BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 16, Number 1, March 1988

ABSTRACT VOLTERRA TYPE INTEGRAL EQUATIONS

BY

D. D. BAINOV, S. I. KOSTADINOV AND A. I. ZAHARIEV

Abstract. The aim of the present paper is to study some properties of an abstract nonlinear analogue of Volterra equation. Sufficient conditions have been obtained guaranteing the existence of solutions in the homogeneous and nonhomogeneous case.

The main results are due to some ideas from [1].

The present paper studies an abstract nonliner analogue of the Volterra equation. The main results are due to some ideas of [1].

Let Ω be a metric space with metric ρ and Borel measure μ while $\phi : x \to M_x$ is a map associating every element $x \in \Omega$ with a closed subset M_x of Ω.

We say that the conditions (A) hold when the following set of suppositions is fulfilled:

A1. The set Ω is compact.

A2. For any $\varepsilon > 0$ and $x \in \Omega$ there exists a number $\delta = \delta(\varepsilon, x) > 0$ such that for every element $y \in \Omega$ for which $\rho(x, y) < \delta$,

$$\mu(\{M_x \backslash M_y\} \cup \{M_y \backslash M_x\}) < \varepsilon.$$

A3. (transitivity) For any $x \in \Omega$ and $y \in M_x$ one has $M_y \subseteq M_x$.

A4. For every choise of $\varepsilon > 0$ and $x \in \Omega$ there exists a number $\delta > 0$ such that for every element $y \in \Omega$ for which $\rho(x, y) < \delta$ the inclusion $M_x \subseteq U(\varepsilon, M_y)$ holds, where $U(\varepsilon, M_y)$ denotes the ε-neighbourhood of the set M_y.

A5. There exists a point $x_0 \in \Omega$ for which $\mu(M_{x_0}) = 0$.

Received by the editors October 21, 1986.

Let B be a Banach space with norm $\|\cdot\|_B$ and let $C(\Omega, B)$ denote the linear space of all continuous maps $f : \Omega \to B$.

REMARK 1. The space $C(\Omega, B)$ is a Banach space with norm $\|f\| = \sup_{x\in\Omega}\|f(x)\|_B$.

Let us denote by w the measure of Ω and let U be the family of all sets M_x, $x \in \Omega$.

We say that the operator $A : \Omega \times C(\Omega, B) \to C(\Omega, B)$ fulfilles the conditions (B) when:

B1. The operator A is continuous.

B2. For every function $f \in C(\Omega, B)$ and for every $x \in \Omega$ the following inequality holds

$$\|A(x, f)\| \leq Q(f) \cdot \|f\|,$$

where the function $Q(f)$ is bounded over every bounded subset of $C(\Omega, B)$.

Consider the equation

$$f + Kf = p, \tag{1}$$

where $f, p \in C(\Omega, B)$ and the operator K is defined by mean of the equation

$$(Kf)(x) = \int_{M_x} A(x, f)(y)\, d\mu_y. \tag{2}$$

Since B1 holds, the integral in (2) exists, see e. g. [2].

REMARK 2. The condition A3 allows for every point $a \in \Omega$ to consider the restriction $K_a : C(M_a, B) \to C(M_a, B)$ where the operator K_a is defined by (2) and $C(M_a, B)$ is the space of the continuous functions from M_a to B. In this case the restiction $f_{|M_a}$ of the solution of the equation $f + Kf = p$, $f \in C(\Omega, B)$ is a solution of the equation $\tilde{\varphi} + K_a\tilde{\varphi} = p_{|M_a}$, $\tilde{\varphi} \in C(M_a, B)$.

LEMMA 1. *Let the conditions* A1, A2 *and* B1 *hold. Then the operator K defined by the equality* (2) *maps continuously* $C(\Omega, B)$ *into* $C(\Omega, B)$.

