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ON THE ZETA FUNCTIONS ASSOCIATED WITH THE TUBE DOMAIN OF EXCEPTIONAL TYPE*

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Abstract. In this paper, we shall derive functional equations for certain zeta functions arising from Selberg trace formula for the vector space of cusp forms on a tube symmetric domain $\mathcal{H} \subset C^{27}$, equivalent to a bounded domain $\mathcal{D}$ with Hol($\mathcal{D}$) a real form $E_{7(-25)}$ of the exceptional Lie group E_7. Contributions from conjugacy classes represented by translations P_B: $Z \to Z + B$ will be computed and expressed in terms of special values of zeta functions defined.

0. **Notation.** We employed the following notations from [1]:

1. $\mathcal{C} = \mathcal{C}_{\mathfrak{f}}$: A Cayley algebra over a field $\mathfrak{f}$; it is an eight dimensional vector space over $\mathfrak{f}$ with basis $e_0, e_1, \cdots, e_7$ and internal law of composition given by the rules:

   (1) $xe_0 = e_0 x = x$ for all $x \in \mathcal{C}$,

   (2) $e_i^2 = -1, \quad i = 1, \cdots, 7,$

   (3) $e_1 e_2 e_4 = e_2 e_3 e_5 = e_3 e_4 e_6 = e_4 e_5 e_7 = e_5 e_6 e_1 = e_6 e_7 e_1$
   $= e_7 e_1 e_3 = -1.$

   For $x = x_0 e_0 + x_1 e_1 + \cdots + x_7 e_7$, we let

   $$\bar{x} = x_0 e_0 - (x_1 e_1 + \cdots + x_7 e_7),$$

   $$T(x) = 2x_0 = x + \bar{x},$$

   $$N(x) = x\bar{x} = x_0^2 + x_1^2 + \cdots + x_2^7.$$

2. $\mathfrak{o}$: ring of integral Cayley number, $\mathfrak{o}$ is generated by the roots system of E_8 as follows:

   $$\alpha_1 = \frac{1}{2}(e_0 + e_7) - \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6)$$

   $$\alpha_2 = e_0 + e_1, \quad \alpha_3 = e_0 - e_1, \quad \alpha_4 = e_2 - e_1,$$

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$\alpha_5 = e_3 - e_2, \quad \alpha_6 = e_4 - e_3, \quad \alpha_7 = e_5 - e_4,$

$\alpha_8 = e_6 - e_5.$

3. For any ring $\mathfrak{f}$ in C, we define

$$\mathfrak{J}_{\mathfrak{f}} = \left\{ \begin{bmatrix} \xi_1 & x_{12} & x_{13} \\ \bar{x}_{12} & \xi_2 & x_{23} \\ \bar{x}_{13} & \bar{x}_{23} & \xi_3 \end{bmatrix} \middle| \xi_1, \xi_2, \xi_3 \text{ in } \mathfrak{f} \text{ and } x_{12}, x_{13}, x_{23} \text{ in } \mathscr{C}_{\mathfrak{f}} \right\}.$$

For X, Y in $\mathfrak{J}_{\mathfrak{f}}$, we let

$$\det X = \xi_1 \xi_2 \xi_3 - \xi_2 N(x_{13}) - \xi_3 N(x_{12}) - \xi_1 N(x_{23}) + T((x_{12} x_{23}) \overline{x_{13}}),$$

$$T(X) = \xi_1 + \xi_2 + \xi_3,$$

$$(X, Y) = \frac{1}{2} T(XY + YX),$$

$$X \times X = X^2 - T(X) X + \frac{1}{2} (T(X)^2 - T(X^2)) E.$$

4. $\mathscr{R}_3 = \{X \in \mathfrak{J}_R \mid \det X \neq 0\}$, $\mathscr{R}_3^+$: square elements of $\mathscr{R}_3$,
$\mathscr{R}_2 = \{X \in \mathfrak{J}_R \mid \det X = 0,\ X \times X \neq 0\}$,
$\mathscr{R}_2^+$: square elements of $\mathscr{R}_2$,
$\mathscr{R}_1 = \{X \in \mathfrak{J}_R \mid \det X = 0,\ X \times X = 0,\ X \neq 0\}$.

5. $\varepsilon(\ \ ) = \exp[2\pi i(\ \ )]$.

