of V/X with dimension at least $r-t$, and

(4) $|\mathcal{G}/X| = |\mathcal{G}| = q^{k(k-r)}$.

It follows, by Theorem 1, that each member of $\mathcal{G}/X$ contains a fixed subspace E/X of V/X of dimension $r-t$ and hence $E \subseteq V$ is a subspace of dimension r with the property that $E \subseteq \bigcap_{F\in\mathcal{G}} F$, a contradiction.

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