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DECOMPOSITION OF L^2-FUNCTIONALS ON HILBERT SPACES WITH POISSON MEASURE

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0. **Introduction.** By using the method of multi-Hilbertian space (see [6, p. 55–62]), one can introduce a Gaussian measure μ on a space $E' \supset L^2(R)$ and two E'-valued processes $X(t)$, $Y(t)$ such that

(0.1) $L^2(E', \mu) = \sum_{n=0}^{\infty} \oplus H_n$.

(0.2) $X(t)$ is an E'-valued Wiener process, μ is the distribution of $X(1)$.

(0.3) $Y(t)$ is an Ornstein-Uhlenbeck process w. r. t. (with respect to) $X(t)$.

(0.4) H_n is an eigenspace of the infinitesimal generator associated with $Y(t)$ for each $n \geq 0$.

(0.5) μ is an invariant measure for $Y(t)$.

Although Poisson measure can be introduced on E' (see [4, p. 148]) and even the Hida calculus can be studied (see [7]), the relations analogous to (0.2)–(0.5) are missing. The purpose of this note is to introduce a different Poisson measure on E' such that all the relations analogous to (0.1)–(0.5) are retained.

In Section 1, a fundamental birth-death process is studied. A Poisson measure will be introduced in Section 2. Processes $X(t)$, $Y(t)$ will be studied in Section 3. Decomposition analogous to (0.1) will be given in Section 4 and, finally, a converging phenomenon will be given in the last section.

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1. **A BD-process.** Let $\lambda > 0$, $S = \{s_n = -\lambda + n: n \geq 0\}$ and let $y(t)$ be a birth-death process on S with birth rate $\alpha_n = \lambda$ and death rate $\beta_n = n$ at each states s_n. With these $\alpha_n, \beta_n, n \geq 0$, $y(t)$ is a Markov process and has an invariant distribution (see [1, Thm. 2, p. 324]). Let A denote the infinitesimal generator of $y(t)$. Then

$$
\begin{aligned}
Af(z) = \lambda f(z+1) - (z+2\lambda) f(z) \\
+ (z+\lambda) f(z-1), \qquad z \in S.
\end{aligned} \tag{1.1}
$$

LEMMA 1.1. *Let ν be the mean centered Poisson distribution with parameter λ and let the Charlie-Poisson polynomials $\{P_n\}$ be defined by*

$$(1+u)^{\lambda+z} e^{-\lambda u} = \sum_{n=0}^{\infty} u^n P_n(z), \qquad u > -1, \quad z \in R. \tag{1.2}$$

Then $\{P_n\}$ are orthogonal and complete in $L^2(R, \nu)$. Furthermore,

$$\lambda P_n(z+1) - (z + 2\lambda - n) P_n(z) + (z+\lambda) P_n(z-1) = 0. \tag{1.3}$$

Proof. Let $K_n(z) = P_n(z-\lambda)$, $n \geq 0$. Then from (1.2), the generating function of $\{K_n\}$ is $(1+u)^z e^{-\lambda u}$. Hence $\{K_n\}$ are orthogonal and complete in $L^2(R, \nu^*)$ where ν^* is the Poisson distribution with parameter λ (see [4, p. 152] or [8, Thm. 4.3, p. 370]). Therefore $\{P_n\}$ are orthogonal and complete in $L^2(R, \nu)$. This proves the first statement of Lemma 1.1. For each $n \geq 0$, K_n satisfies (see [8, Lemma 3.3, p. 369])

$$\lambda K_n(z+1) - (z + \lambda - n) K_n(z) + z K_n(z-1) = 0. \tag{1.4}$$

(1.3) follows from (1.4) and $K_n(z) = P_n(z-\lambda)$.

COROLLARY 1.2. *For each $n \geq 0$, $AP_n(z) = -nP_n(z)$, $z \in R$.*

THEOREM 1.3. *ν is an invariant measure for $y(t)$.*

Proof. It suffices to check that

$$\pi_\nu M_A = 0 \tag{1.5}$$

holds, where $\pi_\nu = e^{-\lambda}(1, \lambda, \cdots, \lambda^n/n!, \cdots)$ is the row vector for ν and where M_A is the infinitesimal matrix of A obtained from (1.1). Let $M_A = [m_{ij}]$. It is easy to see from (1.1) that

$$(1.6)\quad \begin{cases} m_{00} = -\lambda, & m_{10} = 1, \\ m_{n-1,n} = \lambda, & m_{nn} = -(\lambda + n), \quad m_{n+1,n} = n+1, \quad n \geq 1. \end{cases}$$

(1.5) follows from (1.6) by direct computation.