1. Introduction. Let $\mathscr{H}$ be the tube domain defined by

$$\mathscr{H} = \{Z = X + iY \mid X \in \mathfrak{J}_R,\ Y \in \mathscr{R}_3^+\}$$

and $\mathscr{G}_R = \mathrm{Aut}(\mathscr{H})/\{\pm id\}$. It is known that, under the Cayley transform, $\mathscr{G}_R$ is the real form $E_{7(-25)}$ of Lie group of exceptional type E_7[1]. Furthermore, $\mathscr{G}_R$ is generated by

$$P_B : Z \longrightarrow Z + B, \qquad B \in \mathfrak{J}_R$$

and

$$\iota : Z \longrightarrow -(Z)^{-1}.$$

Let Γ be the arithmetic group of $\mathscr{G}_R$ generated by P_B with $B \in \mathfrak{J}_0$ and ι. For $\gamma \in \mathscr{G}_R$ and $Z \in \mathscr{H}$, we let $j(\gamma, Z)$ be the factor of automorph determined by

(1) $j(P_B, Z) = 1, \ \forall B \in \mathfrak{J}_R$,
(2) $j(\iota, Z) = \det(-Z)$,
(3) $j(g_1 g_2, Z) = j(g_1, g_2(Z))\, j(g_2, Z)$.

A holomorphic function $f(Z)$ defined on $\mathscr{H}$ is a cusp form of weight k with respect to Γ if f satisfied the following conditions:

(1) $f(\gamma(Z)) = j(\gamma, Z)^k f(Z), \ \forall \gamma \in \Gamma$,
(2) $(\det Z)^{k/2} f(Z)$ is bounded on $\mathscr{H}$.

Denote by $S(k, \mathscr{H}, \Gamma)$ be the vector space of holomorphic cusp forms of weight k with respect to Γ.

In section 1, we shall derive the Selberg trace formula for the dimension of $S(k, \mathscr{H}, \Gamma)$ when the weight k is sufficiently large.

THEOREM 1. *For even integer $k \geq 36$, we have*

$$\dim_C S(k, \mathscr{H}, \Gamma) = c(k) \int_{\Gamma\backslash\mathscr{H}} \sum_{\gamma\in\Gamma} (\det(Y)^{k-18} \det[(Z - {}^t\overline{\gamma(Z)})/2i]^{-k}\, \overline{j(\gamma, Z)}^{-k} dZ$$

with

$$c(k) = \frac{2^{-30}\pi^{-27}\Gamma(k)\,\Gamma(k-4)\,\Gamma(k-8)}{\Gamma(k-9)\,\Gamma(k-13)\,\Gamma(k-17)}.$$

In section 2 and 3, we shall define zeta functions $\xi_3^+, \xi_3^-, \xi_2^+, \xi_2^-$ associated with various lattices of $\mathfrak{J}_R$ and obtain

THEOREM 2. *Let $\xi_3^+(s)$ and $\xi_3^-(s)$ be the zeta functions defined below in* §3, (A) *and* (B), *respectively. Then we have*

$$\begin{bmatrix}\xi_3^+(9-s)\\ \xi_3^-(9-s)\end{bmatrix} = 2\pi^{12}(2\pi)^{-3s}\,\Gamma(s)\,\Gamma(s-4)\,\Gamma(s-8) \cdot \begin{bmatrix}\cos 3\pi s/2 & \cos \pi s/2\\ f(s) & g(s)\end{bmatrix}\begin{bmatrix}\xi_3^+(s)\\ \xi_3^-(s)\end{bmatrix}$$

with

$$\begin{cases} f(s) = 3\cos \pi s/2,\\ g(s) = \cos \pi s/2 \cdot (3 - 4\sin^2 \pi s/2).\end{cases}$$

THEOREM 3. *Let $\xi_2^+(s)$ and $\xi_2^-(s)$ be the zeta functions defined below in* §3, (C) *and* (D), *respectively. Then we have*

$$\begin{bmatrix} \xi_2^+(5-s) \\ \xi_2^-(5-s) \end{bmatrix} = 2\pi^4(2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s-4)\begin{bmatrix} \cos\pi s & 1 \\ 1 & \cos\pi s \end{bmatrix}\begin{bmatrix} \xi_2^+(s) \\ \xi_2^-(s) \end{bmatrix}.$$

In the final section, we shall compute the contributions from conjugacy classes represented by $P_B : Z \to Z + B$ and write them in terms of special values of zeta functions ξ_3^+ and ξ_2^+ at nonpositive integers.

