BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 16, Number 3, September 1988

ON THE RATE OF MOMENT CONVERGENCE OF SAMPLE SUMS AND EXTREMES

BY

Y. S. CHOW (周元燊)

Abstract. Let X_n be i.i.d., $S_n = X_1 + \ldots + X_n$, $S_n^* = \max_{j\le n}|S_j|$ and $X_n^* = \max_{j\le n}|X_j|$. Let $G(t) = P(|X_1| > t)$ and $I_p = \int_0^\infty (G^{-1+1/p}(t) + \log(1+t))\,G(t)\,dt$. Then for $p \geq 1$, $\sum n^{-1-1/p} EX_n^* < \infty$ if and only if $I_p < \infty$ and for $2 > p \geq 1$, $\sum n^{-1-1/p} ES_n^* < \infty$ if and only if $I_p < \infty$. Some applications are given.

1. Introduction. Let $X, X_1, X_2, \cdots$ be independent, identically distributed, $S_n = \sum_1^n X_j$, $X_n^* = \max_{j\le n}|X_j|$ and $1 \le p < \infty$. The moment convergence has been studied by Pickands (1968) for X_n^*; and by Brown (1970) and Pyke and Root (1968) for S_n. The work of Baum and Katz (1965), Gut (1983), and Tomkins (1986) are devoted to the rate of convergence to S_n and X_n^* respectively, in the sense of complete convergence of Hsu and Robbins (1947). Recently, Lai (1976) and Hall (1982) have given some results on the rate of moment covergence of S_n in terms of series of moments of $|S_n|$.

In this paper, we are interested in the rate of moment convergence of S_n and X_n^*, in terms of some series of $E|S_n|$, $E|S_n|^p$ and EX_n^*. The results are presented in Section 2. Our results on S_n are mainly for the cases $E|X|^p < \infty$, $1 \le p < 2$, and can be considered as a complement of those of Lai (1978) and Hall (1982). We do not assume special distributions for results on X_n^*, as assumed in the work of Pickands (1968) and Tomkins (1986).

Received by the editors February 5, 1988.
AMS 1980 subject classification. 60F10, 60F25, 60J15, 60K15.
Key words and phrases. Moment convergence, complete convergence, rate of convergence, sample maximum, stopping times, split times.

The main results are Theorem 2.1 and 2.2, which establish that for $EX=0$ and $1<p<2$, the covergence of $\sum n^{-1-1/p}E|S_n|$ and $\int_0^\infty P^{1/p}(|X|>t)\,dt$ are equivalent; and for $p>1$, the convergence of $\sum n^{-1-1/p}EX_n^*$ and $\int_0^\infty P^{1/p}(|X|>t)\,dt$ are equivalent. As applications, we extend a result of Athreya and Karlin (1967) for $p=1$ to $1<p<2$ in Theorem 2.4 and give a rate of convergence result in renewal theorems in Theorem 2.6. Theorem 2.5 and 2.7 strengthen some results of Baum and Katz (1965) and Gut (1983) respectively.

2. **Main results.** Let X be a fixed, nonnegative random variables with distribution $F(t)=1-G(t)=P(X\le t)$ and $(X_n, n\ge 1)$ be a sequence of independent random variables, with $S_n=\sum_1^n X_j$. Let $(\mathfrak{F}_n, n\ge 1)$ be a sequence of increasing σ-algebras such that $\mathfrak{F}_n\supset\sigma(X_1,\cdots,X_n)$ and $\mathfrak{F}_n$ and $\sigma(X_{n+1})$ are independent for each $n\ge 1$.

For $p\ge 1$, denote

$$
\begin{aligned}
I_p(X) &= EX\log^+ X, \qquad p=1,\\
&= \int_0^\infty G^{1/p}(t)\,dt, \qquad p>1,
\end{aligned}
\tag{2.1}
$$

and for any sequence of real numbers $(a_n, n\ge 1)$, put

$$
a_n^*=\max_{j\le n}|a_j|,\qquad a_{n_m}^*=\max_{j\le n_m}|a_j|,
\tag{2.2}
$$

where $(n_m, m\ge 1)$ is a subsequence of n.

THEOREM 2.1. *Let $(X_n, n\ge 1)$ be independent, $EX_n=0$, $P(|X_n|>t)\le G(t)$ for $n\ge 1$ and $(T_m, m\ge 1)$ be $\mathfrak{F}_n$-stopping times such that $ET_m=O(m)$ as $m\to\infty$.*

(i) *If $I_p<\infty$ for some $1\le p<2$, then*

$$
\sum n^{-1-1/p}ES_{T_n}^*<\infty .
\tag{2.3}
$$

(ii) *If for $1<p<2$,*

$$
EX^p\log^+X<\infty,
\tag{2.4}
$$

then

$$
\sum n^{-2}ES_{T_n}^{*p}<\infty .
\tag{2.3$'$}
$$

(iii) *Conversely, assume further that* $P(|X_n|>t)=G(t)$ *for* $n\geq 1$ *and* $t>0$, $p\geq 1$, *and for some positive* α *with* $\alpha_m=[\alpha m]$,

$$\lim P(T_m<\alpha_m)=0. \tag{2.5}$$

If either

$$\sum n^{-1-1/p}E|S_n|<\infty \tag{2.6}$$

or

$$\sum n^{-1-1/p}E|S_{T_n}|<\infty, \tag{2.7}$$

then $I_p<\infty$; *and if either*

$$\sum n^{-2}E|S_n|^p<\infty \tag{2.6$'$}$$

or

$$\sum n^{-2}E|S_{T_n}|^p<\infty, \tag{2.7$'$}$$

then (2.4) *holds.*

Theorem 2.1 can be considered as some results on the rate of moment convergence for S_n/n. The cases for $2<p<4$ have been treated by Lai (1976) and Hall (1982).

THEOREM 2.2. *Let* $(X, X_n, n\geq 1)$ *be independent, identically distributed and* $(T_m, m\geq 1)$ *be finite* $\mathfrak{F}_n$*-stopping times.*

(i) *For* $p\geq 1$, $I_p<\infty$ *iff (if and only if)*

$$\sum n^{-1-1/p}EX_n^*<\infty. \tag{2.8}$$

(ii) *If for some positive* α *with* $\alpha_m=[\alpha m]$ *and for some* $p\geq 1$,

$$\lim P(T_m<\alpha_m)=0,\qquad \sum n^{-1-1n/p}EX_{T_n}<\infty, \tag{2.9}$$

then $I_p<\infty$.

(iii) *For* $p>1$ *and* $q>p-1$, *if*

$$EX^p(\log^+X)^q<\infty, \tag{2.10}$$

then $I_p < \infty$. *Conversely, if* $I_p < \infty$ *for some* $p > 1$, *then* $EX^p < \infty$.

Part (i) of Theorem 2.2 follows from Theorem 2.1 (i) when $1 \leq p < 2$. However, for $p \geq 2$ we need a separate proof.

Part (iii) of Theorem 2.2 gives some conditions for I_p being finite. It says that $I_p < \infty$ is slightly stronger than $EX^p < \infty$. The condition $q > p - 1$ can not be improved to $q \geq p - 1$. For example, if for some $p > 1$ and all $t > 10$,

$$G(t) = C \cdot (t \log t \log \log t)^{-p}. \tag{2.11}$$

Then $I_p = \infty$ and by Lemma 3.1

$$EX(\log^+ X)^{p-1} < \infty .$$

However, the proof of Theorem 2.2 (iii) is still valid (with some modifications) if (2.10) is replaced by $EX^p(\log^+ X)^{p-1}(\log^+ \log^+ X)^q < \infty$ for some $q > p - 1$.

From Theorems 2.1 and 2.2, immediately we have Corollary 2.1.