REMARK 1.4. Suppose that $y(0) = s_0 = -\lambda$. Let $b(t)$ and $d(t)$ denote the numbers of births and deaths, respectively, of $y(t)$ up to time t. Then $b(t)$ is a Poisson process with parameter λ, $y(t) = b(t) - d(t) - \lambda$ and $0 \leq d(t) \leq b(t)$. Let $x(t) = b(t) - \lambda t$. Then $x(t)$ is a mean centered Poisson process.

2. **Poisson measure.** Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and an orthonormal basis $\{h_n\}$. Let $e_n = h_n/n$, $n \geq 1$, and for each integer k, let

$$E_k = \left\{ \sum_{n=1}^{\infty} a_n e_n : \sum_{n=1}^{\infty} n^{2k} a_n^2 > \infty \right\}.$$

Then $E_{-1} = H$ and $E_{k+1} \subset E_k$ for all k. Each E_k is a Hilbert space with inner product

$$\left\langle \sum_{n=1}^{\infty} a_n e_n, \sum_{n=1}^{\infty} b_n e_n \right\rangle_k = \sum_{n=1}^{\infty} n^{2k} a_n b_n.$$

For each $k \geq 0$, E_{-k} will be identified with the dual space E_k' in the sense that the pairing for elements $\xi = \sum_{n=1}^{\infty} \xi_n e_n \in E_{-k}$ and $f = \sum_{n=1}^{\infty} f_n e_n \in E_k$ is

$$(\xi, f)_k = \sum_{n=1}^{\infty} \xi_n f_n.$$

Let $E = \bigcap_{k=1}^{\infty} E_k$ and $E' = \bigcup_{k=1}^{\infty} E_{-k}$. Then E is a multi-Hilbertian space (see [6, p. 4]) or a nuclear space (see [5, p. 301]) and E' is the dual of E.

LEMMA 2.1. *Let the functional C on E be defined by*

$$(2.1)\quad C(f) = \prod_{n=1}^{\infty} \exp\{e^{i\lambda\langle f, e_n\rangle_0} - 1 - i\lambda\langle f, e_n\rangle_0\}, \qquad f \in E.$$

Then $|C(f)| < \infty$ *for all* $f \in E$ *and* (i) C *is positive definite,* (ii) $C(0) = 1$, (iii) $C(f) \to 1$ *as* $\|f\|_0 \to 0$.

Proof. It is easily checked from (2.1) that

$$e^{(-1/2)\lambda^2\|f\|_0^2} \leq |C(f)| \leq e^{(1/2)\lambda^2\|f\|_0^2}.$$

This implies $|C(f)| < \infty$ and (iii). Each factor in the product on the right hand side of (2.1) is a characteristic function of a mean centered Poisson distribution and hence is positive definite. Therefore C is positive definite. The assertion (ii) is obvious.

THEOREM 2.2. *There exists a probability measure* μ *on* E' *such that*

$$C(f) = \int_{E'} e^{i\xi(f)}\, d\mu(\xi), \qquad f \in E. \tag{2.2}$$

Furthermore, supp. $\mu \subset E_{-1} = H$.

Proof. The existence of measure μ on E' such that (2.2) holds follows from Lemma 2.1 and the Bochner-Minlos theorem. Let $g_n = e_n/n$, $n \geq 1$. Then $\{g_n\}$ is an orthonormal basis of E_1. And,

$$\sum_{n=1}^{\infty} \|g_n\|_0^2 = \sum_{n=1}^{\infty} 1/n^2 < \infty.$$

This means that the injection mapping from E_1 to E_0 is Hilbert-Schmidt. Thus $E_1' = E_{-1} = H$ is a support for μ (see [6, Thm. 2.6.1, p. 23] or [5, Thm. 3.1, p. 121]).

REMARK 2.3. Since H is separable, the Borel algebra of H coincides with the σ-algebra generated by cylinder sets of the form $\{\xi \in H : \langle \xi, h_k \rangle < a_k,\ 1 \leq k \leq n\}$, $a_1, \cdots, a_n \in R$, $n \geq 1$, or equivalently, of the form $\{\xi \in H : \langle \xi, l_k \rangle_0 < b_k,\ 1 \leq k \leq n\}$, $b_1, \cdots, b_n \in R$, $n \geq 1$. The collection of latter cylinder sets will be denoted by $\mathscr{F}$.