THEOREM 4. *The contributions from elements in Γ, which are conjugate in Γ to $P_B : Z \to Z + B$, to the dimension formula is given by*

$$\frac{\Gamma(k)\,\Gamma(k-4)\,\Gamma(k-8)}{\Gamma(k-9)\,\Gamma(k-13)\,\Gamma(k-17)}\,2^{-30}\pi^{-27}\,\mathrm{vol}\,(\Gamma\backslash\mathscr{H}) + (k-9)\cdot 2^{21}\,3^{-1}\cdot\xi_2^+(-8) + 2^{24}\cdot\xi_3^+(0).$$

From the functional equations for ξ_3^+ and ξ_3^-, we also have

COROLLARY

$$\xi_3^+(0) = 2^{-26}\,\pi^{-14}\,\Gamma(9)\,\Gamma(5)\,\mathrm{vol}(\mathscr{T}_0\backslash\mathscr{R}_3^+).$$

Functional equations for zeta functions had been discussed in [9] in a more general context. However, the result is still far away from an explicit form when the group is a Lie group of type E_6. Here, we began with series arising from Selberg trace formula for an exceptional tube domain and expressed them as linear conbinations of zeta functions similar to those defined in [9]. Subsequently, we obtained the functional equations as shown in Theorem 2 and 3.

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2. **Selberg trace formula.** We need Lemmas which can be proved by using the following well known formulae repeatedly.

$$\text{(A)}\quad \int_{-\infty}^{\infty}\frac{dx}{[(x-a)^2+b^2]^k} = \frac{\Gamma(k-1/2)\,\Gamma(1/2)}{\Gamma(k)}\cdot\frac{1}{b^{2k-1}},\qquad k\geq 2,\quad b>0.$$

(B) $$\int_0^\infty \frac{y^{p-1}\,dy}{(y+a)^q} = \frac{\Gamma(p)\,\Gamma(q-p)}{\Gamma(q)\,a^{q-p}}, \quad a>0, \quad q>p+1, \quad p>0.$$

(C) $$\Gamma(k)\,\Gamma(k-1/2) = 2^{2-2k}\,\Gamma(2k-1)\,\Gamma(1/2), \qquad k>0.$$

(D) $$\Gamma(1/2) = \pi^{1/2}.$$

LEMMA 1. *Let Y' be a 2×2 Hermitian matrix (i.e. ${}^t\bar{Y}' = Y'$) over Cayley number $\mathcal{C}_R$. Then for positive integer k with $k \geq 10$, we have*

$$\begin{aligned} I_1 &= \int_{Y'>0} (\det Y')^{k-10}[\det(Y'+E)]^{-2k+5}\,dY' \\ &= \pi^4\Gamma(k)\,\Gamma(k-4)\,\Gamma(k-5)\,\Gamma(k-9)/\Gamma(2k-5)\,\Gamma(2k-9). \end{aligned}$$

Proof. Let

$$Y' = \begin{bmatrix} y_1 & y_{12} \\ \bar{y}_{12} & y_2 \end{bmatrix}.$$

Then

$$\begin{aligned} \det Y' &= y_1 y_2 - N(y_{12}) \quad \text{and} \\ \det(Y'+E) &= (y_1+1)(y_2+1) - N(y_{12}). \end{aligned}$$

With $u = y_1 y_2 - N(y_{12})$ as a new variable in place of y_2, then

$$I_1 = \int_0^\infty\int_0^\infty\int_{\mathcal{C}_R} \frac{u^{k-10}\,y_1^{-1}\,dy_{12}\,d\mu\,dy_1}{\{u+y_1+1+[u+N(y_{12})\,]/y_1\}^{2k-5}}.$$

Integration with respect to y_{12} by using formula (A) eight times, we get

$$I_1 = \frac{\pi^4\,\Gamma(2k-9)}{\Gamma(2k-5)} \int_0^\infty\int_0^\infty \frac{u^{k-10}\,y_1^3\,du\,dy_1}{(u+y_1+1+u/y_1)^{2k-9}}.$$

Let $u = y_1 v$. Then

$$\begin{aligned} I_1 &= \frac{\pi^4\,\Gamma(2k-9)}{\Gamma(2k-5)} \int_0^\infty \frac{v^{k-10}\,dv}{(v+1)^{2k-9}} \int_0^\infty \frac{y_1^{k-6}\,dy_1}{(y_1+1)^{2k-9}} \\ &= \frac{\pi^4\,\Gamma(2k-9)}{\Gamma(2k-5)} \cdot \frac{\Gamma(k-9)\,\Gamma(k)}{\Gamma(2k-9)} \cdot \frac{\Gamma(k-5)\,\Gamma(k-4)}{\Gamma(2k-9)} \\ &= \frac{\pi^4\,\Gamma(k)\,\Gamma(k-4)\,\Gamma(k-5)\,\Gamma(k-9)}{\Gamma(2k-5)\,\Gamma(2k-9)}. \end{aligned}$$

