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REMARKS ON TRANSFORMATION FORMULAS FOR p-SYMBOLS

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Abstract. For a purely inseparable extension K of exponent 1 and degree p^m over a field F of characteristic p, the Brauer group ${}_p\mathrm{Br}(K/F)$ of classes of p-algebras split by K—being products of m p-symbols—is a homomorphic image of F^m whose kernel is described via transformation formulas is given using Mammone's approach by differential p-crossed products.

1. Introduction. Let F be a field of characteristic $p>0$. For any a, $b\in F$, $b\neq 0$, the p-symbol $[a, b)_F$, first introduced and studied systematically by H. L. Schmid [6], is the class of the following central simple p-algebra in the Brauer group $\mathrm{Br}(F)$,

$$[a, b) = \bigoplus_{0\le i,\, j\le p-1} F x^i y^j$$

$$\mathscr{P}(x) = x^p - x = a, \qquad y^p = b$$

$$yx = (x+1)\, y\,.$$

For any a, $b\in F$, the Jacobson p-symbol $(a, b)_F$ is the following p-algebra class [4]

$$(a, b) = \bigoplus_{0\le i,\, j\le p-1} F x^i y^j$$

$$x^p = a, \qquad y^p = b$$

$$yx = xy + 1\,.$$

If K is a purely inseparable field extension of F of exponent 1 and degree p^m over F, i.e. $K=F(\xi_1,\cdots,\xi_m)$ with $0\neq \xi_i^p = b_i\in F$,

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$i=1,\cdots,m$, it is well-known that the Brauer group ${}_p\mathrm{Br}(K/F)$ of classes of central simple p-algebras A split by K is completely described as sums of p-symbols:

$$A=\begin{cases}\bigoplus_{1\le i\le m}[a_i,\ b_i)\\ \bigoplus_{1\le i\le m}(a_i\,b_i^{-1},\ b_i)\end{cases}$$

for some $a_i\in F$, $i=1,\cdots,m$. (See e.g. Albert [1, Ch. VII, Th. 28]). This fact amounts to saying that the mapping

$$F^m \twoheadrightarrow {}_p\mathrm{Br}(K/F)$$

$$a=(a_1,\cdots,a_m)\longmapsto \bigoplus_{1\le i\le m}[a_i,\ b_i)=\bigoplus_{1\le i\le m}(a_i\,b_i^{-1},\ b_i)$$

is onto (Here we use the known fact that $(a,b)=[ab,b)$, (see e.g. [2, Lemma 1]).

In fact, this mapping is a homomorphism. So what is the kernel? Teichmüller [6, Satz 26] has proved that the kernel is equal to

$$(*)\qquad \{a=(a_1,\cdots,a_m)\in F^m\,|\,a_i=\mathrm{Sp}\,\xi_i(c_i)\ \text{ for some } c=(c_1,\cdots,c_m)\in (K^\times)^m,\ i=1,\cdots,m\},$$

where $\mathrm{Sp}\,\xi_i(c_i)=c_i^p-c_i+(d/dln\xi_i)^{p-1}(c_i)$ is the Teichmüller's trace function from K to F, and the differential $(d/dln\xi_i)$ is formally understood as $\xi_i(d/d\xi_i)$. The above results together with Teichmüller's Satz 9 actually yield the exactness of the following sequence

$$1\to F^{\times m}\to K^{\times m}\xrightarrow{\bigoplus_{1\le i\le m}\frac{dln}{dln\xi_i}} K^m\xrightarrow{\bigoplus_{1\le i\le m}\mathrm{Sp}\xi_i} F^m\to {}_p\mathrm{Br}(K/F)\to 0;$$

for the sequence in the higher exponent case (exponent of K over F is greater than one) we refer to K. Hoechsmann's article [3, p. 9].

The aim of this note is to provide a proof about the kernel (*) (Theorem in Section 3) using the concept of differential p-crossed products initiated by P. Mammone [5]. This approach seems useful in two instances. Firstly, we can treat the transformation formulas (Theorem) for both p-symbols $[\cdot,\cdot)$ and $(\cdot,\cdot)$ simultaneously, depending only on the differential $(d/dln\xi)$ and $(d/d\xi)$, respectively. Secondly, we feel that this approach seems capable of generalization to higher order derivations.

