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ON SOME CLASSES OF CLOSE-TO-CONVEX FUNCTIONS AND ITS APPLICATIONS

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Abstract. In this paper the new classes $G(\alpha)$ $(0 \leq \alpha < 1)$ of functions $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ which are close-to-convex in the unit disk $E = \{z : |z| < 1\}$ are introduced. For $\alpha = 1/2$, $G(1/2)$ is the class defined by M.S. Robertson ([5]). Among other results some interesting estimates of the differences of successive coefficients for starlike and convex functions of order α $(0 \leq \alpha < 1)$, and for k-fold symmetric and starlike of order α $(0 \leq \alpha < 1)$ are shown.

1. **Introduction.** Let A denote the class of functions $f(z)$ which are analytic in the unit disk $E = \{z : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$.

A function $f(z)$ belonging to A is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.1}$$

for some α $(0 \leq \alpha < 1)$, and for all $z \in E$. We denote by $S^*(\alpha)$ the class of all such starlike functions of order α.

Similarly, a function $f(z)$ belonging to A is said to be convex of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$, and for all $z \in E$. Also we denote by $K(\alpha)$ the class of all functions in A which are convex of order α.

We note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, and that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$, $K(\alpha) \subseteq K(0) \equiv K$ and $K(\alpha) \subset S^*(\alpha)$ for $0 \leq \alpha < 1$.

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The classes $S^*(\alpha)$ and $K(\alpha)$ were introduced by Robertson [4].

Let $f(z)$ be analytic in the unit disk E and satisfy $f'(z) \neq 0$ for $z \in E$. Then $f(z)$ is said to be close-to-convex in the unit disk E if and only if there exists a function $\phi(z) \in K$ such that

$$\text{Re}\left\{\frac{f'(z)}{\phi'(z)}\right\} > 0 \qquad (z \in E). \tag{1.3}$$

It is known that every close-to-convex function is univalent in the unit disk E and the class of close-to-convex functions contains the classes S^* and K. The class of close-to-convex functions was introduced by Kaplan [2].

Let $G(\alpha)$ be the class of functions of the form

$$g(z) = 1 + b_1 z + b_2 z^2 + \cdots \tag{1.4}$$

which are analytic and non-vanishing in E, and such that

$$\text{Re}\left\{\frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+z}{1-z}\right\} > 0 \tag{1.5}$$

for some α $(0 \leq \alpha < 1)$, and all $z \in E$.

For $\alpha = 1/2$ we have the class $G(1/2)$ which was introduced by Robertson [5]. In [5] Robertson has proved that $G(1/2)$ is contained in the class of close-to-convex functions in E, and that if $g(z) \in G(1/2)$ (where $g(z)$ is not a constant) then $g(1) = 0$ and $g(z)$ maps $E = \{z : |z| < 1\}$ onto a domain that is starlike with respect to $g(1)$. Robertson [5] also gave many interesting coefficient inequalities for various classes of univalent functions.

2. On the classes $G(\alpha)$ and coefficient inequalities.

We begin with some simple characteristics of the classes $G(\alpha)$ and their relations among themselves and to other classes.

THEOREM 1. *Let $g(z)$ be analytic in the unit disk E with $g(0) = 1$. Then $g(z)$ is in the class $G(\alpha)$ $(0 \leq \alpha < 1)$ if and only if there exists a function $f(z)$ belonging to $S^*(\alpha)$ such that*

$$g(z) = (1-z)^{2(1-\alpha)}\frac{f(z)}{z}. \tag{2.1}$$

Proof. Let $g(z)$ satisfy (2.1) for $f(z) \in S^*(\alpha)$. Then $g(z)$ is analytic and non-vanishing in E, and, after logarithmic differentiation of (2.1), we have

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+z}{1-z}\right\} = \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > 0 \tag{2.2}$$

for $z \in E$. Then, with the definition of $G(\alpha)$, we prove that $g(z)$ belongs to the class $G(\alpha)$.

Conversely, if $g(z) \in G(\alpha)$ and $f(z) = zg(z)/(1-z)^{2(1-\alpha)}$, then $f(0) = 0$, $f'(0) = 1$, and by (2.2) $f(z) \in S^*(\alpha)$. This completes the proof of Theorem 1.