COROLLARY 2.1. *Let* $(X_n, n \geq 1)$ *be independent, identically distributed,* $EX_1 = 0$ *and* $(P|X_1| > t) = G(t)$. *Then* $I_p < \infty$ *iff*

$$\sum n^{-1-1/p} EX_n^* > \infty, \tag{2.12}$$

when $p \geq 1$, *and iff*

$$\sum n^{-1-1/p} E|S_n| < \infty \tag{2.13}$$

when $1 \leq p < 2$.

COROLLARY 2.2. *For any sequence* $(X_n, n \geq 1)$ *of independent, identically distributed random variables,*

$$\sum n^{-3/2} E|S_n| = \infty. \tag{2.14}$$

Proof. Let (2.14) be false. The $E|X_1| < \infty$, $EX_1 = 0$, and by Theorem 2.1 (iii) and Theorem 2.2 (iii), $EX_1^2 < \infty$. Then

$$\lim n^{-1/2} E|S_n| > 0,$$

yielding a contradiction.

Corollary 2.2 is an immediate consequence of Klass (1980). He proves that if $EX_1 = 0$, then

$$A^{-1} K_n \le E|S_n| \le AK_n \tag{2.15}$$

for some $0 < A < \infty$, where

$$n^{-1} K_n^2 = E(X_1^2 \wedge |X_1| K_n). \tag{2.16}$$

Hence $n^{-1/2} K_n \uparrow$ and (2.14) follows by (2.15).

It seems also plausible to apply (2.15) in the derivation of (2.13). However, there are some difficulties. Klass (1980) proves that for some constants $\lambda_n = O(1)$, $G(\lambda_n E|S_n|) \le 1/n$.

Let $\tilde{G}(t)$ be the inverse function of G (see (3.1) for definition). Then

$$\tilde{G}(1/n) = O(E|S_n|) = O(K_n). \tag{2.17}$$

By letting $t = \tilde{G}(1/y)$, for $p > 1$,

$$\begin{aligned} I_p &= \int_0^\infty G^{1/p}(t)\, dt = -p^{-1} \int_0^\infty tG^{-1+1/p}(t)\, dG(t) \\ &= \int_1^\infty y^{-1-1/p}\, \tilde{G}(1/y)\, dy. \end{aligned}$$

Hence $I_p < \infty$ iff

$$\sum n^{-1-1/p}\, \tilde{G}(1/n) < \infty, \qquad p > 1. \tag{2.18}$$

Corollary 2.1 states that (2.18) implies

$$\sum n^{-1-1/p} K_n < \infty. \tag{2.19}$$

If $G(t) = t^{-2}$ for $t \ge 1$, then $n^{-1} K_n^2 \to \infty$ and $\tilde{G}^2(1/n)/n = 1$. Hence $K_n/\tilde{G}(1/n) \to \infty$ and therefore in general, K_n and $\tilde{G}(1/n)$ are not of the same order.

In a related work, Tomkins (1986) gives some necessary and sufficient conditions for the series

$$\sum n^{p-2} P(|X_n^* / \tilde{G}(1/n) - 1| > t) < \infty, \qquad p > 1, \tag{2.20}$$

and proves that if $|X_1|$ is not bounded and (2.20) holds for some $p > 1$, then it holds for all $p > 1$. Hence the series (2.12) and (2.20) are different in nature.

Instead of considering the series, by just considering one term, one obtains the following result.

THEOREM 2.3. *Let* $(X_n, n \geq 1)$ *be independent, identically distributed,* $EX_n = 0$, *and* $P(|X_1| > t) = G(t) = 1 - F(t)$ *for* $t > 0$.

(i) *For* $p > 1$, *as* $n \to \infty$

$$n^{-1/p} EX_n^* = o(1), \tag{2.21}$$

iff as $t \to \infty$

$$t^p G(t) = o(1). \tag{2.22}$$

(ii) *For* $1 < p < 2$, (2.22) *holds iff*

$$n^{-1/p} E|S_n| = o(1). \tag{2.23}$$

The convergence of EX_n^* has been discussed in Pickands (1968). His results are different form ours. Pyke and Root (1968) prove that $n^{-1}E|S_n|^p = o(1)$ for $1 \leq p < 2$ iff $E|X_1|^p < \infty$. Theorem 2.3 (ii) is a variation of their work.

As an application of Theorem 2.1, we will derive the following theorem, which improves some results due to Athreya and Karlin (1967) on split times. They show that for $p = 1$, (2.25) holds iff $EX \log^+ X < \infty$. Under the same conditions, (2.24) holds and (2.25) converges absolutely a. s.

THEOREM 2.4. *Let* $(X_n, n \geq 1)$ *be independent, identically distributed,* $X_1 \geq 0$ *a. s.,* $EX_1 = 1$, $P(|X_1 - 1| > t) = G(t)$, $0 < \beta < 1$ *and* $1 \leq p < 2$.

(i) $I_p < \infty$ *iff*

$$\sum n^{1-1/p} E\left|\frac{1}{S_n + \beta} - \frac{1}{n}\right| < \infty. \tag{2.24}$$

(ii) $EX^p \log^+ X < \infty$ *iff*

$$\sum n^{2p-2} E\left|\frac{1}{S_n + \beta} - \frac{1}{n}\right|^p < \infty. \tag{2.24$'$}$$

(iii) *If* $E|X_1|^p < \infty$ *for some* $1 < p < 2$, *then*

$$\sum n^{1-1/p}\left(\frac{1}{S_n + \beta} - \frac{1}{n}\right) \text{ converges a. s.} \tag{2.25}$$

The following theorem is suggested by the work of Baum and Katz (1965) on the complete convergence for the law of large numbers. It extends their results in connection with ES_n.

THEOREM 2.5. *Let* $(X_n, n \geq 1)$ *be independent, identically distributed,* $EX_1 = 0$, $p \geq 1$, $\alpha > 1/2$ *and* $\alpha p \geq 1$. *Then for* $\varepsilon > 0$,

$$\sum n^{p\alpha-\alpha-2} E(S_n^* - \varepsilon n^\alpha)^+ < \infty, \tag{2.26}$$

if

$$E(|X_1|^p + |X_1| \log^+ |X_1|) < \infty. \tag{2.27}$$

As another application of Theorem 2.1, we have the following results on the rate of convergence in renewal theorems.

THEOREM 2.6. *Let* $(X_n, n \geq 1)$ *be independent,* $EX_n = \mu$, $0 < \mu < \infty$, *and* $P(|X_n - \mu| > t) \leq G(t) = P(X > t)$ *for* $n \geq 1$ *and* $t > 0$. *For* $t > 0$, *let*

$$N_t = \inf\{n \geq 1 : S_n > t\}. \tag{2.28}$$

(i) *Assume* $1 \leq p < 2$. *If* $I_p < \infty$, *then*

$$\sum n^{-1-1/p} E(\mu N_n - n)^* < \infty, \tag{2.29}$$

and

$$\sum n^{-1-1/p} E(S_{N_{n\mu}} - S_n)^* < \infty. \tag{2.30}$$

If $EX^p \log^+ X < \infty$, *then*

$$\sum n^{-2} E(\mu N_n - n)^{*p} < \infty, \tag{2.29$'$}$$

and

$$\sum n^{-2} E(S_{N_{n\mu}} - S_n)^{*p} < \infty. \tag{2.30$'$}$$

(ii) *Suppose further that* $(X_n, n \geq 1)$ *are identically distributed,* $P(|X_1 - \mu| > t) = G(t)$ *and* $\zeta_n = \zeta^{(1)} + \cdots + \zeta^{(n)}$, *where* $\zeta^{(1)} = N_1, \zeta^{(2)}, \cdots$ *are copies of* N_1.