LEMMA 2. *Let X' be a 2×2 Hermitian matrix over Cayley number $\mathscr{C}_R$. Then for $k>6$, we have*

$$I_2=\int \det(X'+iE)^{-k}\det(X'-iE)^{-k}\,dX'$$
$$=2^{12-4k}\pi^6\,\Gamma(2k-5)\,\Gamma(2k-9)/\Gamma^2(k)\,\Gamma^2(k-4).$$

Proof. Let

$$X'=\begin{bmatrix} x_1 & x_{12}\\ \bar{x}_{12} & x_2\end{bmatrix}.$$

Then

$$\det(X'+iE)\det(X'-iE)$$
$$=(x_1^2+1)[(x_2-x_1N(x_{12})/(x_1^2+1))^2$$
$$+(1+N(x_{12})/(1+x_1^2))^2].$$

with $x'_{12}=x_{12}/\sqrt{x_1^2+1}$ as a new variable in place of x_{12}, then

$$I_2=\int_{-\infty}^{\infty}\int_{\mathscr{C}_R}\int_{-\infty}^{\infty}\frac{dx_2\,dx'_{12}\,dx_1}{(x_1^2+1)^{k-4}[(x_2-x_1N(x'_{12}))^2+(1+N(x'_{12}))^2]^k}.$$

Integration with respect to x_2 by using formula (A), then with respect to x'_{12} and x_1 simultaneously; we finally obtain

$$I_2=\frac{\Gamma(k-1/2)\,\Gamma(1/2)}{\Gamma(k)}\cdot\frac{\Gamma(2k-5)\pi^4}{\Gamma(2k-1)}\cdot\frac{\Gamma(k-9/2)\,\Gamma(1/2)}{\Gamma(k-4)}.$$

Our assertion for I_2 then follows from the following duplication formulae:

$$\Gamma(k)\,\Gamma(k-1/2)=2^{2-2k}\,\Gamma(2k-1)\,\pi^{1/2},$$

$$\Gamma(k-4)\,\Gamma(k-9/2)=2^{10-2k}\,\Gamma(2k-9)\,\pi^{1/2}.$$

LEMMA 3. *Let Y be a variable of 3×3 Hermitian matrix over Cayley number $\mathscr{C}_R$. Then for $k\geq 18$, we have*

$$I_3=\int_{\mathscr{R}_3^+}(\det Y)^{k-18}[\det(Y+E)]^{-2k+9}\,dY$$
$$=\pi^{12}\,\Gamma(k)\,\Gamma(k-4)\,\Gamma(k-8)\,\Gamma(k-9)\,\Gamma(k-13)$$
$$\cdot\,\Gamma(k-17)/\Gamma(2k-9)\,\Gamma(2k-13)\,\Gamma(2k-17)$$

Proof. Since Y is positively definite, we can let $Y = {}^t\bar{T}T$ with

$$T = \begin{bmatrix} t_1 & t_{12} & t_{13} \\ 0 & t_2 & t_{23} \\ 0 & 0 & t_3 \end{bmatrix},\ t_1,\ t_2,\ t_3 > 0 \quad \text{and} \quad t_{12},\ t_{13},\ t_{23} \in \mathcal{C}_R.$$

Then a direct calculation shows that

$$\det Y = (t_1 t_2 t_3)^2,$$

$$\begin{aligned}\det(Y+E) = {} & (t_1^2+1)(t_2^2+1)(t_3^2+1) + (t_3^2+1)\,N(t_{12}) \\ & + (t_2^2+1)\,N(t_{13}) + (t_1^2+1)\,N(t_{23}) \\ & + N(t_{12})\,N(t_{23}) - t_2\,T[\bar{t}_{23}(\bar{t}_{13}\,t_{12})] \\ \equiv {} & Q(T)\end{aligned}$$

and the Jacobian for the transformation from Y to T is $8t_1^{17}t_2^9t_3$. Consequently, we have

$$I_3 = \int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathcal{C}_R}\int_{\mathcal{C}_R}\int_{\mathcal{C}_R} \cdot \frac{8(t_1^2t_2^2t_3^2)^{k-18}\,t_1^{17}\,t_2^9\,t_3\,dt_{12}\,dt_{23}\,dt_{13}\,dt_1\,dt_2\,dt_3}{Q(T)^{2k-9}}.$$