Proof. Let $x_0 \in \Omega$ be an arbitrary element and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points convergent to x_0. Then, if $f \in C(\Omega, B)$ is an arbitrary fixed element we have

$$
\begin{aligned}
&\|(Kf)(x_n)-(Kf)(x_0)\|_B \\
&\quad=\left\|\int_{M_{x_n}} A(x_n, f)(y)\,d\mu_y-\int_{M_{x_0}} A(x_0, f)(y)\,d\mu_y\right\|_B \\
&\quad=\Bigg\|\int_{M_{x_n}\cap M_{x_0}}[A(x_n, f)(y)-A(x_0, f)(y)]\,d\mu_y \\
&\qquad+\int_{M_{x_n}\setminus M_{x_0}} A(x_n, f)(y)\,d\mu_y \\
&\qquad-\int_{M_{x_0}\setminus M_{x_n}} A(x_0, f)(y)\,d\mu_y\Bigg\|_B \\
&\quad\le\int_{M_{x_n}\cap M_{x_0}}\|A(x_n, f)(y)-A(x_0, f)(y)\|_B\,d\mu_y \\
&\qquad+\int_{M_{x_n}\setminus M_{x_0}}\|A(x_n, f)(y)\|_B\,d\mu_y \\
&\qquad+\int_{M_{x_0}\setminus M_{x_n}}\|A(x_0, f)(y)\|_B\,d\mu_y.
\end{aligned}
\tag{3}
$$

Let $\varepsilon>0$ be an arbitrary number. Condition B1 implies that a number $n_0=n_0(\varepsilon, x_0, f)$ exists so that for $n\ge n_0$ and $y\in M_{x_0}$ the following inequality holds

$$
\|A(x_n, f)(y)-A(x_0, f)(y)\|_B<\frac{\varepsilon}{3\mu(M_{x_0})}. \tag{4}
$$

On the other hand, the condition A2 yields that there exists a number $\delta=\delta(\varepsilon, x_0)>0$ so that for any x from the ball $\rho(x, x_0)<\delta$ one has

$$
\mu(\{M_{x_0}\setminus M_x\}\cup\{M_x\setminus M_{x_0}\})<\frac{\varepsilon}{3\tilde{A}} \tag{5}
$$

where $\tilde{A}=\sup_{x,y\in\Omega}\|A(x, f)(y)\|_B$.

Besides, there exists a number $n_1=n_1(\varepsilon)$ such that for $n\ge n_1$ it holds $\rho(x_n, x_0)<\delta$. Then, if $n\ge\max(n_1, n_0)$ from (5) we get

$$
\begin{aligned}
&\int_{M_{x_n}\setminus M_{x_0}}\|A(x_n, f)(y)\|_B\,d\mu_y \\
&\qquad+\int_{M_{x_0}\setminus M_{x_n}}\|A(x_0, f)(y)\|_B\,d\mu_y<\frac{2\varepsilon}{3}.
\end{aligned}
\tag{6}
$$

For $n\ge\max(n_1, n_0)$ using (5), (4) and (6) we finally obtain

$$
\|(Kf)(x_n)-(Kf)(x_0)\|_B<\varepsilon.
$$

We will prove that the operator K is continuous. Let $\{f_n\}_{n=1}^{\infty}$ be a convergent sequence from $C(\Omega, B)$ and let f_0 be its limit. For $\varepsilon > 0$, in view of B1, it follows that there exists a number $n_0 = n_0(\varepsilon, f_0)$ such that for $n \geq n_0$ the following inequality holds

$$\|A(x, f_n) - A(x, f_0)\| < \frac{\varepsilon}{w}, \qquad x \in \Omega.$$

Then $\|Kf_n - Kf_0\| < \varepsilon$ provided $n \geq n_0$.

This completes the proof of Lemma 1.

LEMMA 2. *Suppose that the conditions* A1, A2 *and* (B) *be fulfilled. Then the set* $K(B(0, R))$ *is equicontinuous for every central ball* $B(0, R)$.