THEOREM 2. *Let* $0 \leq \alpha_1 \leq \alpha_2 < 1$. *Then*

$$G(\alpha_2) \subseteq G(\alpha_1). \tag{2.3}$$

Proof. Let $g(z)$ be in the class $G(\alpha_2)$. Then we have

$$\begin{aligned}\operatorname{Re}&\left\{\frac{zg'(z)}{g(z)} + (1-\alpha_1)\frac{1+z}{1-z}\right\}\\ &= \operatorname{Re}\left\{\frac{zg'(z)}{g(z)} + (1-\alpha_2)\frac{1+z}{1-z}\right\}\\ &\qquad + (\alpha_2 + \alpha_1)\operatorname{Re}\left\{\frac{1+z}{1-z}\right\}\\ &> 0\end{aligned}$$

for $z \in E$, since $\operatorname{Re}\{(1+z)/(1-z)\} > 0$ for $z \in E$. Hence $g(z) \in G(\alpha_1)$, which implies the assertion of the theorem.

THEOREM 3. *Let $g(z)$ be in the class $G(\alpha)$ $(0 \leq \alpha < 1)$. Then either $g(z)$ is close-to-convex in E or $g(z)$ is the constant* 1.

Proof. In view of Theorem 2, we know that

$$G(\alpha) \subseteq G(0) \qquad (0 \leq \alpha < 1), \tag{2.4}$$

where $G(0)$ is the class of functions of the form

$$g(z) = (1-z)^2 \frac{f(z)}{z}, \tag{2.5}$$

where $f(z) \in S^*(0) \equiv S^*$. But, in the proof of Theorem 8 in [5], Robertson has proved that $g(z)$ given by (2.5) is close-to-convex in

E or $g(z)$ is the constant 1. The proof of Theorem 3 follows from the relation (2.4).

REMARK. From Theorem 2 we have that

$$G(\alpha) \subseteq G(\tfrac{1}{2}) \qquad (\tfrac{1}{2} \leq \alpha < 1),$$

and from there if $g(z) \in G(\alpha)$ $(\frac{1}{2} \leq \alpha < 1)$, and if $g(z)$ is not a constant, then $g(z)$ is starlike with respect to a boundary point (see [5]).

In the next theorems we use the following results.

LEMMA 1. (Reade [3]), *Let* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be a close-to-convex function in* E. *Then*

$$|b_n| \leq n|b_1| \qquad (n = 2, 3, 4, \cdots). \tag{2.6}$$

LEMMA 2. *For real* β *and non-negative* k *we have*

(i) $\binom{-\beta}{k} = (-1)^k \binom{\beta + k - 1}{k}$,

(ii) $\binom{k+\beta}{k} = \dfrac{1+\beta}{k} \binom{k+\beta}{k-1}$ $\quad (k \geq 1)$,

(iii) $\displaystyle\sum_{k=0}^{n} \binom{k+\beta}{k} = \binom{n+1+\beta}{n}$.

Applying the above lemmas, we prove.

THEOREM 4. *Let the function* $f(z)$ *defined by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

belong to the class $S^*(\alpha)$ $(0 \leq \alpha < 1)$. *Then*

$$\begin{aligned} |a_{n+1} - a_n| \leq & \left|\binom{n-2\alpha}{n}\right| \\ & + \left\{n - n \operatorname{sgn}(1-2\alpha)\left(\binom{n-2\alpha}{n-1} - 1\right)\right. \\ & \left. - |1-2\alpha|\binom{n-2\alpha}{n-2}\right\}|a_2 - 2(1-\alpha)|, \end{aligned} \tag{2.7}$$

where $n \geq 2$ *and* $\operatorname{sgn} x$ *denotes the signum function.*

Proof. From Theorem 1 and Theorem 3, we have that the function $g(z)$ given by (2.1) is close-to-convex in E for $f(z)\in S^*(\alpha)$ $(0\leq\alpha<1)$. From there we have

$$(1-z)\frac{f(z)}{z}=(1-z)^{2\alpha-1}g(z),$$

and if put $g(z)=1+b_1z+b_2z^2+\cdots$ we obtain

$$\begin{aligned}&(1-z)\left(1+\sum_{n=2}^{\infty}a_nz^{n-1}\right)\\ &=\left(1+\sum_{n=1}^{\infty}(-1)^n\binom{2\alpha-1}{n}z^n\right)\left(1+\sum_{n=1}^{\infty}b_nz^n\right).\end{aligned}\tag{2.8}$$