LEMMA 2.4. *Let* $z_n(\xi) = \xi(e_n) = \langle \xi, e_n \rangle_0$, $n \geq 1$, *be a sequence of functionals on* H. *Then they are i.i.d. w.r.t.* μ *and have* ν *as their distribution.*

Proof. For each n, the characteristic function of z_n is, by (2.1) and Theorem 2.2,

$$\begin{aligned} &\int_H e^{itz_n(\xi)}\, d\mu(\xi) \\ &\quad = \int_H e^{i\xi(te_n)}\, d\mu(\xi) = C(te_n) = \exp\{e^{i\lambda t} - 1 - i\lambda t\}. \end{aligned} \tag{2.3}$$

This is the characteristic function of the distribution ν. Therefore,

ν is the distribution function of z_n. To show the independence of $\{z_n\}$, let $m>0$ be an arbitrary integer, $t_k \in R$, $1 \leq k \leq m$. Then by (2.1) and (2.3),

$$\int_H \exp\left\{i\sum_{k=1}^m t_k z_k(\xi)\right\} d\mu(\xi) = \int_H \exp\left\{i\xi\left(\sum_{k=1}^m t_k z_k\right)\right\} d\mu(\xi)$$
$$= \prod_{k=1}^m \exp\{e^{i\lambda t_k} - 1 - i\lambda t_k\} = \prod_{k=1}^m \int_H e^{it_k z_k(\xi)} d\mu(\xi).$$

Therefore, $\{z_n\}$ are independent w. r. t. μ.

REMARK 2.5. For $\xi \in H$, $\xi = \sum_{n=1}^\infty \langle \xi, h_n \rangle h_n = \sum_{n=1}^\infty \langle \xi, e_n \rangle_0 e_n = \sum_{n=1}^\infty z_n(\xi) e_n$.

COROLLARY 2.6. *Let $B_k \in \mathcal{B}(R)$, $1 \leq k \leq n$. Then*

$$\mu\{\xi \in H : \langle \xi, e_k \rangle_0 \in B_k,\ 1 \leq k \leq n\} = \prod_{k=1}^n \nu(B_k).$$

3. **Markov process.** Let $\{x_k(t), y_k(t)\}$, $k \geq 1$, be independent copies of $\{x(t), y(t)\}$ given in Section 1 and let

$$X(t) = \sum_{n=1}^\infty x_n(t) e_n, \qquad Y(t) = Y(0) + \sum_{n=1}^\infty y_n(t) e_n, \qquad t \geq 0$$

where $Y(0) \in H$.

LEMMA 3.1. *For each $t \geq 0$, $X(t)$, $Y(t) \in H$ a.s. and μ is the distribution of $X(1)$.*

Proof. Since

$$E\|X(t)\|^2 = \lim_{n\to\infty} E\left\|\sum_{k=1}^n x_k(t) e_n\right\|^2 = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{k^2} Ex_k^2(t)$$
$$= \lambda t \sum_{n=1}^\infty \frac{1}{n^2} < \infty,$$

it is seen that $X(t) \in H$ a.s.. Form Remark 1.4, one has $\lambda \leq y_n(t) \leq x_n(t) + \lambda t$. This fact together with $X(t) \in H$ imply that $Y(t) - Y(0) \in H$ a.s. Therefore, $Y(t) \in H$ a.s. since $Y(0) \in H$. The characteristic functional of $X(1)$ is

$$\begin{aligned} Ee^{iX(1)(f)} &= E\exp\Big\{i\sum x_n(1)\langle f, e_n\rangle_0\Big\} \\ &= \lim_{m\to\infty} E\exp\Big\{i\sum_{n=1}^{m} x_n(1)\langle f, e_n\rangle_0\Big\} \\ &= \prod \exp\{e^{i\lambda\langle f, e_n\rangle_0} - 1 - i\lambda\langle f, e_n\rangle_0\} \\ &= \int_H e^{i\xi(f)}\, d\mu(\xi). \end{aligned}$$

Therefore, μ is the distribution of $X(1)$.