Rewrite $Q(T)$ as

$$\begin{aligned}& (t_1^2+1)(t_2^2+1)(t_3^2+1) + (t_3^2+1)\,N(t_{12}) \\ & + (t_2^2+1)\,N(t_{13}) + N(t_{12})\,N(t_{23})/(1+t_2^2) \\ & + (t_2^2+1)\,N\left(t_{13} - \frac{t_2}{t_2^2+1}\,t_{23}\,t_{12}\right).\end{aligned}$$

Integration with respect to t_{13}, then t_{23} and then t_{12}; we get

$$I_3 = \frac{\pi^4\,\Gamma(2k-13)}{\Gamma(2k-9)}\cdot\frac{\pi^4\,\Gamma(2k-17)}{\Gamma(2k-13)}\cdot\frac{\pi^4\,\Gamma(2k-17)}{\Gamma(2k-13)} \cdot\int_0^\infty\int_0^\infty\int_0^\infty \frac{8(t_1^2t_2^2t_3^2)^{k-18}\,t_1^{17}t_2^9\,t_3^2\,dt_1\,dt_2\,dt_3}{[(t_1^2+1)(t_2^2+1)(t_3^2+1)]^{2k-17}}.$$

With $u = t_1^2$, $v = t_2^2$, $w = t_3^2$ as new variables, then the integral in the above formula can be evaluated by using formula (B) three times. Our assertion then follows.

LEMMA 4. *Let X be a variable of 3×3 Hermitian matrix over Cayley number $\mathcal{C}_R$. Then for $k \geq 18$, we have*

$$I_4 = \int_{\mathcal{G}_R} \det(X + iE)^{-k} \det(X - iE)^{-k} dX$$
$$= 2^{30-6k} \pi^{15} \Gamma(2k-9)\,\Gamma(2k-13)$$
$$\cdot\, \Gamma(2k-17)/\Gamma^2(k)\,\Gamma^2(k-4)\,\Gamma^2(k-8).$$

Proof. Let

$$A = \{(x_2 + i)\,N(x_{13}) + (x_1 + i)\,N(x_{23}) - T[(x_{12}\,x_{23})\,\bar{x}_{13}]\}/\{(x_1 + i)(x_2 + i) - N(x_{12})\},$$

$$B = [(x_1 + i)(x_2 + i) - N(x_{12})][(x_1 - i)(x_2 - i) - N(x_{12})].$$

Then

$$\det(X + iE)\det(X - iE) = B\left[\left(x_3 - \frac{A + \bar{A}}{2}\right)^2 + \left(1 + \frac{A - \bar{A}}{2i}\right)^2\right].$$

Integration with respect to x_3, we get

$$I_4 = \frac{\Gamma(k - 1/2)\,\Gamma(1/2)}{\Gamma(k)} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathcal{C}_R}\int_{\mathcal{C}_R}\int_{\mathcal{C}_R} \frac{dx_{12}\,dx_{13}\,dx_{23}\,dx_1\,dx_2}{B^k(1 + (A - \bar{A})/2i)^{2k-1}}.$$

Note that $(A - \bar{A})/2i$ is a linear combination of $N(x_{13})$, $N(x_{23})$ and $T[(x_{12}\,x_{23})\,\bar{x}_{13}]$ with coefficients depending only on x_1, x_2 and x_{12}. Indeed, we have

$$\frac{A - \bar{A}}{2i} = \alpha N(x_{13}) + \beta N(x_{23}) + \gamma T[(x_{12}\,x_{23})\,\overline{x_{13}}\,]$$

with

$$\alpha = -\frac{x_2^2 + 1 + N(x_{12})}{B},$$
$$\beta = -\frac{x_1^2 + 1 + N(x_{12})}{B}, \quad \gamma = \frac{x_1 + x_2}{B}.$$

Use formula (A), we get

$$\int_{\mathcal{C}_R}\int_{\mathcal{C}_R} \frac{dx_{13}\,dx_{23}}{(1 + (A - \bar{A})/2i)^{2k-1}} = \frac{\pi^8\,\Gamma(2k-9)}{\Gamma(2k-1)} \cdot \frac{1}{[\alpha\beta - \gamma^2 N(x_{12})]^4}.$$

A direct calculation shows that

$$\alpha\beta - \gamma^2 N(x_{12}) = B^{-1}.$$

Thus our assertion follows from Lemma 2 since

$$I_4 = \frac{2^{2-2k}\pi^9\Gamma(2k-9)}{\Gamma^2(k)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathcal{C}_R}\frac{dx_{12}\,dx_1\,dx_2}{B^{k-4}}.$$

Now we shall obtain an integral formula for holomorphic functions on $\mathscr{H}$.