(α) *If* $I_p < \infty$ *for some* $1 \leq p < 2$, *then*

(2.31) $$\sum n^{-1-1/p} E(\zeta_n - nEN_1)^* < \infty,$$

(2.32) $$\sum n^{-1-1/p} E(\mu N_{nEN_1} - \zeta_n)^* < \infty,$$

and

(2.33) $$\int_1^\infty (P^{1/p}(N_1 > t) + P(N_1 > t) \log t)\, dt < \infty .$$

(β) *If* $EX^p \log^+ X < \infty$ *for some* $1 \le p < 2$, *then*

(2.31)′ $$\sum n^{-2} E(\zeta_n - nEN_1)^{*p} < \infty,$$

(2.32)′ $$\sum n^{-2} E(\mu N_{nEN_1} - \zeta_n)^{*p} < \infty,$$

and

(2.33)′ $$EN_1^p \log N_1 < \infty .$$

(γ) *If for some* $p \ge 1$,

(2.34) $$\sum n^{-1-1/p} E(\mu N_n - n)^+ < \infty,$$

then $I_p < \infty$.

In the i. i. d. case, Theorem 2.6 states that for $1 \le p < 2$, (2.34) holds if and only if $I_p < \infty$, which is obviously equivalent to

$$\int_1^\infty (P^{1/p}(|X_1| > t) + P(|X_1| > t) \log t)\, dt < \infty,$$

which is a two-sided conditions. Gut (1974A) proves that for $p \ge 1$ $EN_1^p < \infty$, if and only if $E(X_1^-)^p < \infty$, a one-sided condition. The following result gives some one-side conditions in the form of I_p when $1 \le p < 2$. However, when $p \ge 2$, we are not able to prove it.

COROLLARY 2.3. *Let* $(X, X_n, n \ge 1)$ *be independent, identically distributed,* $EX = \mu \in (0, \infty)$ *and for* $t \ge 0$

$$N_t = \inf \{n \ge 1 : S_n > t\}.$$

Then for any $\theta > \mu$,

(2.35) $$\liminf_{y\to\infty} P(N_0 > y)/P(X^- > \theta y) \ge 1 .$$

In particular, for $1 \le p < 2$

$$\int_1^\infty (P^{1/p}(X^- > t) + P(X^- > t)\log t)\,dt < \infty \tag{2.36}$$

iff

$$\int_1^\infty (P^{1/p}(N_0 > t) + P(N_0 > t)\log t)\,dt < \infty; \tag{2.37}$$

and

$$E(X^-)^p \log^+ X^- < \infty \tag{2.36$'$}$$

iff

$$EN_0^p \log N_0 < \infty. \tag{2.37$'$}$$

In (2.37) *and* (2.37)′, N_0 *can be replaced by* N_ξ, $\xi > 0$.

Proof. For $y > 1$, by conditioning X_1,

$$P(N_0 > y) = \int_{0+}^\infty P(1 + N_t > y)\,dP(0 \le -X < t).$$

Hence for $\xi > 0$,

$$\begin{aligned} P(N_0 > y) &\ge \int_\xi^\infty P(N_t > y-1)\,dP(0 \le -X < t) \\ &\ge P(N_\xi > y-1)\,P(X^- > \xi). \end{aligned} \tag{2.38}$$

Since $N_{\theta y}/y \to \theta/\mu > 1$ a.s. for $\theta > \mu$, (2.35) holds by letting $\xi = \theta y$ in (2.38), and (2.36) is implied by (2.37). Now assume that (2.36) is valid. By truncation, we can assume that $P(X \le c) = 1$ for some $\infty > c > 0$. Then by Theorem 2.6 (ii) (α), (2.37) holds. Similarly for (2.36) and (2.37)′.

From (2.38), if $P(X^- > \xi) > 0$, then N_0 can be replaced by N_ξ in (2.37) and (2.37)′. If $P(X^- > \xi) = 0$, choose $\zeta > 0$ so that $P(X^- > \zeta) > 0$ and $n\zeta > \xi$. Then

$$P(N_\xi > y) \le P(N_{n\zeta} > y) \le nP(N_\zeta > y/n).$$

Hence the general case follows. The next theorem improves a result of Gut (1983) on the rate of convergence for first passage times when $p > 1$.

THEOREM 2.7. *Let* $(X, X_n, n \geq 1)$ *be independent, identically distributed,* $EX_1 = \mu$, $0 < \mu < \infty$, $p \geq 1$, $\alpha > 1/2$, $\alpha p \geq 1$ *and* $\varepsilon > 0$. *For* $t > 0$, *let*

$$N_t = \inf\{n \geq 1 : S_n > t\}.$$

If $p \geq 1$ *and*

$$E(|X|^p + |X| \log^+|X|) < \infty, \tag{2.39}$$

then

$$\int_1^\infty t^{\alpha p - \alpha - 2} E\left(\left|N_t - \frac{t}{\mu}\right| - \varepsilon t^\alpha\right)^+ dt < \infty. \tag{2.40}$$

3. **Proof of Theorems 2.1–2.7.** For $1 - F(t) = G(t) = P(X > t)$ for $t \geq 0$ and $G(t) = 1$ for $t < 0$, define for $0 < y \leq 1$,

$$\tilde{G}(y) = \sup\{t > 0 : G(t) > y\}. \tag{3.1}$$

Then for $0 < y \leq 1$ and $t \geq 0$.

$$\tilde{G}(t) \leq y \quad \text{iff} \quad t \geq \tilde{G}(t). \tag{3.2}$$

LEMMA 3.1. *For* α, $\beta > 0$, *the following relations are equivalent.*

$$EX^\alpha (\log^+ X)^\beta < \infty, \tag{3.3}$$

$$A \equiv \int_1^\infty t^{\alpha-1} G(t) |\log G(t)|^\beta dt < \infty, \tag{3.4}$$

$$\int_1^\infty t^\alpha |\log G(t)|^\beta dF(t) < \infty. \tag{3.5}$$

Proof. Assume (3.3). Then

$$B \equiv \int_1^\infty t^{\alpha-1} G(t) \log^\beta t dt < \infty.$$

Put $S = \{t \geq 1 : t^{\alpha+1} G(t) \leq 1\}$. Then

$$\begin{aligned} A &= \int_1^\infty t^{\alpha-1} G(t) |(\alpha+1)\log t - \log(t^{\alpha+1} G(t))|^\beta dt \\ &\leq 2^\beta (\alpha+1)^\beta B + 2^\beta \int_S t^{\alpha-1} G(t) |\log(t^{\alpha+1} G(t))|^\beta dt \\ &\leq 2^\beta (\alpha+1)^\beta B + 2^\beta \sup_{0 \leq y \leq 1} y |\log y|^\beta \int_1^\infty t^{-2} dt < \infty, \end{aligned}$$

yielding (3.4).

Assume (3.4). For $F(K+)=F(K-)$ by integration by parts,

$$\int_0^K t^\alpha |\log G(t)|^\beta dF(t)$$
$$\leq \alpha \int_0^K t^{\alpha-1} G(t)|\log G(t)|^\beta dt + \beta \int_0^K t^\alpha |\log G(t)|^{\beta-1} dF(t).$$

As $K\to\infty$, (3.5) follows from (3.4).

Finally, assume (3.5). Then $t^\alpha G(t)=o(1)$ as $t\to\infty$ and (3.3) follows.