2. **Mammone's differential p-crossed products.** Let F be a field of characteristic $p>0$ and $K=F(\xi_1,\cdots,\xi_m)$ a purely inseparable field extension of F of exponent 1 and degree p^m. We consider the derivations

$$D_i := \frac{d}{dln\xi_i} = \xi_i \frac{d}{d\xi_i}$$

defined by

$$D_i \xi_j = \xi_i \delta_{ij}, \quad i, j = 1,\cdots, m. \tag{1}$$

Then the p-Lie algebra of F-derivations of K

$$\mathrm{Der}_F(K) = \langle D_1,\cdots, D_m \rangle_F$$

where $D_i^p - D_i = 0$, $i=1,\cdots,m$; which means that the minimal polynomial for each D_i is $m(X) = X^p - X$.

In this section we use $\mathcal{D} := \{D_1,\cdots, D_n\}$—analogous as Mammone did in [5] using his $\mathcal{G} = \{\tau_1,\cdots,\tau_m\}$—to construct differential p-crossed products with expectedly analogous properties.

To this end, let A be a p-algebra containing K as a maximal subfield, i.e. $[A:K]=[K:F]=p^m$, and, by virtue of Noether-Skolem's Theorem, let, for $i=1,\cdots,m$, $D_i = [\cdot, z_i]$ for some $z_i \in A$, i.e. for all $c \in K$,

$$D_i(c) = [c, z_i] = cz_i - z_i c. \tag{2}$$

Then it is easily seen that, for $i, j = 1,\cdots, m$,

$$\begin{aligned} [z_i, z_j] &= u_{ij} \in K \\ z_i^p - z_i &= a_i \in K \end{aligned} \tag{3}$$

and the (u_{ij}) and (a_i) satisfy the following properties

(i) $\qquad u_{ii} = 0; \quad u_{ij} = -u_{ji};$

(ii) $\qquad D_i(u_{jk}) + D_j(u_{ki}) + D_k(u_{ij}) = 0;$

(iii) $\qquad D_j(a_i) = (D_i^{p-1} - 1)(u_{ij}).$

The proof of (3) is almost identical with Mammone's, whereas (i) is trivial and (ii) is the Jacobi's identity. (iii) is also immediate, since

$$D_j(a_i) = [a_i, z_j] = [z_i^p - z_i, z_j] = [z_i^p, z_j] - [z_i, z_j]$$
$$= D_i^{p-1}(u_{ij}) - u_{ij} = (D_i^{p-1} - 1)(u_{ij}).$$

Following Mammone, the differential p-crossed product

$$A = \bigoplus_{\substack{0 \leq i_j \leq p-1 \\ 1 \leq j < m}} K z_1^{i_1} \cdots z_m^{i_m}$$

$$\mathcal{P}(z_i) = z_i^p - z_i = a_i \in K, \qquad [z_i, z_j] = u_{ij} \in K$$

$$D_i(\xi_i) = [\xi_i, z_i] = \xi_i$$

is denoted by $(K/F, \mathcal{D}, U, B)$, $U = (u_{ij})$, $B = (a_i)$.

Next, we mimic Mammone's construction of $(K/F, \mathcal{D}, U, B)$, for a given pair of (U, B) which satisfy (i), (ii) and (iii). However, proofs concerning the centrality and the simplicity of $(K/F, \mathcal{D}, U, B)$, will be given directly using only standard arguments without referring to Cauchon's thesis as Mammone did in the proof of his theorem (2.1).

Construction. Let

$$A_1 := F(\xi_1)[z_1; D_1] = \bigoplus_{0 \leq i_1 \leq p-1} F(\xi_1) z_1^{i_1}$$

$$z_1^p - z_1 = a_1, \qquad [\alpha, z_1] = D_1(\alpha), \qquad \alpha \in F(\xi_1)$$

be the differential polynomial ring. For $k \leq m$ define inductively,

$$A_k := F(\xi_1, \cdots, \xi_k)[z_1, \cdots, z_k; D_1, \cdots, D_k]$$

$$z_i^p - z_i = a_i; \qquad D_j^*(z_i) = [z_i, z_j] = u_{ij} \tag{4}$$

$$[\alpha, z_i] = D_i(\alpha), \qquad \alpha \in F(\xi_1, \cdots, \xi_k), \qquad i \leq j \leq k.$$

where D_j^* is the extended derivation of D_j to A_{j-1}. By abuse of notation we write $D_j^* = D_j$.