PROPOSITION 1. *Let $f(Z)$ be a square integrable holomorphic function defined on $\mathscr{H}$ and k be an even integer with $k \geq 18$. Then we have*

$$f(W) = c(k)\int_{\mathscr{H}}(\det Y)^{k-18}\det\left[\frac{1}{2i}(W - {}^t\bar{Z})\right]^{-k} f(Z)\,dZ, \qquad W \in \mathscr{H},$$

with

$$c(k) = \frac{2^{-30}\pi^{-27}\Gamma(k)\Gamma(k-4)\Gamma(k-8)}{\Gamma(k-9)\Gamma(k-13)\Gamma(k-17)}.$$

Proof. Let $H(k, \mathscr{H})$ be the vector space of square integrable holomorphic function defined on $\mathscr{H}$. In other words, $H(k, \mathscr{H})$ consists of all holomorphic function $f(Z)$ defined on $\mathscr{H}$ and it satisfies the square integrable condition:

$$\|f\|_2^2 = \int_{\mathscr{H}}(\det Y)^{k-18}|f(Z)|^2\,dZ < +\infty.$$

It is well known that the Bergmann kernel function $K(W, Z)$ of $H(k, \mathscr{H})$, is a constant multiple of $\det[(1/2i)(W - {}^t\bar{Z})]^{-k}$. Now it suffices to determine the constant.

Let the constant be $c(k)$ and

$$g(Z) = \det(Z + iE)^{-k}.$$

of course, $g(Z)$ is a holomorphic functions defined on $\mathscr{H}$ and

$$\|g\|_2^2 = \int_{\mathscr{H}}(\det Y)^{k-18}\det(Z + iE)^{-k}\det({}^t\bar{Z} + iE)^{-k}\,dZ.$$

By changes of variables in the real part of Z, we get

$$\|g\|_2^2 = \int_{\mathscr{R}_3^+} (\det Y)^{k-13} \det(Y+E)^{-2k+5}\, dY$$

$$\cdot \int_{\mathscr{G}_R} \det(X+iE)^{-k} \det(X-iE)^{-k}\, dX$$

$$= I_3 \times I_4$$

$$= 2^{30-6k} \pi^{27} \Gamma(k-9)\, \Gamma(k-13)$$

$$\cdot \Gamma(k-17)/\Gamma(k)\, \Gamma(k-4)\, \Gamma(k-8).$$

This proves $g(Z)$ is an element of $H(k, \mathscr{H})$ and hence we can use it to determine the constant $c(k)$. But it follows easily from the evaluation at $Z = iE$, i.e.

$$g(iE) = (2i)^{-3k} = c(k) \cdot (2i)^{3k} \|g\|_2^2.$$

PROPOSITION 2. *For even integer $k \geq 36$ and function $f(Z)$ in $S(k, \mathscr{H}, \Gamma)$, we have*

$$f(W) = c(k) \int_{\Gamma \backslash \mathscr{H}} (\det Y)^{k-18}$$

$$\cdot \sum_{\gamma \in \Gamma} \det \left[\frac{1}{2i} (W - \overline{{}^t\gamma(Z)}) \right]^{-k} \overline{j(\gamma, Z)}^{-k} f(Z)\, dZ.$$

Proof. Let $J(\gamma, Z)$ be the determinant of the functional (Jacobian) matrix of γ at Z. By the argument of [1], we have

$$J(\iota, Z) = \pm \det(Z)^{-18}$$

where $\iota : Z \to -Z^{-1}$. Thus we have

$$|j(\gamma, Z)|^{-18} = |J(\gamma, Z)|$$

for all $\gamma \in \Gamma$. Note that $\mathscr{H}$ can be transformed into a bounded domain of C^{27}, by Proposition 1, page 44 of [2]; we obtained that the series $L(Z)$ defined by

$$L(Z) = (\det Y)^k \sum_{\gamma \in \Gamma} \det \left[\frac{1}{2i} (W - \overline{{}^t\gamma(Z)}) \right]^{-k} \overline{j(\gamma, Z)}^{-k} f(Z)$$

converges uniformly and absolutely on each compact subset of $\Gamma \backslash \mathscr{H}$ if $k \geq 36$. Thus our proposition follows from the previous proposition and $f(\gamma(Z)) = j(\gamma, Z)^k f(Z)$.