Before we show that $Y(t)$ is a Markov process, we note that $B=\{\xi\in H : \langle\xi, h_n\rangle < a\}$ is a Borel set for each n and $a\in R$. And, $\{\omega : Y(s,\omega)\in B\} = \{\omega : y_n(s,\omega) < na\}$. This amounts to say that $\sigma(Y(s)) = \sigma(y_n(s), n\geq 1)$ and $\sigma(Y(r), r\leq s) = \sigma(y_n(r), r\leq s, n\geq 1)$.

LEMMA 3.2. *$Y(t)$ is a Markov process.*

Proof. Let $0\leq s<t_1<\cdots<t_m$, $n_k\geq 1$, $1\leq k\leq m$, $B\in\mathcal{B}(R^{\sum_{k=1}^m n_k})$ $F=\{(y_j(t_k), 1\leq j\leq n_k, 1\leq k\leq m)\in B\}$. From the independence and Markov property of $y_m(t)$, $m\geq 1$, one obtains that

$$\begin{aligned} E\{I_F | Y(r), r\leq s\} &= E\{I_F | y_k(r), r\leq s, k\geq 1\} \\ &= E\{I_F | y_k(r), r\leq s, 1\leq k\leq m\} \\ &= E\{I_F | y_k(s), 1\leq k\leq m\} = E\{I_F | y_k(s), k\geq 1\} \\ &= E\{I_F | Y_k(s)\}. \end{aligned} \tag{3.1}$$

Since every indicator function of a measurable set in $\sigma(Y(r), r\geq s)$ is the limit in probability (see the proof in [2, p. 309]) of a sequence of indicator functions of cylinder sets like F. (3.1) holds for every $F\in\sigma(Y(t), t\geq s)$ by Dominated Convergence Theorem. This shows that $Y(t)$ is a Markov process.

THEOREM 3.3. *μ is an invariant measure of the process $Y(t)$.*

Proof. Let T_t denote the semigroup of operators associated with the transition probability $P_t(\xi, d\eta)$ of $Y(t)$ and let T_t^* be the adjoint of T_t. For each n, let p_n denote the projection operator on the span of $h_1,\cdots,h_n$ $(e_1,\cdots,e_n)$. Then, by independence of $\{y_k(t)\}$, Theorem 1.3 and Corollary 2.6, one has for $F=\{\xi\in H : \langle\xi, e_k\rangle_0 \in B_k\in\mathcal{B}(R), 1\leq k\leq n\}\in\mathscr{F}$,

$$
\begin{aligned}
(T_t^* \mu)(F) &= \int_H P_t(\xi, F)\, d\mu(\xi) \\
&= \int_H d\mu(\xi)\, E_\xi\{y_k(t) \in B_k,\ 1 \le k \le n\} \\
&= \int_{p_n H} d(\mu p_n^{-1})(\eta)\, E_\eta\{y_k(t) \in B_k,\ 1 \le k \le n\} \\
&= \int_{p_n H} d(\mu p_n^{-1})(\eta) \prod_{k=1}^{n} E_{\eta_k}\{y_k(t) \in B_k\}, \\
&\qquad\qquad\qquad\qquad \eta = \sum_{k=1}^{n} \eta_k e_k, \\
&= \prod_{k=1}^{n} \int_R E_{\eta_k}\{y_k(t) \in B_k\}\, d\nu(\eta_k) \\
&= \prod_{k=1}^{n} \nu(B_k) = \mu(F).
\end{aligned}
$$

Therefore, $T_t^* \mu$ agrees with μ on $\mathscr{F}$. Since both $T_t^* \mu$ on μ are measures, Remark 2.3 implies that $T_t^* \mu = \mu$. Hence μ is invariant for $Y(t)$.

4. **Decomposition.** Let $L = L^2(H, \mu)$ and let $\mathscr{T}_n$ be the collection of all tame functionals $F(\xi) = u(\langle \xi, e_1 \rangle_0, \cdots, \langle \xi, e_k \rangle_0) = u(z_1(\xi), \cdots, z_n(\xi))$, $n \ge 1$, where u is a tame function on R^n, $\mathscr{T} = \bigcup_{n=1}^{\infty} \mathscr{T}_n$. To each $n^* = (n_1, n_2, \cdots)$ with $|n^*| = \sum_{k=1}^{\infty} n_k < \infty$, let

$$
P_{n^*}^*(\xi) = \prod_{k=1}^{\infty} P_{n_k}(z_k(\xi)), \qquad \xi \in H. \tag{4.1}
$$

Let $H_0 = R$ and for each $n \ge 1$, let $H_n =$ the closed span of $P_{n^*}^* (\xi)$, $|n^*| = n$.