Compatibility. D_j^* is well-defined on A_{j-1}, since the equations (i) and (ii) guarantee the compatibility of D_j^* with the second and the third equation of (4) whereas (iii) that of the first one.

Now, we show that

$$A_m = (K/F, \mathcal{D}, U, B)$$

where

$$A_m = \bigoplus_{\substack{0\le i_j \le p-1 \\ 1\le j\le m}} K z_1^{i_1}\cdots z_m^{i_m} = \bigoplus_{0\le i\le p-1} [A_{m-1}\otimes_F F(\xi_m)]\, z_m^i$$

which, by (i) and (ii), is an associative algebra. By induction over k, $1\le k\le m$, we prove that A_m is central simple over F.

Centrality. Let $A_k = \bigoplus_{0\le i\le p-1}[A_{k-1}\otimes_F F(\xi_k)]\, z_k^i$. By induction hypothesis the center of A_{k-1}, $C(A_{k-1})=F$ hence $C(A_{k-1}\otimes_F F(\xi_k)) = F(\xi_k)$. Now, let $f(z_k)=\sum_{0\le i\le l} c_i z_k^i \in C(A_k)$ with $c_l\neq 0$, $l\le p-1$ and $c_0,\cdots, c_l\in A_{k-1}\otimes_F F(\xi_k)$. First we claim that $l=0$. Otherwise, using the equality $[z_k^i, \xi_k]=\xi_k[(z_k-1)^i - z_k]$ we would get a contradiction $0=[f,\xi_k]=[\sum_{0\le i\le l} c_i z_k^i, \xi_k]=\xi_k[f(z_k-1)-f(z_k)]$ since $f(z_k-1)-f(z_k)$ has degree exactly $l-1\ge 0$. Thus $f=c_0 = \sum_{0\le i\le p-1} a_i \xi_k^i \in C(A_{k-1}\otimes_F F(\xi_k)) = F(\xi_k)$. But now $0=[f, z_k] = D_k f = \sum_{1\le i\le p-1} i a_i \xi_k^i$ hence $a_1=a_2=\cdots=a_{p-1}=0$. This proves $f=a_0\in F$.

Simplicity. Let I be a non-zero two-sided ideal of A_k and $f(z_k)=\sum_{0\le i\le l} c_i z_k^i \in I$ with $c_l\neq 0$, $c_0, c_1,\cdots, c_l\in A_{k-1}\otimes_F F(\xi_k)$ such that l is minimal. As in the proof of centrality we claim that $l=0$. If not, as before, we would get $[f,\xi_k]=\xi_k[f(z_k-1) - f(z_k)]\in I$ with degree equal to $l-1$, a contradiction. Thus $0\neq f=c_0\in I\cap[A_{k-1}\otimes_F F(\xi_k)] = A_{k-1}\otimes_F F(\xi_k)$ since $A_{k-1}\otimes_F F(\xi_k)$ is central simple over $F(\xi_k)$, by induction. But this implies $1\in I$.

Altogether we have proved the following

THEOREM A. *Let K be a purely inseparable extension over F of exponent* 1 *and degree p^m, $K=F(\xi_1,\cdots,\xi_m)$ with $\xi_i^p=b_i\in F$, $i=1,\cdots,m$, and the p-Lie algebra of derivations.* $\mathrm{Der}_F(K) = \langle D_1,\cdots, D_m\rangle$. *For $U=(u_{ij})\in M_m(K)$ and $B=(a_i)\in K^m$ satisfying the properties* (i), (ii) *and* (iii) *there exists a differential p-crossed product of the form*

$$A_m = (K/F, \mathcal{D}, U, B).$$

3. **Transformation formulas.** By Theorem A the Brauer group ${}_p\mathrm{Br}(K/F)$ of classes of p-algebras split by a purely inseparable

splitting field $K=F(\xi_1\cdots, \xi_m)$ of exponent 1 and degree p^m can be completely described as follows: The mapping

$$\begin{aligned}\mathscr{Z}=M_m(K)\times K^m &\twoheadrightarrow {}_p\mathrm{Br}(K/F)\\ (U,B)&\longmapsto (K/F,\mathscr{D},U,B)\end{aligned}$$

is an onto homomorphism [5, Th. 3.4] with the kernel equal to

$$(**)\qquad \mathscr{B}=\{(U,B)\,|\,u_{ij}=D_j(c_i)-D_i(c_j),\ a_i=D_i^{p-1}(c_i)+c_i^p-c_i \text{ for some } c_i\in K,\ i,j=1,\cdots,m\}.$$