Note that $S(k, \mathscr{H}, \Gamma)$ is a subspace of the Hilbert space $H(k, \Gamma \backslash \mathscr{H})$ consisting of holomorphic functions defined on $\Gamma \backslash \mathscr{H}$

and square integrable on $\Gamma\backslash\mathscr{H}$ with respect to the measure $(\det Y)^{-18}\,dX\,dY$. By Lemma 12, p. 181 of [2], we get that $\dim_C S(k, \mathscr{H}, \Gamma) < +\infty$.

THEOREM 1. *Suppose k is an even integer with $k \geq 36$. Then*

$$\dim_C S(k, \mathscr{H}, \Gamma) = c(k)\int_{\Gamma\backslash\mathscr{H}} (\det Y)^{k-18} \cdot \sum_{\gamma\in\Gamma}\left[\det\left(\frac{1}{2i}(Z - \overline{{}^t\gamma(Z)})\right)\right]^{-k}\overline{j(\gamma, Z)}^{-k}\,dZ$$

with

$$c(k) = \frac{2^{-30}\pi^{-27}\Gamma(k)\Gamma(k-4)\Gamma(k-8)}{\Gamma(k-9)\Gamma(k-13)\Gamma(k-17)}.$$

Proof. Let $\dim_C S(k, \mathscr{H}, \Gamma) = N$ and let $\psi_1(Z), \psi_2(Z), \cdots, \psi_N(Z)$ be an orthonomal basis of $S(k, \mathscr{H}, \Gamma)$ with respect to the inner product

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathscr{H}} (\det Y)^{k-18} f(Z)\overline{g(Z)}\,dZ.$$

Suppose that $f(Z) = \sum_{j=1}^{N} c_j\psi_j(Z)$, $c_j \in C$, is an element of $S(k, \Gamma)$. By the orthonomality of the basis, we have

$$f(W) = \int_{\Gamma\backslash\mathscr{H}} (\det Y)^{k-18}\sum_{j=1}^{N}\psi_j(W)\overline{\psi_j(Z)}f(Z)\,dZ.$$

On the other hand, we also have the integral formula for $f(W)$ as shown in the previous Proposition. Hence, by the uniqueness of Bergmann kernel function, we have

$$\sum_{j=1}^{N}\psi_j(W)\overline{\psi_j(Z)} = c(k)\sum_{\gamma\in\Gamma}\left[\det\left(\frac{1}{2i}(W - \overline{{}^t\gamma(Z)})\right)\right]^{-k}\overline{j(\gamma, Z)}^{-k}.$$

Consequently, we have

$$N = \int_{\Gamma\backslash\mathscr{H}} (\det Y)^{k-18}\sum_{j=1}^{N}|\psi_j(Z)|^2\,dZ$$
$$= c(k)\int_{\Gamma\backslash\mathscr{H}} (\det Y)^{k-18}\sum_{\gamma\in\Gamma}\left[\det\left(\frac{1}{2i}(Z - \overline{{}^t\gamma(Z)})\right)\right]^{-k}\overline{j(\gamma, Z)}^{-k}\,dZ.$$

REMARK. By the fundamental set constructed on p. 533 of [1], it is easy to prove $(\det Y)^{-18}\,dX\,dY$ is a finite measure on $\Gamma\backslash\mathscr{H}$. In other words, we have

$$\mathrm{vol}\,(\Gamma\backslash\mathscr{H}) = \int_{\Gamma\backslash\mathscr{H}} (\det Y)^{-18}\, dX\, dY < +\infty.$$

2. **Zeta functions associated with Hermitian forms of rank three.** We define two subgroups of $\mathrm{GL}(\mathfrak{J}_R)$ as follows:

$$\mathscr{G}_R = \{g \mid g \in \mathrm{GL}(\mathfrak{J}_R),\ \det(g \cdot X) = \det(X)\},$$

$$\mathscr{G}_{\mathfrak{o}} = \{g \mid g \in \mathscr{G}_R,\ g \cdot \mathfrak{J}_{\mathfrak{o}} \subset \mathfrak{J}_{\mathfrak{o}}\}.$$

Then $\mathscr{G}_{\mathfrak{o}}$ is a Lie group of type E_6 and $\mathscr{G}_{\mathfrak{o}}$ is generated by unipotent transformations $(y)_{ij}$, $i \neq j$, $1 \leq i$, $j \leq 3$, $y \in \mathfrak{o}$ as defined in page 517 of [1].