Comparing the coefficients of both sides in (2.8), and using (i) of Lemma 2, we see that

$$\begin{aligned}a_{n+1}-a_n&=\sum_{k=0}^{n}(-1)^k\binom{2\alpha-1}{k}b_{n-k}\\ &=\sum_{k=0}^{n}\binom{k-2\alpha}{k}b_{n-k},\end{aligned}\tag{2.9}$$

where $b_1=a_2-2(1-\alpha)$, $b_0=1$. With the help of Lemma 2, it is easily proved that

$$\begin{aligned}\left|\binom{k-2\alpha}{k}\right|&=\frac{|1-2\alpha|}{k}\binom{k-2\alpha}{k}\\ &=\operatorname{sgn}(1-2\alpha)\binom{k-2\alpha}{k}\qquad(k\geq1).\end{aligned}\tag{2.10}$$

Using Lemma 1, (2.10), and (iii) of Lemma 2, (2.9) gives

$$\begin{aligned}|a_{n+1}-a_n|&\leq\sum_{k=0}^{n}\left|\binom{k-2\alpha}{k}\right||b_{n-k}|\\ &\leq\left|\binom{n-2\alpha}{n}\right|+\sum_{k=0}^{n-1}(n-k)\left|\binom{k-2\alpha}{k}\right||b_1|\\ &=\left|\binom{n-2\alpha}{n}\right|+\left\{n+n\operatorname{sgn}(1-2\alpha)\sum_{k=1}^{n-1}\binom{k-2\alpha}{k}\right.\\ &\qquad\left.-|1-2\alpha|\sum_{k=1}^{n-1}\binom{k-2\alpha}{k-1}\right\}|b_1|\\ &=\left|\binom{n-2\alpha}{n}\right|+\left\{n+n\operatorname{sgn}(1-2\alpha)\left(\binom{n-2\alpha}{n-1}-1\right)\right.\\ &\qquad\left.-|1-2\alpha|\binom{n-2\alpha}{n-2}\right\}|a_2-2(1-\alpha)|,\end{aligned}$$

which was to be proved.

Taking $\alpha = 0$ or $a = 1/2$ in Theorem 4, we have

COROLLARY 1. *If $f(z)$ belongs to the class $S^*(0)$, then*

$$|a_{n+1} - a_n| \leq 1 + \frac{n(n+1)}{2}|a_2 - 2| \quad (n \geq 2). \tag{2.11}$$

COROLLARY 2. *If $f(z)$ belongs to the class $S^*(1/2)$, then*

$$|a_{n+1} - a_n| \leq n|a_2 - 1| \quad (n \geq 2). \tag{2.12}$$

Corollary 2 is the earlier result by Robertson [5].

Next we derive

THEOREM 5. *Let the function $h(z)$ defined by*

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

belongs to the class $K(\alpha)$ $(0 \leq \alpha < 1)$. Then, for $n \geq 2$, we have

$$\begin{aligned} |c_{n+1} - c_n| \leq {} & \frac{1}{n(n+1)}\left|n\binom{-2(1-\alpha)}{n} + (n+1)\binom{-2(1-\alpha)}{n-1}\right| \\ & + \frac{2}{n(n+1)}\left\{n^2 + \sum_{k=1}^{n-1}(n-k)\left|n\binom{-2(1-\alpha)}{k} + (n+1)\binom{-2(1-\alpha)}{k-1}\right|\right\}|c_2 - (1-\alpha)| \end{aligned} \tag{2.13}$$

where $|c_2| \leq 1 - \alpha$.

Proof. From the relation (2.1) we have that

$$\frac{f(z)}{z} = (1-z)^{-2(1-\alpha)} g(z). \tag{2.14}$$

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\alpha)$ $(0 \leq \alpha < 1)$ and $g(z)$ is given by (1.4), then from (2.14), similarly as in the proof of Theorem 4, we obtain that