LEMMA 3.2. *Let* $(X_n, n\geqslant 1)$ *be independent and* $(T_m, m\geqslant 1)$ *be finite* $\mathfrak{F}_n$*-stopping times such that for some positive integers* α_m,

$$\lim_{m\to\infty} P(T_m<\alpha_m)=0. \tag{3.6}$$

(i) *As* $n\to\infty$,

$$EX^*_{T_n\vee\alpha_n} \leq (1+o(1))EX^*_{T_n}. \tag{3.7}$$

(ii) *Suppose that for each* n, $EX_n=0$ *and*

$$E|S_{T_n}|<\infty, \qquad \liminf_m \int_{\{T_n>m\}} |S_m|\, dP=0. \tag{3.8}$$

Then for $p\geqslant 1$, *as* $n\to\infty$

$$E|S_{T_n\vee\alpha_n}|^p=(1+o(1))\, E|S_{T_n}|^p. \tag{3.9}$$

Proof. (i) We can assume that $EX^*_{T_n}<\infty$. By independence and (3.6),

$$\begin{aligned} E|X^*_{T_n\vee\alpha_n}-X^*_{T_n}| &= \sum_{j<\alpha_n} E(I_{\{T_n=j\}}|X^*_{\alpha_n}-X^*_j|) \\ &\leq \sum_{j<\alpha_n} E(I_{\{T_n=j\}} \max_{j<i\leq\alpha_n}|X_i|) \\ &\leq P(T_n<\alpha_n)\, EX^*_{\alpha_n}=o(EX^*_{T_n\vee\alpha_n}), \end{aligned}$$

yielding (3.7).

(ii) Put $\Delta=|S_{T_n\vee\alpha_n}-S_{T_n}|$. Then

$$\begin{aligned} E\Delta^p &= \sum_{j<\alpha_n} E(I_{\{T_n=j\}}|S_{\alpha_n}-S_j|)^p \\ &= \sum_{j<\alpha_n} P(T_n=j)E|S_{\alpha_n}-S_j|^p \\ &\leq P(T_n<\alpha_n)E|S_{\alpha_n}|^p=o(E|S_{\alpha_n}|^p), \end{aligned}$$

by independence, martingale property and (3.6). By a result of Doob (1953, p. 302), (3.8) implies that $(S_{\alpha_n}, |S_{T_n \vee \alpha_n}|)$ is a two term submartingale, and by Jensen inequality,

$$E\left(|S_{T_n \vee \alpha_n}|^p \middle| |S_{\alpha_n}|\right) \geq |S_{\alpha_n}|^p \quad \text{a.s.},$$

$$E|S_{\alpha_n}|^p \leq E|S_{T_n \vee \alpha_n}|^p, \qquad \|\Delta\|_p = o(\|S_{T_n \vee \alpha_n}\|_p).$$

Hence

$$\|S_{T_n \vee \alpha_n}\|_p \leq \|S_{T_n}\|_p + \|\Delta\|_p = \|S_{T_n}\|_p + o(\|S_{T_n \vee \alpha_n}\|_p)$$

yielding (3.9).

REMARK. In Lemma 3.2 (ii), if in (3.8) and (3.9), $|S_{T_n}|$, $|S_m|$, $S_{T_n \vee \alpha_n}$ and $|S_{T_n}|$ are replaced by $S^+_{T_n}$, S^+_m, $S^+_{T_n \vee \alpha_n}$ and $S^+_{T_n}$ respectively the result is still valid by the same proof.

Proof of Theorem 2.1. (i) For each fixed $n \geq 1$, let $h_n = \tilde{G}(1/n)$,

$$X'_j = (X_j \wedge h_n) \vee (-h_n) - E\{(X_j \wedge h_n) \vee (-h_n)\},$$

$$X''_j = X_j - X'_j, \quad S'_m = \sum_1^m X'_j, \quad S''_m = S_m - S'_m. \tag{3.10}$$

By the maximum inequality of Doob (1953, p. 317) and the Wald equation for the second moment (cf. Chow-Teicher (1978), p. 241).

$$E(S'^*_{T_n})^2 = O(1)\, E(S'_{T_n})^2 = O(1)\, E \sum_1^{T_n} EX'^2_j$$

$$= O(ET_n) \int_0^{h_n} tG(t)\, dt,$$

$$ES'^*_{T_n} = O(n^{1/2}) \int_0^{h_n} tG(t)\, dt \cdot E^{-1/2}(X \wedge h_n)^2.$$

Hence

$$\sum_1^\infty n^{-1-1/p} ES'^*_{T_n}$$

$$= O(1) \sum_1^\infty n^{-1/2-1/p} \int_0^{h_n} tG(t)\, dt \cdot E^{-1/2}(X \wedge h_n)^2$$

$$= O(1) \int_0^\infty tG(t)\, dt \cdot \sum_{h_n > t} n^{-1/2-1/p} E^{-1/2}(X \wedge h_n)^2.$$

When $h_n > t$, $E(X \wedge h_n)^2 \geq E(X \wedge t)^2 \geq t^2 G(t)$ and $nG(t) > 1$ by (3.2). Therefore for $1 \leq p < 2$

$$\begin{aligned} \sum_1^\infty n^{-1-1/p} ES'^*_{T_n} &= O(1) \int_0^\infty G^{1/2}(t)\, dt \cdot \sum_{nG(t)>1} n^{-1/2-1/p} \\ &= O\left(\int_0^\infty G^{1/p}(t)\, dt\right) < \infty . \end{aligned} \tag{3.11}$$

Now

$$\begin{aligned} ES''^*_{T_n} &\leq E \sum_1^{T_n} |X''_j| \leq E \sum_1^{T_n} E(|X_j| - h_n)^+ \\ &= O(n) \int_{h_n}^\infty G(t)\, dt, \end{aligned}$$

$$\begin{aligned} \sum_1^\infty n^{-1-1/p} ES''^*_{T_n} &= O(1) \sum_1^\infty n^{-1/p} \int_{h_n^+}^\infty G(t)\, dt \\ &= O(1) \int_0^\infty G(t)\, dt \sum_{t > h_n} n^{-1/p}. \end{aligned}$$

Since $t > h_n$ implies that $G(t) \leq 1/n$ by (3.2),

$$\begin{aligned} \sum_{t>h_n} n^{-1/p} &= O(|\log G(t)|), \qquad p = 1 \\ &= O(G^{-1-1/p}(t)), \qquad p > 1 . \end{aligned}$$

Hence

$$\begin{aligned} \sum_1^\infty n^{-1-1/p} ES''^*_{T_n} &= O(1) \int_0^\infty G^{1/p}(t)\, dt < \infty, \qquad p > 1 \\ &= O(1) \int_0^\infty G(t) |\log G(t)|\, dt < \infty, \qquad p = 1, \end{aligned} \tag{3.12}$$

by Lemma 3.1 when $p = 1$. (3.11) and (3.12) yield the proof of (2.3).