Proof. Let $B(0, R)$ be an arbitrary ball centered at zero with radius R. Then for $g \in B(0, R)$ and $x, y \in \Omega$,

$$\begin{aligned} &\|(Kg)(x) - (Kg)(y)\| \\ &\quad \leq \int_{M_x \cap M_y} \|A(x, g)(z) - A(y, g)(z)\|_B \, d\mu_z \\ &\quad\quad + \int_{M_x \setminus M_y} \|A(x, g)(z)\|_B \, d\mu_z \\ &\quad\quad + \int_{M_y \setminus M_x} \|A(y, g)(z)\|_B \, d\mu_z. \end{aligned} \tag{7}$$

In view of B1, for any fixed $x \in \Omega$ and $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, x) > 0$ such that for $\rho(x, y) < \delta$ and $z \in \Omega$ the following inequality holds

$$\|A(x, g)(z) - A(y, g)(z)\|_B < \frac{\varepsilon}{2w}. \tag{8}$$

Moreover, there exists a number $\delta^* \in (0, \delta)$ such that for $\rho(x, y) < \delta^*$ we have

$$\mu(\{M_x \setminus M_y\} \cup \{M_y \setminus M_x\}) < \frac{\varepsilon}{4QR} \tag{9}$$

where $Q = \sup_{g \in B(0,R)} Q(g)$.

Then, from (7), (8) and (9) we obtain that for $\rho(x, y) < \delta^*$

$$\|(Kg)(x) - (Kg)(y)\|_B < \varepsilon$$

i. e. the set $K(B(0, R))$ is equicontinuous at any point $x \in \Omega$ which is to be proved.

Observe that B2 implies that the set $K(B(0, R))$ is uniformly bounded.

REMARK. The set $K(B(0, R))$ is relatively compact if and only if the sets

$$H_{x_0} = \left\{ \int_{M_{x_0}} A(x_0, f)(y)\, d\mu_y \,|\, f \in B(0, R) \right\}$$

are relatively compact for, $x_0 \in \Omega$ see [3].

REMARK 4. Let the conditions of Lemma 2 hold supposing that $Q(f)$ in the condition B2 does not depend on f, and let $wQ < 1$. Then the equation

$$Kf = f \tag{10}$$

possesses the trivial solution $f \equiv 0$ only.

Remark 4 implies that the condition $Qw \geq 1$ is necessary for (10) to possess a nonzero solution. Moreover, this condition is necessary for the existence of a nonzero element $\tilde{f} \in C(\Omega, B)$ with the property $\|\tilde{f}\| = \|K\tilde{f}\|$.

THEOREM 1. *Suppose that the following conditions hold*:

1. *The conditions* (A) *and* (B) *hold.*
2. *The space* Ω *is connected.*
3. *For every* $x_0 \in \Omega$ *and* $f \in C(\Omega, B)$ *the following inequality holds*

$$\sup_{y \in M_x} \|A(x, f)(y)\|_B \leq Q_1(f, x_0) \sup_{y \in M_x} \|f(y)\|_B, \qquad x \in M_{x_0},$$

where $Q_1(f, x_0)$ *is continuous with respect to* x_0 *for fixed* f.

Then the equation

$$f = \lambda K f \tag{11}$$

possesses the trivial solution $f \equiv 0$ *only for* $\lambda \in C$.

Proof. Let $\lambda \neq 0$, $\lambda \in C$ be an arbitrary fixed complex number and let $f \not\equiv 0$ be a solution of the equation (11). Consider the set

$$N_f = \{x \mid x \in \Omega,\ f(y) = 0,\ y \in M_x\}.$$

We will prove that $N_f \neq \emptyset$. A5 implies that there exists a point $x_0 \in \Omega$ with the property $\mu(M_{x_0}) = 0$. Let us assume that $M_{x_0} \neq \emptyset$ (The case $M_{x_0} = \emptyset$ is trivial). Then, in view of A3, it follows that for any $x \in M_{x_0}$, $M_x \subseteq M_{x_0}$ and hence $\mu(M_x) = 0$, whence, with the help of (11) we obtain that $f(x) = 0$. Thus, we proved that $x_0 \in N_f$.

The set N_f is closed. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence of elements of N_f and let $x^* = \lim_{n\to\infty} x_n \in \Omega$ be its limit. We will prove that $x^* \in N_f$.