$$a_{n+1} = \sum_{k=0}^{n} (-1)^k \binom{-2(1-\alpha)}{k} b_{n-k}, \tag{2.15}$$

and from there

$$
\begin{aligned}
\left|\frac{a_{n+1}}{n+1}-\frac{a_n}{n}\right| &= \frac{1}{n(n+1)}\,|na_{n+1}-(n+1)\,a_n| \\
&= \frac{1}{n(n+1)}\Bigg|nb_n+\sum_{k=1}^{n-1}(-1)^k\Bigg\{n\binom{-2(1-\alpha)}{k} \\
&\qquad + (n+1)\binom{-2(1-\alpha)}{k-1}\Bigg\}b_{n-k} \\
&\qquad + (-1)^n\Bigg\{n\binom{-2(1-\alpha)}{n} \\
&\qquad + (n+1)\binom{-2(1-\alpha)}{n-1}\Bigg\}\Bigg| \\
&\leq \frac{1}{n(n+1)}\Bigg|n\binom{-2(1-\alpha)}{n} \\
&\qquad + (n+1)\binom{-2(1-\alpha)}{n-1}\Bigg| \\
&\quad + \frac{1}{n(n+1)}\Bigg\{n^2+\sum_{k=1}^{n-1}(n-k)\Bigg|n\binom{-2(1-\alpha)}{k} \\
&\qquad + (n+1)\binom{-2(1-\alpha)}{k-1}\Bigg|\Bigg\}|a_2-2(1-\alpha)|.
\end{aligned}
\tag{2.16}
$$

By using the fact that $h(z)\in K(\alpha)$ if and only if $f(z)=zh'(z)\in S^*(\alpha)$, we have that $a_n=nc_n$ $(n\geq 2)$, and that $|c_2|\leq 1-\alpha$ by [4]. Now, the proof of Theorem 5 is derived from previous estimates (2.14) for $f(z)\in S^*(\alpha)$.

Letting $\alpha=0$, Theorem 5 derives

COROLLARY 3. *Let the function $h(z)$ defined in Theorem 5 be in the class K. Then we have*

$$|c_{n+1}-c_n|\leq\frac{2n+1}{3}\,|1-c_2|\qquad(n\geq 2),$$

which is the former result by Robertson [5].

Further, taking $\alpha=1/2$ in Theorem 5, we have

COROLLARY 4. *Let the function $h(z)$ defined in Theorem 5 be in the class $K(\frac{1}{2})$. Then*

$$|c_{n+1}-c_n|\leq\frac{1}{n(n+1)}+\frac{3n-1}{2(n+1)}\,|2c_2-1|\qquad(n\geq 2).$$

Schild [6] has proved that for $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in S^*(\alpha)$ $(0\leq\alpha<1)$ the following inequality

$$|a_n| \leq \binom{n-2\alpha}{n-1} \qquad (n \geq 2) \tag{2.17}$$

is valid. In the next theorem we give the estimate for difference $a_n - \binom{n-2\alpha}{n-1}$ depending on the difference $a_2 - 2(1-\alpha)$.

THEOREM 6. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *belong to the class* $S^*(\alpha)$ $(0 \leq \alpha < 1)$. *Then*

$$\begin{aligned} &\left|a_n - \binom{n-2\alpha}{n-1}\right| \\ &\quad \leq \left\{(n-1)\binom{n-2\alpha}{n-2} - 2(1-\alpha)\binom{n-2\alpha}{n-3}\right\}|a_2-2(1-\alpha)| \end{aligned} \tag{2.18}$$

for $n \geq 3$.

Proof. From the relation (2.15), we know that

$$\begin{aligned} a_n &= \sum_{k=0}^{n-1} (-1)^k \binom{-2(1-\alpha)}{k} b_{n-1-k} \\ &= \sum_{k=0}^{n-1} \binom{k+1-2\alpha}{k} b_{n-1-k} \\ &= \binom{n-2\alpha}{n-1} + \sum_{k=0}^{n-2} \binom{k+1-2\alpha}{k} b_{n-1-k}. \end{aligned} \tag{2.19}$$

Applying Lemma 1 and Lemma 2, (2.19) shows that

$$\begin{aligned} \left|a_n - \binom{n-2\alpha}{n-1}\right| &\leq \sum_{k=0}^{n-2} (n-1-k)\binom{k+1-2\alpha}{k} |b_1| \\ &= \left\{(n-1)\sum_{k=0}^{n-2}\binom{k+1-2\alpha}{k} \right. \\ &\qquad \left. - 2(1-\alpha)\sum_{k=1}^{n-2}\binom{k+1-2\alpha}{k-1}\right\} |b_1| \\ &= \left\{(n-1)\binom{n-2\alpha}{n-2} \right. \\ &\qquad \left. - 2(1-\alpha)\binom{n-2\alpha}{n-3}\right\}|a_2 - 2(1-\alpha)|. \end{aligned}$$

Taking $\alpha = 0$, Theorem 6 gives

COROLLARY 5. *Let the function* $f(z)$ *defined in Theorem* 6 *be in the class* $S^*(0)$. *Then*

$$|a_n - n| \leq \frac{n(n^2-1)}{6}\,|a_2 - 2| \qquad (n \geq 3),$$

which is the former result by Hummel [1].