(ii) Put $h_n = n^{1/p}$ define X'_j, X''_j, S'_m and S''_m by (3.10). For $1 < p < 2$, by Doob inequality (1953, p. 317)

$$ES''^{*p}_{T_k} = O(E|S''_{T_k}|^p)$$

and by Burkholder (1966) inequality

$$\begin{aligned} E|S''_{T_n}|^p &= O(1) E\left(\sum_1^{T_n} X_j^2\right)^{p/2} = O(1)\, E \sum_1^{T_n} |X''_j|^p \\ &= O(ET_n) \int_{h_n}^\infty t^{p-1} G(t)\, dt . \end{aligned}$$

Similarly,

$$E(S_{T_n}'^*)^2 = O(ET_n)\int_0^{h_n} tG(t)\,dt\,.$$

Hence

$$\begin{aligned}\sum_2^\infty n^{-1-2/p}\log nE(S_{T_n}'^*)^2 &= O(1)\sum n^{-2/p}\log n\int_0^{h_n} tG(t)\,dt\\ &= O(1)\int_1^\infty tG(t)dt\sum_{n\ge t^p} n^{-2/p}\log n\\ &= O(1)\int_1^\infty t^{p-1}G(t)\log t dt<\infty,\end{aligned}$$

by (2.4), and by Hölder inequality for $1<p<2$

$$\begin{aligned}\sum_2^\infty n^{-2}E|S_{T_n}'^*|^p &\le \Big\{\sum n^{-1}(\log n)^{-p/(2-p)}\Big\}^{(2-p)/2}\\ &\quad\cdot\Big\{\sum n^{-1-2/p}\log nE(S_{T_n}'^*)^2\Big\}^{p/2}\\ &<\infty,\end{aligned}\tag{3.11$'$}$$

$$\begin{aligned}\sum_1^\infty n^{-2}E|S_{T_n}''^*|^p &= O(1)\sum n^{-1}\int_{h_n}^\infty t^{p-1}G(t)\,dt\\ &= O(1)\int_1^\infty t^{p-1}G(t)\,dt\sum_{n\le t^p} n^{-1}\\ &= O(1)\int_1^\infty t^{p-1}G(t)\log t dt<\infty,\end{aligned}\tag{3.12$'$}$$

yielding (2.3)′ by (3.11)′.

(iii) Since $(X_n, n\ge 1)$ are independent with $P(|X_n|>t)=G(t)$ and $ET_n=O(n)$, (3.8) is satisfied by (2.7) or (2.7)′. By Lemma 3.2 (ii), (2.7) and (2.7)′ imply (2.6) and (2.6)′ respectively with S_n being replaced by S_{α_n} where $\alpha_n=[\alpha n]$. Since $(X_n, n\ge 1)$ are independent, by a theorem of Doob (1953, p. 337),

$$ES_n^{*p}\le 8E|S_n|^p,\qquad p\ge 1,\ \ n\ge 1.\tag{3.13}$$

Therefore (2.7) and (2.7)′ imply

$$\sum n^{-1-1/p}ES_{\alpha_n}^*<\infty\tag{3.14}$$

and

$$\sum n^{-2}ES_{\alpha_n}^*<\infty\tag{3.14$'$}$$

respectively. Hence if (2.6) or (2.7) holds, (3.15) $$\sum n^{-1-1/p} ES_n^* < \infty,$$

and if (2.6)′ or (2.7)′ holds,

(3.15)′ $$\sum n^{-2} ES_n^{*p} < \infty$$

From the fact that

(3.16) $$X_n^* \le S_n^* + S_{n-1}^* \le 2S_n^*,$$

(3.15) and (3.15)′ imply

(3.17) $$\sum n^{-1-1/p} EX_n^* < \infty \quad \text{and} \quad \sum n^{-2} EX_n^{*p} < \infty$$

respectively. Hence $I_p < \infty$ and (2.4) hold respectively by Theorem 2.2 (ii).

In the following, we will use the notation

$$f(t) = |O(1)| g(t)$$

to denote that there exists $0 < A < \infty$ such that

(3.18) $$A^{-1} |f(t)| \le |g(t)| \le A |f(t)|, \quad \text{all } t.$$

Proof of Theorem 2.2. We can assume that $X \ge 0$ a.s., $EX < \infty$, $F(t) > 0$ and $G(t) > 0$ for all $t > 0$, where $F(t) = 1 - G(t) = P(X < t)$. For $n \ge 1$,

(3.19) $$EX_n^* = n \int_0^\infty t F^{n-1}(t)\, dF(t).$$

Hence (2.8) holds iff

(3.20) $$\mathrm{II}_p \equiv \int_1^\infty y^{-1/p} dy \int_1^\infty t F^y(t)\, dF(t) < \infty.$$

(i) Since $|\log F(t)| = (1 + o(1))\, G(t)$ as $t \to \infty$,

$$\begin{aligned} \int_1^\infty y^{-1/p} F^y(t)\, dy &= \int_{-\log F(t)}^\infty y^{-1/p} e^{-y} dy\, |\log F(t)|^{-1+1/p} \\ &= |O(1)|\, G^{-1+1/p}(t), \qquad p > 1 \\ &= |O(1)| \cdot |\log G(t)|, \qquad p = 1. \end{aligned}$$

Therefore

$$\mathrm{II}_1 = |O(1)| \int_1^\infty t\, |\log G(t)|\, dF(t),$$

and

$$\begin{aligned}\text{II}_p &= |O(1)| \int_1^\infty tG^{-1+1/p}(t)\, dF(t) \\ &= |O(1)| \int_1^\infty G^{1/p}(t)\, dt, \qquad p > 1.\end{aligned}$$

By (3.20) and Lemma 3.1 (for $p=1$), (2.8) holds iff $I_p < \infty$.

(ii) Assume that (2.9) holds. By Lemma 3.2 (i),

$$EX_{\alpha_n}^* \le (1 + o(1))\, EX_{T_n}^* \tag{3.21}$$

as $n \to \infty$. Hence (2.9) implies

$$\sum n^{-1-1/p}\, EX_{\alpha_n}^* < \infty .$$

Since $\alpha_n = [\alpha n]$,

$$\sum n^{-1-1/p}\, EX_n^* < \infty .$$

By part (i), $I_p < \infty$.

(iii) Let $q > p - 1 > 0$ and (2.10) hold. By Lemma 3.1,

$$\int_0^\infty tG(t)\, |\log G(t)|^q\, dt < \infty . \tag{3.22}$$

Since $tG(t) = o(1)$ and $q > p - 1$,

$$\int_1^\infty t^{-1}\, |\log G(t)|^{-q/(p-1)}\, dt < \infty . \tag{3.23}$$

Put

$$A = \{t \ge 1 : G^{1/p}(t) \le t^{p-1} G(t)\, |\log G(t)|^q\}.$$

Then

$$B = [1, \infty) - A = \{t \ge 1 : G^{1/p}(t) < t^{-1}\, |\log G(t)|^{-q/(p-1)}\}.$$

By (3.22) and (3.23),

$$\begin{aligned}\int_1^\infty G^{1/p}(t)\, dt &\le \int_A t^{p-1} G(t)\, |\log G(t)|^q\, dt \\ &\quad + \int_B t^{-1}\, |\log G(t)|^{-q/(p-1)}\, dt < \infty .\end{aligned}$$

Hence $I_p < \infty$. Conversely, if $I_p < \infty$, then

$$\infty > p\int_0^\infty G^{1/p}(t)\,dt = \int_0^\infty y^{-1+1/p}\,G^{1/p}(y^{1/p})dy$$
$$= \int_0^\infty \{y \cdot G(y^{1/p})\}^{-1+1/p} \cdot G(y^{1/p})\,dy.$$

Since $yG^{1/p}(2y) \le \int_y^{2y} G^{1/p}(t)\,dt = o(1)$, $EX^p < \infty$.

Proof of Theorem 2.3. (i) Assume (2.21). Then

(3.24) $$n^{-1/p} X_n^* \xrightarrow{P} 0,$$

which is equivalent to

$$F^n(n^{1/p}) = p(X_n^{*p} < n) = 1 + o(1),$$

i. e.,

(3.25) $$nG(n^{1/p}) = (1 + o(1)\, n\, |\log F(n^{1/p})| = o(1),$$

yielding (2.22). Now assume (2.22). By (3.25), (3.24) holds. To prove (2.21), it is sufficient to establish that $(n^{-1/p} X_n^*,\ n \ge 1)$ is uniformly integrable. Let $\varepsilon > 0$ and choose $K > 1$ so that $t^p G(\varepsilon) < \varepsilon$ for $t \ge K$. Then for $n \ge 1$,

$$\int_K^\infty P(n^{-1/p} X_n^* > t)\,dt \le n\int_K^\infty P(X > n^{1/p}t)\,dt$$
$$= \int_K^\infty nt^p \cdot G(n^{1/n}t)t^{-p}\,dt \le \varepsilon\int_K^\infty t^{-p}\,dt.$$

Since for $p > 1$ and $n \ge 1$,

$$KP(n^{-1/p}X_n^* > K) \le nKP(X > n^{1/p}K) \le \sup_{t \ge K^{1/p}} t^pG(t^{1/p}),$$

$(n^{-1/p} X_n^*,\ n \ge 1)$ is uniformly integrable.