It is sufficient to consider the case $\mu(M_{x^*}) \neq 0$ only, then, infinitely many terms of the sequence $\{x_n\}_{n=1}^{\infty}$ exist for which $\mu(M_{x_n}) \neq 0$. Let $z \in M_{x^*}$ be an arbitrary fixed point, while $\varepsilon > 0$ is a number. Then there exists a number $\delta_1 = \delta_1(\varepsilon) > 0$ such that for $x \in \Omega$ and $\rho(x, z) < \delta_1$ the following inequality holds

$$\|f(x) - f(z)\|_B < \varepsilon .$$

If we denote $\rho(z, M_{x_n}) = \inf_{q \in M_{x_n}} \rho(z, q)$, $n = 1, 2, \cdots$, then for any n there exists an element $z_n \in M_{x_n}$ for which

$$\rho(z, M_{x_n}) + \frac{\delta_1}{2} > \rho(z, z_n).$$

Condition A4 implies that there exists a number $\delta_2 = \delta_2(\delta_1) > 0$ such that if $y \in \Omega$ and $\rho(y, x^*) < \delta_2$ then $M_{x^*} \subseteq U(\delta_1/2, M_y)$. Let the number n_0 be so large that the following $\rho(x_n, x^*) < \delta_2$ holds for $n \geq n_0$. Then $M_{x^*} \subseteq U(\delta_1/2, M_{x_n})$ and hence $z \in U(\delta_1/2, M_{x_n})$ i.e. $\rho(z, M_{x_n}) < \delta_1/2$ whence we obtain $\delta_1 > \rho(z, M_{x_n}) + \delta_1/2 > \rho(z, z_n)$. Taking into account the continuity of f we get

$$\|f(z_n) - f(z)\|_B < \varepsilon .$$

Since $z_n \in M_{x_n}$ then $f(z_n) = 0$ and hence $\|f(z)\|_B < \varepsilon$, i.e. $f(z) = 0$. Thus, we proved that $x^* \in N_f$ i.e. N_f is closed. We will show that N_f is an open set as well. Let a be an element of N_f and let $\varepsilon > 0$ be a number satisfying the condition

$$\varepsilon|\lambda| \cdot Q_1(f, a) < \frac{1}{4}.$$

Condition A2 implies that there exists a number $\delta = \delta(\varepsilon, a) > 0$ such that for $x \in \Omega$ and $\rho(a, x) < \delta$ the following inequality holds

$$\mu(\{M_x \backslash M_a\} \cup \{M_a \backslash M_x\}) < \varepsilon.$$

There exists a number $\delta_1 \in (0, \delta)$ such that for any x for which $\rho(a, x) < \delta_1$ one has

$$|Q_1(f, a) - Q_1(f, x)| < \frac{1}{4|\lambda|\varepsilon}.$$

Let $b \in \Omega$, $\rho(a, b) < \delta_1$ and let $\varphi(x)$ denotes the restriction of $f(x)$ over M_b i.e. $\varphi(x) = f(x)|_{M_b}$. Then from the conditions (A) and from the condition 3 of Theorem 1 the following inequalities are fulfilled

$$\begin{aligned}
\sup_{x \in M_b} \|\varphi(x)\|_B &= |\lambda| \sup_{x \in M_b} \left\| \int_{M_x} A(x, \varphi)(y)\, d\mu_y \right\|_B \\
&\leq |\lambda| \sup_{x \in M_b} \int_{M_x \backslash M_a} \|A(x, \varphi)(y)\|_B\, d\mu_y \\
&\leq |\lambda| Q_1(f, b) \sup_{x \in M_b} \mu(M_x \backslash M_a) \sup_{x \in M_b} \|\varphi(x)\|_B \\
&\leq \varepsilon|\lambda| \left(Q_1(f, a) + \frac{1}{4|\lambda|\varepsilon} \right) \sup_{x \in M_b} \|\varphi(x)\|_B \\
&\leq \frac{1}{2} \sup_{x \in M_b} \|\varphi(x)\|_B.
\end{aligned}$$

Hence $\varphi(x) \equiv 0$ for $x \in M_b$ i.e. $f(x) \equiv 0$ for $x \in M_b$. Thus we proved that $b \in N_f$ and hence N_f is open set.