Letting $\alpha = \frac{1}{2}$ in Theorem 6, we have

COROLLARY 6. *Let the function* $f(z)$ *defined in Theorem* 6 *be in the class* $S^*(\frac{1}{2})$. *Then*

$$|a_n - 1| \leq \frac{n(n-1)}{2}\,|a_2 - 1| \qquad (n \geq 3).$$

3. **Class of k-fold symmetric and starlike functions of order α.** A function $f(z)$ belonging to A is said to be k-fold symmetric if it satisfies $f(e^{2\pi i/k}z) = e^{2\pi i/k}f(z)$ for $k = 2, 3\cdots$. In particular, every odd function $f(z)$ is 2-fold symmetric. A simple argument shows that $f(z)$ which is k-fold symmetric is characterised by having a power series of the form

$$(3.1) \qquad f(z) = z + \sum_{n=1}^{\infty} a_{nk+1} z^{nk+1} \qquad (z \in E).$$

We denote by $S_k^*(\alpha)$ $(0 \leq \alpha < 1;\ k = 2, 3, \cdots)$ the class of functions which are k-fold symmetric and starlike of order α in the unit disk E.

THEOREM 7. *Let the function* $f(z)$ *defined by* (3.1) *be in the class* $S_k^*(\alpha)$ $(0 \leq \alpha < 1;\ k = 2, 3, \cdots)$. *Then*

$$(3.2) \qquad \begin{aligned} &|a_{nk+1} - a_{(n-1)k+1}| \\ &\leq \left|\binom{n-2\beta}{n}\right| + \left\{n\binom{n-2\beta}{n-1} - (1-2\beta)\binom{n-2\beta}{n-2}\right\} \\ &\quad \cdot |a_{k+1} - 2(1-\beta)|, \end{aligned}$$

where $\beta = 1 - (1-\alpha)/k$ *and* $n = 2, 3, \cdots$.

Proof. For $f(z)$ belonging to the class $S_k^*(\alpha)$, define the function $g(z)$ by

$$(3.3) \qquad g(z) = z^{1-1/k} f(z^{1/k}).$$

Then we have

$$g(z) = z + a_{k+1} z^2 + a_{2k+1} z^3 + \cdots = z + b_2 z^2 + b_3 z^3 + \cdots, \tag{3.4}$$

where

$$b_n = a_{(n-1)k+1} \qquad (n \geq 2). \tag{3.5}$$

Also we have that

$$\begin{aligned} \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} &= \operatorname{Re}\left\{1 - \frac{1}{k} + \frac{1}{k}\,\frac{z^{1/k} f'(z^{1/k})}{f(z^{1/k})}\right\} \\ &> 1 - \frac{1}{k} + \frac{\alpha}{k} \\ &= \beta, \end{aligned}$$

that is, that $g(z) \in S^*(\beta)$. Because of that, from Theorem 4 and the fact that $\frac{1}{2} \leq \beta < 1$, we get

$$\begin{aligned} |b_{n+1} - b_n| \leq \left|\binom{n-2\beta}{n}\right| &+ \left\{n\binom{n-2\beta}{n-1}\right. \\ &\left.+ (1-2\beta)\binom{n-2\beta}{n-2}\right\}|b_2 - 2(1-\beta)| \end{aligned} \tag{3.7}$$

for $n \geq 2$. Now, the result of Theorem 7 follows from (3.7) and (3.5).

For $k = 2$, that is, for odd functions in E, and for $\alpha = 0$ or $\alpha = \frac{1}{2}$, from Theorem 7 we have the following results.

COROLLARY 7. *Let* $f(z) = z + a_3 z^3 + \cdots + a_{2n+1} z^{2n+1} + \cdots$ *be in the class* $S^*(0)$. *Then*

$$|a_{2n+1} - a_{2n-1}| \leq n|a_3 - 1| \qquad (n \geq 2).$$

Corollary 7 is the former result due to Robertson [5].

COROLLARY 8. *Let the function* $f(z)$ *defined in Corollary* 7 *be in the class* $S^*(\frac{1}{2})$. *Then*

$$|a_{2n+1} - a_{2n-1}| \leq \frac{1}{2^n}\,\frac{((2n-3)!)!}{n!}\left\{1 + 2n\left|a_3 - \frac{3}{4}\right|\right\} \qquad (n \geq 3).$$

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