(ii) Let (2.22) hold. For simplicity, we can assume that X_n's are symmetric. For $n = 1, 2, \cdots$ and $j = 1, 2, \cdots, n$, put

$$X_j' = (X_j \wedge n^{1/p}) \vee (-n^{1/p}), \qquad X_j'' = X_j - X_j',$$
$$S_n' = \sum_1^n X_j', \qquad S_n'' = S_n - S_n'.$$

Then

(3.26)
$$E|S_n''| \le n\int_{n^{1/p}}^\infty G(t)\,dt = o(n)\int_{n^{1/p}}^\infty t^{-p}\,dt = o(n^{1/p}),$$
$$ES_n'^2 = O(n)\int_0^{n^{1/p}} tG(t)\,dt = o(n)\int_a^{n^{1/p}} t^{1-p}\,dt = o(n^{2/p}),$$

(3.27) $$E|S_n'| = o(n^{1/p}).$$

Hence (2.23) holds. Since (2.23) implies (2.21) by (3.13) and (3.16), by part (ii) (2.22) is valid if (2.23) holds.

Proof of Theorem 2.4. (i) By Theorem 2.1, $I_p < \infty$ iff

(3.28) $$W \equiv \sum_1^\infty n^{-1-1/p} E(n - S_n)^+ < \infty .$$

Put

$$A_{n,1} = \{S_n < n/2\}, \qquad A_{n,2} = \{S_n \geq n/2\},$$
$$U_j = \sum_1^\infty n^{-1/p} E\left| \frac{(S_n - n + \beta)I(A_{n,j})}{(S_n + \beta)} \right|,$$
$$W_j = \sum_1^\infty n^{-1-1/p} E(n - S_n)^+ I(A_{n,j}),$$

where I is the indicator. Since $X \geq 0$ and $EX = 1$, by the Chernoff (1952) exponential bounds, for every $\varepsilon > 0$, there exist $\theta = \theta_\varepsilon > 0$ and $m = m_\varepsilon \geq 1$ such that

(3.29) $$P(S_n - n \leq -\varepsilon n) \leq e^{-n\theta}, \qquad n \geq m .$$

Hence

(3.30) $$U_1 \leq \beta^{-1} \sum n^{1-1/p} P(S_n < n/2) < \infty,$$

and

(3.31) $$W_1 \leq \sum n^{-1/p} P(S_n < n/2) < \infty .$$

If (3.28) holds, then

$$U_2 \leq 2 \sum n^{-1-1/p} E|S_n - n + \beta| < \infty,$$

yielding (2.24) by (3.30).

If (2.24) holds,

$$\begin{aligned} W_2 &= \sum n^{-1-1/p} E(n - S_n - \beta)^+ I(A_{n,2}) + O(1) \\ &= \sum n^{-1-1/p} E(n - S_n - \beta)^+ I(n > S_n > n/2) + O(1) \\ &\leq 2 \sum n^{-1/p} E(n - S_n - \beta)^+/(S_n + \beta) + O(1) < \infty . \end{aligned}$$

Since $W = W_1 + W_2$, by (3.31) $I^p < \infty$.

(ii) By part (i), we can assume that $1<p<2$. By Theorem 2.1, $EX_1^p \log^+ X_1 < \infty$ iff

$$\sum_1^\infty n^{-2} E|S_n - n|^p < \infty . \tag{3.28)'$$

As in part (i),

$$\sum_1^\infty n^{2p-2} E\left|\frac{1}{S_n+\beta} - \frac{1}{n}\right|^p I(S_n < n/2) < \infty, \tag{3.30)'$$

$$\sum_1^\infty n^{-2} E|S_n - n|^p I(S_n < n/2) < \infty . \tag{3.31)'$$

If (3.28)′ holds, then

$$\sum_1^\infty n^{2p-2} E\left|\frac{1}{S^n+\beta} - \frac{1}{n}\right|^p I(S_n \geq n/2)$$
$$= O(1) \sum n^{-2} E|S_n - n + \beta|^p < \infty,$$

yielding (2.24)′ by (3.30)′.

Next, assume that (2.24)′ holds. For $\varepsilon > 0$ by (2.24)′

$$\sum_1^\infty n^{p-2} P(|S_n - n| > \varepsilon n)$$
$$= O(1) \sum_1^\infty n^{2p-2} E\left|\frac{n - S_n - \beta}{(S_n+\beta)\, n}\right|^p I(|S_n - n| > \varepsilon n) < \infty$$

and then $EX_1^p < \infty$ by Baum and Katz (1965). Now

$$\sum_1^\infty n^{-2} E|S_n - n|^p I\left(\frac{n}{2} \leq S_n \leq 2n\right)$$
$$= O(1) \sum n^{p-2} E\left|\frac{n - Sn}{S_n + \beta}\right|^p > \infty,$$

yielding with (3.31)′

$$\sum n^{-2} E|S_n - n|^p I(S_n \leq 2n) < \infty . \tag{3.32}$$

For $n = 1, 2, \cdots$, and $j = 1, \cdots, n$, put

$$X_j' = X_j I(X_j \leq 2n), \qquad S_n' = \sum_1^n X_j' .$$

Then (3.32) implies

$$\sum n^{-2} E|S_n' - n|^p I(S_n \leq 2n) < \infty.$$

Since $EX_1^p < \infty$, $n - ES_n' = nEX_1 I(X_1 > 2n) = O(1)\, n^{2-p} EX_1^p$ and

$$\sum n^{-2}(n - ES_n')^p = O(1) \sum n^{-2+2p-p^2} < \infty,$$

$$\sum n^{-2} E|S_n' - ES_n'|^p I(S_n \leq 2n) < \infty . \tag{3.33}$$

Set $\alpha = p(4-p)/4$ and $\beta = (2-p)^2/4$. Then $\alpha + \beta = 1$,

$$\sum_1^\infty n^{-4\alpha/p} E(S_n' - ES_n')^2$$

$$= O(1) \sum_1^\infty n^{p-3} EX_1^2 I(X_1 \leq 2n) = O(EX_1^p) < \infty,$$

$$\sum_1^\infty n^{-4\beta/(2-p)} P(S_n > 2n) = \sum n^{p-2} P\{(S_n - n) > n\} < \infty,$$

and by Hölder inequality

$$\sum_1^\infty n^{-2} E|S_n' - ES_n'|^p I(S_n > 2n)$$

$$\leq \sum_1^\infty n^{-2} E^{p/2}(S_n' - ES_n')^2 P^{(2-p)/2}(S_n > 2n)$$

$$\leq \left\{\sum n^{-4\alpha/p} E(S_n' - ES_n')^2\right\}^{p/2}$$

$$\cdot \left\{\sum n^{-4\beta/(2-p)} P(S_n > 2n)\right\}^{(2-p)/2} < \infty,$$

yielding with (3.33)

$$\sum n^{-2} E|S_n' - ES_n'|^p < \infty .$$

By (3.13) and (3.16),

$$\sum n^{-2} E \max_{1\leq j\leq n} |X_j I(X_j \leq 2n) - EX_1 I(X_1 \leq 2n)|^p < \infty .$$