Taking into account that Ω is a connected set we conclude that $N_f = \Omega$ i.e. for every $\lambda \in \boldsymbol{C}$ the equation (11) possesses the trivial solution only.

REMARK 5. We note that in the proof of the closeness of the set N_f the compactness of the space Ω is not used.

REMARK 6. It is not difficult to see that the assertion of Theorem 1 remains valid replacing A1 by the following condition A1′. For $x \in \Omega$ the sets M_x are compact, connecred, $\bigcup_{x \in \Omega} M_x = \Omega$ and every set M_x contains a point $y = y(x)$ for which $\mu(M_y) = 0$.

While proving the solvability of the equation (1) we will employ the following theorem:

THEOREM 2 [4]. *Let D be a bounded open subset of the real Banach space $\mathcal{L}$ and let $K: \bar{D} \to \mathcal{L}$ be a compact operator. If the point $p \in D$ is such that $f + tKf \neq p$ for $f \in \partial D$ and $t \in [0, 1]$ then the equation $f + Kf = p$ has at least one solution in D.*

It will be assumed further that B is a real Banach space.

Let $p \in C(\Omega, B)$, $p \neq 0$ and let $f \in C(\Omega, B)$ be such an element for which the inequality $\|p - f\| < \|p\|$ holds.

DEFINITION 1. We say that the element f is well dislocated to the ball $B(p, R_f)$,

$$R_f \in (\|p - f\|, \|p\|) \quad \text{if for}$$

any element $g \in \partial B(p, R_f)$ there exist a point $\bar{x} = \bar{x}(f, g) \in \Omega$ and a number $\bar{\alpha} = \bar{\alpha}(f, g) \in (0, \mu(M_{\bar{x}}))$ for which the following inequality holds

$$\max \{\sup_{y \in M_{\bar{x}}} \|A(\bar{x}, f)(y)\|_B, \sup_{y \in M_{\bar{x}}} \|A(\bar{x}, g)(y)\|_B, \sup_{y \in M_{\bar{x}}} \|A(\bar{x}, f)(y) - A(\bar{x}, g)(y)\|_B\} \leq \bar{\alpha} \|f(\bar{x}) - g(\bar{x})\|_B. \tag{12}$$

DEFINITION 2. The element $p \in C(\Omega, B)$, $p \neq 0$ is called regular for the operator A if there exist a radius $R_0 \in (0, \|p\|)$ and a number $\varepsilon \in (0, \|p\| - R_0)$ such that every element $f \in \partial B(p, R_0)$ is well dislocated with respect to some ball $B(p, R_f)$, where $R_f \in (R_0 + \varepsilon, \|p\|)$.

THEOREM 3. *Let the conditions* A1, A2 *and* (B) *be fulfilled and besides let the set $K(B(p, \|p\|))$ be relatively compact. Then for every regular point $p \in C(\Omega, B)$ the equation* (1) *has at least one solution $f(x) \not\equiv 0$ in the ball $B(p, \|p\|)$.*

Proof. The condition B2 implies that $f(x) \equiv 0$ is not a solution of the equation (1).

Let p be a regular point of the operator A, and let $\{B(p, R)\}$, $R \in (0, \|p\|)$ be a family of closed balls. Assume that the condition of Theorem 2 is violated on the boundary of one of these balls.

Then for every ball $B(p, R)$ there exist a number $t_R = [0, 1]$ and an element $f_R \in \partial B(p, R)$ for which the following equality holds

$$f_R + t_R K f_R = p. \tag{13}$$

Then (13) implies that $t_R \neq 0$ for $R > 0$. Moreover, one can consider $t_R \neq 1$ for every R since the opposite means that the desired solution is found.

Let $R_0 \in (0, \|p\|)$ and $\varepsilon \in (0, \|p\| - R_0)$ be the radius and the number from Definition 2 and let $f_0 \in \partial B(p, R_0)$ $t_0 \in (0, 1)$ satisfy (13).

