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SOME RESULTS ON ASCENDING SUBGRAPH DECOMPOSITION

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1. Introduction. In [1] the authors give the following decomposition conjecture.

CONJECTURE. *Let G be a graph with $\binom{n+1}{2}$ edges. Then the edge set of G can be partitioned into n sets generating graphs $G_1, G_2, \cdots, G_n$ such that $|E(G_i)| = i$ (for $i = 1, 2, \cdots, n$) and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \cdots, n-1$.*

A graph G which can be decomposed as described in the conjecture will be said to have an asceding subgraph decmposition (abbreviate *ASD*). The graphs $G_1, G_2, \cdots, G_n$ are said to members of such a decomposition.

In [1, 2], the conjecture has been verified for the graphs which are star forest. Also in [2], they prove that if G is a graph of maximum degree d on $\binom{n+1}{2}$ edges and $n \geq 4d^2 + 6d + 3$, then G has an *ASD* with each member a matching. The second result has been improved by Fu in [3] which shows, if G is a graph of maximum degree $d \leq [(n-1)/2]$ on $\binom{n+1}{2}$ edges, then G has an *ASD* with each member a matching. Moreover, in [3] the author prove that every graph of maximum degree $d \leq [(n+1)/2]$ has an *ASD*, and subsequently every regular graph of degree a prime power has an *ASD*.

In this paper, we apply the idea on the decomposition of the set $\{1, 2, \cdots, n\}$ into subsets with prescribed sum to verify the conjecture for some classes of graphs.

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2. **Main results.** Let N be the set $\{1, 2, \cdots, n\}$, and $A_1, A_2, \cdots, A_k$ be mutually disjoint subsets of N such that $\bigcup_{i=1}^{k} A_i = N$. Let $s(A_i)$ be the sum of all elements in A_i. We will say that N can be decomposed into subsets of type $\langle s_1, s_2. \cdots, s_k\rangle$ if there exists a collection of mutually disjoint subsets of N, $A_1, A_2, \cdots, A_k$, such that their union is N and $s(A_i) = s_i$, $i = 1, 2, \cdots, k$. Obviously, $\sum_{i=1}^{k} s_i = \binom{n+1}{2}$.

In [3], the above idea has been applied to show that a graph of maximum degree $d \leq [(n-1)/2]$ on $\binom{n+1}{2}$ edges has an *ASD*. But the idea on the decomposition of N into subsets with prescribed sum is still not well developed. In what follows, we are going to list some of the results obtained.

LEMMA 2.1. [4] *N can be decomposed into subsets of type $\langle m, m, \cdots, m\rangle$ (k-tuple) where $m \geq n$ and $km = \frac{1}{2} n(n+1)$.*

Proof. The proof is by induction on n. Assume it's true for the case $n' < n$. Now we consider the relationship between m and n. (i) $m > 2n$. We can pair off the elements in $\{n-2k+1, n-2k+2, \cdots, n\}$ into k pairs, each summing to $2n-2k+1$. Let $m' = m-(2n-2k+1) = m-2n+2k-1$. It is not difficult to check $m' \geq n-2k = n'$. By induction, we can decompose the set $\{1, 2, \cdots, n-2k\}$ into subsets of type $\langle m-2n+2k-1, m-2n+2k-1, \cdots, m-2n+2k-1\rangle$ (k-tuple). Hence, we can decompose the set N into subsets of type $\langle m, m, \cdots, m\rangle$ (k-tuple). (ii) $m \leq 2n$ and m is odd. In this case, we can pair off the elements in $\{m-n, m-n+1, \cdots, n\}$ into $(2n-m+1)/2$ pairs each summing to m. Since $m-n-1 < n$, by induction $\{1, 2, \cdots, m-n-1\}$ can be decomposed into subsets of type $\langle m, m, \cdots, m\rangle$ ($k-(2n-m+1)/2$-tuple). We conclude this case. (iii) $m \leq 2n$ and m is even. Let $m = 2t$. We can pair off the elements in $\{m-n, m-n+1, \cdots, t-1, t+1, \cdots, n\}$ into $(2n-m)/2$ pairs, each summing to m. By induction, the set $\{1, 2 \cdots, m-n-1\}$ can be decompoed into subsets of type $\langle t, t, \cdots, t\rangle$ ($(m-2n+2k-1)$-tuple). Since $m-2n+2k-1$ is odd, we can pair off two parts to obtain the sum $m = 2t$, and the only part left can be paired with $\{t\}$. This concludes the proof.

As you can easily see, Lemma 2.1 is a very special case of a decomposition of the set N. An obvious necessary condition for the set N to be decomposed into subsets of type $\langle m_1, m_2, \cdots, m_k\rangle$ is $m_i \geq n$ for some $i \in \{1, 2, \cdots, k\}$. The author believes that the classification of types in which N can be decomposed is a hard problem, but it seems possible that we can decompose the set N into subsets of type $\langle m_1, m_2, \cdots, m_k\rangle$ provided $m_i \geq n$ for each $i \in \{1, 2, \cdots, k\}$ and $\sum_{i=1}^{k} m_i = \binom{n+1}{2}$. So far this question is still unsolved. Recently, D. Hoffman [4] has proved that if $m_i = m$ or $m+1$, where $m \geq n$, then N can be decomposed into subsets of type $\langle m_1, m_2, \cdots, m_k\rangle$. With this proposition we are able to find some classes of graphs which have an *ASD*. The idea is quite simple to obtain. If G is a graph with $\binom{n+1}{2}$ edges which can be partitioned into edge-disjoint paths with the lengths equal to $m_1, m_2, \cdots, m_k$ respectively, and the set N can be decomposed into subsets of type $\langle m_1, m_2, \cdots, m_k\rangle$. Then we can find an *ASD* for G by letting G_i, $i = 1, 2, \cdots, n$, be a path of length i which is obtained from the path of length m_j if $i \in A_j$, where $s(A_j) = m_j$ and $N = \bigcup_{i=1}^{k} A_i$. Hence, we have the following theorem.