Hence

$$\infty > \sum n^{-2} E \max_{1\leq j\leq n} X_j^p I(X_j \leq 2n)$$

$$\geq \sum n^{-2} \int_{n^{1/p}}^{2n} t^{p-1} P\{\max_{1\leq j\leq n} X_j I(X_j \leq 2n) > t\}\, dt$$

$$\geq \sum n^{-1} \int_{n^{1/p}}^{2n} t^{p-1} P(t < X_1 \leq 2n)\, P^{n-1}(X_1 \leq t)\, dt .$$

Since $EX_1^p < \infty$, $P^n(X_1 \leq n^{1/p}) \rightarrow 1$ and therefore

$$\sum n^{-1} \int_{n^{1/p}}^{2n} t^{p-1} P(t < X_1 \le 2n)\, dt < \infty . \tag{3.34}$$

Since

$$\sum n^{-1} \int_{n^{1/p}}^{2n} t^{p-1} P(X_1 > 2n)\, dt < \infty,$$

$$\sum n^{-1} \int_{n^{1/p}}^{2n} t^{p-1} P(X_1 > t)\, dt < \infty ,$$

yielding $EX_1^p \log^+ X_1 < \infty$.

(iii) For $1 < p < 2$, let $E|X_1|^p < \infty$. By Marcinkiewicz and Zygmund strong law of large numbers (cf. Chow-Teicher (1978), p. 115)

$$\sum_1^\infty (X_n - 1)\, n^{-1/p} \quad \text{converges a.s.,} \quad S_n - n = o(n^{1/p}) \quad \text{a.s.} \tag{3.35}$$

Hence

$$\sum (S_n - n)\, n^{-1-1/p} \quad \text{converges a.s.} \tag{3.36}$$

By (3.35),

$$\frac{1}{S_n + \beta} - \frac{1}{n} - \frac{n - S_n - \beta}{n^2} = \frac{(n - S_n - \beta)^2}{n^2(S_n + \beta)} = o(n^{-3+2/p}) \quad \text{a.s.,}$$

yielding (2.25) by (3.36).

Proof of Theorem 2.5. We can assume that $\alpha p > 1$, $\varepsilon = 1$ and $E|X_1| > 0$. Since if $\alpha p = 1$, then $p = 1 = \alpha$ and (2.26) follows Theorem 2.1 (i). For $t > 0$ and $k = 1, 2, \cdots$, put

$$A = \{X_n^* \le t,\ S_n^* > 2kt\} .$$

Define

$$\zeta^{(1)} = \inf\{j \ge 1 : |S_j| > t\}, \qquad \zeta_0 = 0 = S_0,$$

$$\zeta^{(m)} = \inf\{j \ge 1 : |S_{j+\zeta_{m-1}} - S_{\zeta_{m-1}}| > t\} \quad \text{on} \quad |\zeta_{m-1} > \infty| ,$$

where $\zeta_m = \zeta^{(1)} + \cdots + \zeta^{(m)}$. Then $(\zeta^{(m)}, X_{\zeta_{m-1}}+1, \cdots, X_{\zeta_n})$, $m \ge 1$, are independent and identically distributed (cf. Chow-Teicher (1978), p. 136). Then

$$A \subset \{\zeta_k \le n\} \subset \bigcap_{m=1}^{k} \{\zeta^{(m)} \le n\}$$

Assume that (2.27) holds and let $\beta=\max(1, p/2)$. By Marcinkiewicz and Zygmund inequality

$$P(\zeta_1 \leq n) \leq t^{-p} E|S_n|^p = O(1)\, t^{-p} n^{\beta} E|X_1|^p.$$

Hence as $t \to \infty$,

$$\begin{aligned} &PA = O(1)\, t^{-kp} n^{k\beta}, \\ (3.37) \qquad &P(S_n^* > t) \leq P(X_n^* > t/(2k)) + O(1)\, t^{-kp} n^{k\beta}. \end{aligned}$$

Now,

$$\begin{aligned} \Pi_1 &\equiv \sum_1^{\infty} n^{p\alpha-\alpha-2} \int_{n^\alpha}^{\infty} P(2kX_n^* > t)\, dt \\ &\leq \int_1^{\infty} P(2k|X_1| > t) \sum_{n^\alpha < t} n^{p\alpha-\alpha-1}\, dt \\ &= O(1) \int_1^{\infty} t^{p-1} P(2k|X_1| > t)\, dt > \infty \quad \text{if} \quad p > 1, \\ &= O(1) \int_1^{\infty} \log t P(2k|X_1| > t)\, dt < \infty \quad \text{if} \quad p = 1. \end{aligned}$$

$$\Pi_2 \equiv \sum_1^{\infty} n^{p\alpha-\alpha-2} \int_{n^\alpha}^{\infty} t^{-kp} n^{k\beta}\, dt = O(1) \sum_1^{\infty} n^{p\alpha \cdot 2+(\beta-\alpha p)k} < \infty,$$

by choosing k large enough, since $\alpha p > \beta$.

Therefore by (3.37)

$$\begin{aligned} &\sum_1^{\infty} n^{p\alpha-\alpha-2} E(S_n^* - n^\alpha)^+ \\ &\qquad = \sum_1^{\infty} n^{p\alpha-\alpha-2} \int_{n^\alpha}^{\infty} P(S_n^* > t)\, dt \leq \Pi_1 + O(\Pi_2) < \infty. \end{aligned}$$

Proof of Theorem 2.6. (i) Since $0 < \mu < \infty$, by an elementary renewal theorem of Chow and Robbins (1963),

$$(3.38) \qquad EN_n/n \to \mu^{-1}.$$

Since

$$\begin{aligned} (3.39) \qquad \mu N_n - S_{N_n} &\leq \mu N_n - n \\ &\leq |\mu N_n - S_{N_n}| + X_{N_n}^* \leq 3(\mu N_n - S_{N_n})^* + \mu, \end{aligned}$$

(2.29) and (2.29)′ follow from (2.3) and (2.3)′, respectively, by (3.38). Since

(3.40) $$S_{N_{n\mu}} - S_n = S_{N_{n\mu}} - \mu N_{n\mu} + \mu N_{n\mu} - n\mu + n\mu - S_n,$$

(2.30) follows from (2.3) and (2.29), and (2.30)′ from (2.3)′ and (2.29)′.

(ii) Put $Y_n = S_{\zeta_n} - S_{\zeta_{n-1}}$ for $n \geq 1$ where $\zeta_0 = S_0 = 0$. Then $(\zeta^{(n)}, Y_n, n \geq 1)$ are independent and identically distributed (cf. Chow-Teicher (1978), p. 136) and $Y_1 \geq 1$ a.s. For $t \geq 1$,

(3.41)
$$\begin{aligned} P(Y_1 > t) &= \sum_{j=1}^{\infty} P(N_1 = j, S_j > t) \\ &\leq \sum_{j=1}^{\infty} P(N_1 = j, X_j > t-1) \\ &\leq \sum P(N_1 \geq j, X_j > t-1) \leq EN_1 G(t-1-\mu). \end{aligned}$$

(α) Let $I_p < \infty$. Since $EY_1 = \mu EN_1 < \infty$ and $S_{\zeta_n} = \sum_1^n Y_j$, by (3.41), Y_1 satisfies the condition $I_p < \infty$ also. Hence by Theorem 2.1 (i),

(3.42) $$\sum n^{-1-1/p} E(S_{\zeta_n} - \mu n EN_1)^* < \infty .$$

Because $(\zeta_m, m \geq 1)$ is a sequence of (X_n)-stopping times (cf. Chow-Teicher (1978), p. 134) and $E\zeta_n = nEN_1$, we have by Theorem 2.1 (i) again,

(3.43) $$\sum n^{-1-1/p} E(S_{\zeta_n} - \mu\zeta_n)^* < \infty,$$

yielding (2.31) by (3.42). (2.32) follows from (2.29) and (2.31). By Theorem 2.1 (iii), (2.31) implies (2.33). Similarly we have (β).