To prove (**) we follow Mammone's argument used in the proof of his Theorem 3.3. So suppose $(K,\mathscr{D},U,B)\simeq(K,\mathscr{D},U',B')$, and $(y_i)_{1\le i\le m}$ and $(z_i)_{1\le i\le m}$ are their systems of generators, respectively. Then $y_i=z_i+c_i$ for some $c_i\in K$, $i=1,\cdots,m$. Obviously, we have

$$u_{ij}=u'_{ij}+D_j(c_i)-D_i(c_j).$$

and

$$D_i+l_{c_i}=r_{y_i}-l_{z_i}$$

where l_x, r_x are the left and the right multiplication function by x, respectively. Raising to the p-power we have

$$(D_i+l_{c_i})^p=(r_{y_i}-l_{z_i})^p$$

and by Jacobson's formula

$$l_{(D_i^{p-1}(c_i)+c_i^p)}=r_{y_i^p}-l_{z_i^p};$$

hence

$$\begin{aligned}a_i-a'_i&=(y_i^p-y_i)-(z_i^p-z_i)\\ &=D_i^{p-1}(c_i)+c_i^p-c_i.\end{aligned}$$

This essentially proves (**); the rest of Mammone's arguments carries over analogously,

To formulate the transformation formulas we specialize to the case (U,B) where $U=(u_{ij})=0$. By (iii) this implies $B=(a_i)\in F^m$. Note that $\mathrm{Sp}\,\xi_i(c)=D_i^{p-1}(c_i)+c_i^p-c_i$ (see Introduction). Thus (**) yields the following

THEOREM (Transformation formula for p-symbols). *Let K be a purely inseparable extension over F of exponent* 1 *and degree*

p^m, $K=F(\xi_1,\cdots,\xi_m)$ *with* $\xi_i^p=b_i\in F$, $i=1,\cdots,m$, *and the p-Lie algebra of derivations* $\mathrm{Der}_F(K)=\langle D_1,\cdots,D_m\rangle_F$. *For* $(a_i)_{1\le i\le m}$, $(a_i')_{1\le i\le m}\in F^m$ *we have*:

(a) *If* $D_i=\dfrac{d}{dln\xi_i}$, *then*

$$\bigoplus_{1\le i\le m}[a_i,b_i)=\bigoplus_{1\le i\le m}[a_i',b_i)\Leftrightarrow a_i-a_i'=\mathrm{Sp}\,\xi_i(c_i),$$

for some $(c_i)_{1\le i\le m}\in K^m$.

(b) *If* $D_i=\dfrac{d}{d\xi_i}$, *then*

$$\bigoplus_{1\le i\le m}(a_i,b_i)=\bigoplus_{1\le i\le m}(a_i',b_i)\Leftrightarrow a_i-a_i'=T\xi_i(c_i),$$

for some $(c_i)_{1\le i\le m}\in K^m$.

(*Here* $T\xi_i(c_i)=D_i^{p-1}(c_i)+c_i^p$ *is the Jacobson's trace function from* K *to* F).

References

1. A. A. Albert, *Structure of Algebras*, Amer. Math. Soc. Colloquium Publ. vol 24, Providence, 1961.
2. H. Chu, M. C. Kang and E.-T. Tan, *The corestriction of p-symbols*, Communications in Algebra, **16** (1988), 735–741.
3. K. Hoechsmann, *Simple algebras and derivations*, Trans. Amer. Math. Soc. **108** (1963), 1–12.
4. N. Jacobson, *p-algebras of exponent p*, Bull. Amer, Math. Soc. **43** (1937), 667–670.
5. P. Mammone, *Remarques sur les produits croisés différentiels*, Acad. Roy. Belg. Bull. Cl. Sci (5) **68** (1982), no. 10, 651–664.
6. O. Teichmüller, *p-Algebren*, Deutsche Math. 1 (1936), 362–388.

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