The regularity of p implies that we can find a ball $B(p, R_{f_0})$, $R_{f_0} \in (R_0 + \varepsilon, \|p\|)$ so that f_0 should be well dislocated with respect to $B(p, R_{f_0})$. Then, if for $g^* \in \partial\ B(p, R_{f_0})$ and $t^* \in (0, 1)$ the equality (13) holds, then there exist a point $\bar{x} = \bar{x}(f_0, g^*) \in \Omega$ and a number $\bar{\alpha} = \bar{\alpha}(f_0, g^*) \in (0, 1/\mu(M_{\bar{x}}))$ such that (12) implies the estimate

$$\sup_{y \in M_{\bar{x}}} \|t_0 A(\bar{x}, f_0)(y) - t^* A(\bar{x}, g^*)(y)\|_B \leq \bar{\alpha} \|f_0(\bar{x}) - g^*(\bar{x})\|_B.$$

The above inequality and (13) yield

$$\begin{aligned} \|f_0(\bar{x}) - g^*(\bar{x})\|_B &= \|t_0(Kf_0)(\bar{x}) - t^*(Kg^*)(\bar{x})\|_B \\ &\leq \mu(M_{\bar{x}}) \sup_{y \in M_{\bar{x}}} \|t_0 A(\bar{x}, f_0)(y) - t^* A(\bar{x}, g^*)(y)\|_B \\ &< \|f_0(\bar{x}) - g^*(\bar{x})\|_B, \end{aligned}$$

which is a contradiction.

This completes the proof of Theorem 3.

We given an example for an operator $A : \Omega \times C(\Omega, B) \to C(\Omega, B)$ for which every element $p \in C(\Omega, B)$, $p \neq 0$ is regular.

Suppose that the following conditions are fulfilled:

1. $A(x, 0) = 0$ for $x \in \Omega$.
2. For $f, g \in C(\Omega, B)$ there exists a point $\bar{x} = \bar{x}(f, g) \in \Omega$ so that the following inequality holds

$$\sup_{y \in M_{\bar{x}}} \| A(\bar{x}, f)(y) - A(\bar{x}, g)(y)\|_B \leq T(f, g)\|f(\bar{x}) - g(\bar{x})\|_B$$

where $T(f, g) \in (0, 1/2\mu(M_{\bar{x}}))$.

Let $p \in C(\Omega, B)$ be an arbitrary element ($p \neq 0$). For all functions $f, g \in C(\Omega, B)$ we set $\bar{x}(f, g) = x^*$ where x^* is a point for which $\|f - g\| = \|f(x^*) - g(x^*)\|_B$. Choose R_0 so small that $T(f, g) < \varepsilon_p/\|p\| \cdot 1/2\mu(M_{x^*})$ for suitable choise of $\varepsilon_p \in (0, \|p\| - R_0)$. Then the following inequality is fulfilled

$$\begin{aligned}
&\sup_{y \in M_{x^*}} \|A(x^*, f)(y)\|_B \\
&\quad \leq T(f, g)\|f(x^*)\|_B < \frac{\varepsilon_p}{\|p\|} \cdot \frac{1}{2\mu(M_{x^*})} \|f(x^*)\|_B \\
&\quad \leq \frac{\varepsilon_p}{\|p\|} \cdot \frac{\|f\|}{2\mu(M_{x^*})} \leq \frac{\varepsilon_p}{\mu(M_{x^*})} = \frac{\varepsilon_p \|f(x^*) - g(x^*)\|_B}{\mu(M_{x^*})\|f - g\|} \\
&\quad \leq \frac{1}{\mu(M_{x^*})} \|f(x^*) - g(x^*)\|_B.
\end{aligned}$$

Analogically we get

$$\sup_{y \in M_{x^*}} \|A(x^*, g)(y)\|_B < \frac{1}{\mu(M_{x^*})} \|f(x^*) - g(x^*)\|_B.$$

It is not difficult to see that Remark 5 remains valid for Theorem 3 as well if the equation (1) has unique solution for every regular $p \in C(\Omega, B)$.