(γ) Assume that (2.34) holds. Then

$$\sum n^{-1-1/p} E(\mu N_n - S_{N_n})^+ \leq \sum n^{-1-1/p} E(\mu N_n - n)^+ < \infty .$$

By Wald equation, $ES_{N_n} = \mu EN_n$ and hence

(3.44) $$\sum n^{-1-1/p} E|\mu N_n - S_{N_n}| < \infty .$$

Since $N_n/n \to \mu^{-1}$ a.s. and (3.28) holds, by Theorem 2.1 (iii), (3.44) implies that $I_p < \infty$.

Proof of Theorem 2.7 For simplicity, we can assume that $\mu = 1$ and $\varepsilon < 1$. Put $W_n = S_n - n$, $W_t^* = \max_{1 \leq j \leq t} |W_j|$, $\zeta_n = \zeta^{(1)}$

$+\cdots+\zeta^{(n)}$, where $\zeta^{(1)}=N_1$, $\zeta^{(2)},\cdots$ are copies of N_1, $\zeta_t=\zeta_{[t]}$, and $\theta=EN_1$. Then $\theta\geq 1$ and

$$\begin{aligned}&P(|N_t-t|>y)\\ &\quad=P(|N_t-t|>y,\ N_t\leq 3\theta t)+P(|N_t-t|>y,\ N_t>3\theta t)\\ &\quad=I(t,y)+II(t,y),\ \text{say.}\end{aligned}\tag{3.45}$$

Since

$$|N_t-t|\leq|N_t-S_{N_t}|+|S_{N_t}-t|\leq 3W^*_{N_t}+1,$$

for $M>(2/\varepsilon)^{1/\alpha}$ and $t\geq M$,

$$\int_{\varepsilon t^\alpha}^\infty I(t,y)\,dy\leq\int_{\varepsilon t^\alpha}^\infty P(6W^*_{3\theta t}>y)\,dy=E(6W^*_{3\theta t}-\varepsilon t^\alpha)^+.\tag{3.46}$$

Since $\theta\geq 1$ and $N_t\leq\zeta_{2t}$ for $t>1$,

$$\begin{aligned}II(t,y)&\leq P(N_t>y,\ N_t>3\theta t)\\ &\leq P(\zeta_{2t}>y,\ \zeta_{2t}>3\theta t),\qquad t>1.\end{aligned}\tag{3.47}$$

First, let $\alpha\leq 1$. Then $\alpha p-\alpha\leq p-1$ and for $t\geq M$, $\varepsilon t^\alpha\leq 3\theta t$.

$$\begin{aligned}&\int_{\varepsilon t^\alpha}^\infty II(t,y)\,dy\\ &\leq(3\theta t-\varepsilon t^\alpha)P(\zeta_{2t}>3\theta t)+\int_{3\theta t}^\infty P(\zeta_{2t}>y)\,dy\\ &\leq O(t)P(\zeta_{2t}-2\theta t>\theta t)+E(\zeta_{2t}-3\theta t)^+.\end{aligned}\tag{3.48}$$

Hence be (3.45), (3.46) and (3.48)

$$\begin{aligned}&\int_M^\infty t^{\alpha p-\alpha-2}E(|N_t-t|-\varepsilon t^\alpha)^+\,dt\\ &\quad=\int_M^\infty t^{\alpha p-\alpha-2}\,dt\int_{\varepsilon t^\alpha}^\infty(I(t,y)+II(t,y))\,dy\\ &\quad\leq\int_M^\infty t^{\alpha p-\alpha-2}\{E(W^*_{3\theta t}-\varepsilon t^\alpha)^++E(\zeta_{2t}-3\theta t)^+\}\,dt\\ &\qquad\quad+O(1)\int_M^\infty t^{p\alpha-\alpha-1}P(\zeta_{2t}-2\theta t>\theta t)\,dt.\end{aligned}$$

By Theorem 2.5 (and Theorem 2.6 (ii) (β), (2.33)$'$, when $p=1$) the first integral is finite and by Baum and Katz (1965) so is the second integral. Hence (2.40) holds when $\alpha\leq 1$.

Now assume that $\alpha>1$. Choose M so large that $\varepsilon M^\alpha>4\theta M>2$. Then for $t>M$, $\varepsilon t^\alpha>4\theta t$ and by (3.47)

$$(3.49)\quad \int_{\varepsilon t^\alpha}^\infty \Pi(t, y)\, dy \le \int_{\varepsilon t^\alpha}^\infty P(\zeta_{2t} > y)\, dy = E(\zeta_{2t} - \varepsilon t^\alpha)^+ \le E(\zeta_{2t} - 2\theta t - \varepsilon t^\alpha/2)^+.$$

Hence by (3.46), (3.49) and Theorem 2.5

$$\int_M^\infty t^{\alpha p-\alpha-2} E(|N_t - t| - \varepsilon t^\alpha)^+ dt \le \int_M^\infty t^{\alpha p-\alpha-2}\{E(W^*_{3\theta t} - \varepsilon t^\alpha)^+ + E(\zeta_{2t} - 2\theta t - \varepsilon t^\alpha/2)^+\}\, dt < \infty .$$

References

1. K. Athreya and S. Karlin, *Limit theorems for the split times of branching processe,* J. Math. Mech. 17 (1967), 257-277.

2. L. E. Baum and M. Katz, *Convergence rates in the law of large numbes,* Trans. Amer. Math. Soc. 120 (1965), 108-123.

3. B. M. Brown, *Characteristic functions, moments and the central limt theorem,* Ann. Math. Statist. 41 (1970), 658-664.

4. D. L. Burkholder, *Martingale tranforms,* Ann. Math. Statist. 37 (1966), 1494-1504.

5. H. Chernoff, *A measure of asymptotic efficiency for tests of a hypothessi based on the sum of obserpations,* Ann. Math. Statist. 23 (1952), 493-507.

6. Y. S. Chow and H. Teicher, *Probability Theory,* Springer-Verlag (1978).

7. Y. S. Chow and H. Robbins, *A renewal theorem for random variables with are dependent or non-identically distributed,* Ann. Math. Statist. 34 (1963), 390-395.

8. J. L. Doob, *Stochastic Processes.* Jonn Wiley (1953).

9. A. Gut, *On the moments and limit disttributions of some first passge times,* Ann. Probability, 2, (1974A), 277-308.

10. _______, *On convergence in γ-mean of some first passage times and randomly indexed sums,* Ann. Probability, 2 (1974B), 321-323.

11. _______, *Complete comvegence and convergence rates for randomly indexed partial sums with an application to some first passage times,* Acta. Math. Hung. 42 (1983), 225-232.

12. P. Hall, *Bounds on the rate of convergence of moments in the central limit theorem,* Ann. Probability, 10 (1982), 1004-1018.

13. M. Klass, *Precision bounds for the relative error in approximation of $E|S^n|$ and extensions,* Ann. Probability, 8 (1980), 350-367.

14. T. L. Lai, *Asymptotic moments of random walks with applications to ladder varibles and renewal theory,* Ann. Probability, 4 (1976), 51-66.

15. J. Pickands, *Moment convergence of sample extremes,* Ann. Math. Statist. 39 (1968) 881-889.

16. R. J. Tomkins, *Regular variation and the stability of maxima,* Ann. Probability, 14 (1986), 984-995.

Department of Statistics
Columbia University
New York, NY 10027
U.S.A.