THEOREM 2.2. *If a graph G on $\binom{n+1}{2}$ edges can be partitioned into edge disjoint paths with the length of each path either m or $m+1$, $m \geq n$, then G has an ASD with each member a path.*

THEOREM 2.3. *If a graph G on $\binom{n+1}{2}$ edges has a vertex covering pattern $\langle m, m, \cdots, m+1, \cdots, m+1\rangle$, $m \geq n$, then G has an ASD with each member a star.*

Proof. It is similar to the proof of Theorem 2.2.

We note here, $\langle \beta_1, \beta_2, \cdots, \beta_k\rangle$ is a vertex covering pattern for a graph G if we can find a vertex covering $\{v_1, v_2, \cdots, v_k\}$ such that there are β_i edges incident with the vertex v_i, $i = 1, 2, \cdots, k$ and each edge is counted only once.

COROLLARY 2.4. *If $G=(A, B)$ is a bipartite graph on $\binom{n+1}{2}$ edges and for each $v \in A$, the degree of v is either m or $m+1$, $m \geq n$, then G has an ASD with each member a star.*

Now we are ready to prove the following theorem.

THEOREM 2.5. *Every complete bipartite graph on $\binom{n+1}{2}$ edges has an ASD.*

Proof. Let $G=(A, B)$ be a complete bipartite graph on $\binom{n+1}{2}$ edges with the degrees of vertices in A and B being d, e respectively. If $d \geq n$ or $e \geq n$, then by Corollary 2.4, G has an *ASD* with each member a star. The only case left is $(n+1)/2 < d$, $e < n$. We consider $v_1, v_2, \cdots, v_e$, and let $t=n-d$. Denote the star induced by the edges which are incident with v_i as S_i, $i=1, 2, \cdots, e$. Since $e > 2t+1$ and $d \cdot (2t+1) = \binom{n+1}{2} - \binom{2d-n}{2}$, $\{1, 2, \cdots, 2d-n-1\}$ can be decomposed into subsets of type $\langle d, d, \cdots, d\rangle$($(e-2t-1)$-tuple). This implies that the subgraph of G induced by the union of stars $S_1, S_2, \cdots, S_{e-2t-1}$ has an *ASD*, $G_1, G_2, \cdots, G_{2d-n-1}$, where G_i is a star with i edges, $i=1, 2, \cdots, 2d-n-1$. Now we can construct an *ASD* for G by letting G_{2d-n+i} be the subgraph obtained by deleting $t-i$ edge(s) from the star S_{e-2t+i}, $i=0, 1, \cdots, t-1$. Let $G_d = S_{e-t}$, and G_{d+j} be the graph obtained by adding the j edges deleted from $S_{e-2t+t-j}$ to S_{e-t+j}, $j=1, 2, \cdots, t$. Because G is a complete bipartite graph, this construction gives an *ASD* for G. This concludes the proof. (See Fig. 2.1 for G_{d+j}.)

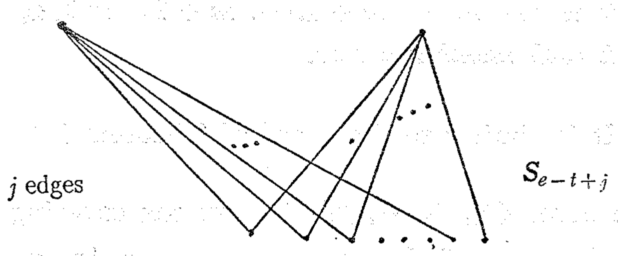

FIG. 2.1.

One more result is worthy of mention.

THEOREM 2.6. *Let G be a graph on $\binom{n+1}{2}$ edges. If G can be partitioned into n edge disjoint subgraphs G_i, $i=1, 2, \cdots, n$, such that $|E(G_i)|=i$ and for each $k \in \{2, 3, \cdots, n\}$, there is at most one edge of G_k which is incident with some vertex of the edge induced subgraph $\bigcup_{i=1}^{k-1} G_i$, then G has an ASD.*

Proof. We prove it by induction. Obviously, it is true $n=1$. Assume it is true for $n=k$, i.e. $\bigcup_{i=1}^{k} G_i$ has an *ASD* with its members $H_1, H_2, \cdots, H_k$. Since G_{k+1} has $k+1$ edges, let them be $e_0, e_1, e_2, \cdots, e_k$. Without loss of generrality, let e_0 be the (possible) edge which is incident with some vertex of the edge induced subgraph $\bigcup_{i=1}^{k} G_i$. It is not difficult to see that $\{e_0\}$, $\{e_1\} \cup H_1, \cdots$, $\{e_k\} \cup H_k$ forms an *ASD* for $\bigcup_{i=1}^{k+1} G_i$. We conclude the proof.

A bit of reflection, we have the following corollary.

COROLLARY 2.7. *If G is a disjoint (vertex and edge) union of n graphs $G_1, G_2, \cdots, G_n$ such that $|E(G_i)|=i$, $i=1, 2, \cdots, n$, then G has an ASD.*

3. **Remarks.** We have included in this paper some results on the ascending subgraph decomposition problem, but, the general problem is far from being solved. We do believe more work could be done in this direction. Finally, I would like to appreciate the helpful comments of Professor D. G. Hoffman, P. Frankl and the referees.

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