We present two corollaries of Theorem 3 for the case when the equation (1) has unique solution:

COROLLARY 1. *Suppose that the following conditions are fulfilled:*

1. *The conditions of Theorem* 3 *hold where the condition* A1 *is replaced with* A1′.

2. *For* $x \in \Omega$ *there exists a number* $L(x) \in (0, 1/\mu(M_x))$, *such that*

$$\sup_{y \in M_x} \|A(x, f)(y) - A(x, g)(y)\| \leq L(x) \sup_{y \in M_x} \|f(y) - g(y)\|_B,$$

for all $f, g \in C(\Omega, B)$.

Then for every regular element $p \in C(\Omega, B)$ *the equation* (1) *has unique solution.*

DEFINITION 3. We say that the function $p \in C(\Omega, B)$ is weakly regular if the restriction $p|_{M_x}$ over every M_x, $x \in \Omega$ is regular element in the sense of the Definition 2 for the Banach space $C(M_x, B)$.

COROLLARY 2. *Suppose that the following conditions are fulfilled:*

1. *The conditions* A1′, A2–A5, B *hold.*

2. *There exists a sequence* $\{M_{x_n}\}_{n=1}^{\infty}$, $M_{x^{n+1}} \supset M_{x_n}$ *such that* $\bigcup_{n=1}^{\infty} M_{x_n} = \Omega$.

3. *There exists a continuous function* $L : \Omega \to [0, \infty)$ *such that for* $x \in \Omega$ *and for* $f, g \in C(M_x, B)$ *the following inequality holds*

$$\|A(x, f)(z) - A(x, g)(z)\|_B \leq L(x)\|f(z) - g(z)\|_B, \quad z \in M_x.$$

4. *The sets* $K_x(B(p|_{M_x}, \|p|_{M_x}\|))$ *are relatively compact in* $C(M_x, B)$, $x \in \Omega$.

Then the equation (1) *has unique solution for every weakly regular element* $p \in C(\Omega, B)$.

Proof. Let $x \in \Omega$ be arbitrary point. Then from Theorem 3 it follows that there exists $f \in C(M_x, B)$ for which $f + K_x f = p|_{M_x}$. Suppose that there exists another function $g \in C(M_x, B)$ for which $g + K_x g = p|_{M_x}$. According to the condition 3 of Corollary 2 we have

$$\|f(y) - g(y)\|_B \leq \int_{M_y} L(y)\|f(z) - g(z)\|_B \, d\mu_z, \qquad y \in M_x.$$

Consider the equation

$$\phi(y) = \int_{M_y} L(y)\, \phi(z)\, d\mu_z, \qquad y \in M_x,$$

where $\phi : M_x \to \mathbf{R}$.

Using Theorem 3 from [1] we conclude that for $y \in M_x$ the following inequality holds

$$\|f(y) - g(y)\|_B \leq \phi(y).$$

Taking into account Theorem 1 we get that

$$\|f(y) - g(y)\|_B = 0, \qquad y \in M_x$$

which contradicts our supposition.

The assertion of Corollary 2 follows from the condition 2 of Corollary 2, the condition A1 and the proved uniqueness of the equation $f + K_x f = p$.

Acknowledgement. The authors wishes to thank to Prof. A. D. Myshkis and Prof. G. S. Skordev for their kindness in discussing thes results.

References

1. D. D. Bainov, A. D. Myshkis and A. I. Zahariev, *On an abstract analogue of the Bellman-Gronwall inequality*, Publications of the Research Institute for Mathematical Sciences, Kyoto Uuiversity, 20, 1984.
2. E. Hille and R. Phillips, *Functional analysis and semi-groups*, AMS, Colloquium Publications Providence 1957.
3. R. E. Edwards, *Functional analysis*, Holt, Rinehart and Winston, 1965, 1070 pp.
4. V. C. L. Hutson and J. S. Pym, *Applications of functional analysis and operator theory* 1980, Academic Press, 431 pp.