LEMMA 4.1. *T_t, $t \ge 0$, can be extended contractively to L.*

Proof. Let $F \in L$. By Cauchy-Schwarz inequality and Theorem 3.3, one has

$$
\begin{aligned}
\|T_t F\|_L^2 &= \int_H \left\{ \int_H P_t(\xi, d\eta) F(\eta) \right\}^2 d\mu(\xi) \\
&\le \int_H \int_H P_t(\xi, d\eta)\, F^2(\eta)\, d\mu(\xi) \\
&= \int_H F^2(\eta)\, d\mu(\eta) = \|F\|_L^2.
\end{aligned}
$$

Hence T_t can be extended contractively to L for each $t \ge 0$.

LEMMA 4.2. *Let G be the infinitesimal generator of T_t. Then, for $F \in \mathscr{F}_n$ or a polynomial of $z_k(\xi)$, $1 \leq k \leq n$,*

$$(GF)(\xi) = \sum_{k=1}^{n} A_k u(z_1(\xi), \cdots, z_n(\xi)), \tag{4.2}$$

where $F(\xi) = u(z_1(\xi), \cdots, z_n(\xi))$ and where A_k is the operator A acting on the k-th variable.

Proof. This lemma follows from the fact that $\{y_k(t)\}$ are independent copies of $y(t)$ which has A as infinitesimal generator.

LEMMA 4.3. *For each $n \geq 0$, H_n is the eigenspace of G corresponding to the eigenvalue $-n$.*

Proof. This lemma follows from the definition of H_n, (4.1), (4.2) and Corollary 1.2.

LEMMA 4.4. *For $n^* = (n_1, n_2, \cdots) \neq m^* = (m_1, m_2, \cdots)$,*

$$\int_H P^*_{m^*}(\xi) P^*_{n^*}(\xi) \, d\mu(\xi) = 0.$$

Proof. This lemma follows from (4.1) and Lemma 1.1.

Since every $F \in L$ can be approximated by tame functionals in $\mathscr{T}$ which, by Lemma 1.1, in turn can be approximated by linear combination of elements in H_n, $n \geq 0$, Lemma 4.3 and Lemma 4.4 imply

THEOREM 4.5. *The space $L^2(H, \mu)$ has decomposition*

$$L^2(H, \mu) = \sum_{n=0}^{\infty} \oplus H_n,$$

where H_n, $n \geq 0$, are eigenspaces of the infinitesimal generator G of $Y(t)$. Indeed, $-G$ is the number operator such that $(-G)(F) = nF$ for $F \in H_n$, $n \geq 0$.

5. **Convergence.** In the provious discussion, the process $y(t)$, $x(t)$ have jumps equal to ± 1. If we consider jumps of $\pm h$ and let μ_h, $X_h(t)$, $Y_h(t)$, A_h, G_h, $T_h(t)$ denote the corresponding μ, $X(t)$, $Y(t)$, A, G, T_t, respectively. Then

$$A_h f(z) = \lambda(f(z+h) - 2f(z) + f(z-h)) - z/h(f(z) - f(z-h)).$$

Therefore,

$$\lim_{\lambda=(2h^2)^{-1}\to\infty} A_h f(z) = \frac{1}{2} f''(z) - zf'(z)$$

for nice function f. This implies that

$$\lim_{\lambda=(2h^2)^{-1}\to\infty} G_h u(z_1(\xi), \cdots, z_k(\xi)) = \frac{1}{2}\sum_{k=1}^{n} \frac{\partial^2}{\partial z_k^2} u(z_1(\xi), \cdots, z_n(\xi)) - \sum_{k=1}^{n} z_k(\xi) \frac{\partial}{\partial z_k} u(z_1(\xi), \cdots, z_n(\xi)).$$

Then, by the Approximation Theorem in [3, p. 190],

$$\lim_{\lambda=(2h^2)^{-1}\to\infty} T_h(t) F = T_t F$$

for each tame function F. This shows that, as $\lambda = (2h^2)^{-1} \to \infty$, $Y_h(t)$ converges weakly in H to an Ornstein-Uhlenbeck process $Y_0(t)$ for which $\lim_{\lambda=(2h^2)^{-1}\to\infty} \mu_h$ is an invariant (Ganssian) measure.

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