Let $\Lambda = \mathfrak{J}_{\mathfrak{o}} \cap \mathscr{R}_3$ and $\Lambda^+ = \mathfrak{J}_{\mathfrak{o}} \cap \mathscr{R}_3^+$. Then Λ is a selfdual lattice with respect to the product $(X, Y) = \mathrm{trace}\ \frac{1}{2}(XY + YX)$. Define an equivalence relation $\sim$ on Λ under the operation of $\mathscr{G}_{\mathfrak{o}}$ by

$$T_1 \sim T_2 \text{ iff there exists } g \in \mathscr{G}_{\mathfrak{o}} \text{ such that } g \cdot T_1 = T_2.$$

If T is an element of Λ^+, then the set

$$\{g \mid g \in \mathscr{G}_{\mathfrak{o}},\ g \cdot T = T\}$$

is a finite subgroup of $\mathscr{G}_{\mathfrak{o}}$. We let $\omega(T)$ be the order of this group. Now define a zeta function $\xi_3^+(s)$ as

$$\text{(A)} \qquad \xi_3^+(s) = \sum_{\Lambda^+/\sim} \frac{1}{\omega(T)(\det T)^s}.$$

Let $\Lambda_1 = \Lambda^+ \cup (-\Lambda^+)$, $\Lambda_2 = \Lambda - \Lambda_1$ and $\bar{\Lambda}_2$ be the subset of Λ_2 with signature $+, +, -$.

For those T in Λ_2, we can replace $\omega(T)$ by $\mu(T)$, a constant multiple of the density defined in [9 or 10] and choose the constant so that $\mu(T) = \omega(T)^{-1}$ when $T \in \Lambda_1$, and define

$$\text{(B)} \qquad \xi_3^-(s) = \sum_{\bar{\Lambda}_2/\sim} \frac{\mu(T)}{|\det T|^s}.$$

For Re $s > 9$, the series in right hand sides of (A) and (B) are absolutely convergent and hence $\xi_3^+(s)$ and $\xi_3^-(s)$ are holomorphic functions in the complex half plane $\mathrm{Re}\, s > 9$. Furthermore, $\xi_3^+(s)$ and $\xi_3^-(s)$ have analytic continuations which are memerophic functions in the whole complex plane and holomorphic except possible simple poles at $s=9$, 5 or 1 [9].

PROPOSITION 3. *For $s>0$ and $k\geq 18+s$, we have*

$$L_3(s)\equiv\int_{\mathcal{S}_0\backslash\mathcal{R}_3^+}\sum_{S\in\Lambda}(\det Y)^{k-18-s}\det(Y+iS/2)^{-k}\,dY$$
$$=\tilde{\eta}(k,s)(\cos 3\pi(1+s)/2\cdot\xi_3^+(9+s)$$
$$+\cos\pi(1+s)/2\cdot\xi_3^-(9+s))$$

with

$$\tilde{\eta}(k,s)=2^{28+3s}\pi^{12}\prod_{j=0}^{2}\Gamma(k-s-9-4j)\,\Gamma(1+s+4j)/\Gamma(k-4j).$$

Proof. Suppose $L_3^+(s)$ and $L_3^-(s)$ are subseries of $L_3(s)$ with the summation over Λ_1 and Λ_2, respectively.

By changes of variables and an elementary calculation, we obtain

$$L_3^+(s)=\sum_{T\in\Lambda^+/\sim}\frac{1}{\omega(T)(\det T)^{9+s}}$$
$$\cdot 2\,\mathrm{Re}\left(\int_{\mathcal{R}_3^+}(\det Y)^{k-18-s}\det(Y+iE/2)^{-k}\,dY\right)$$
$$=\tilde{\eta}(k,s)\cdot\cos 3\pi(1+s)/2\cdot\xi_3^+(9+s),$$

with

$$\tilde{\eta}(k,s)=2^{28+3s}\pi^{12}\prod_{j=0}^{2}\Gamma(k-s-9-4j)\Gamma(1+s+4j)/\Gamma(k-4j).$$

In the same way, we also have

$$L_3^-(s)=\xi_3^-(s)\cdot 2\,\mathrm{Re}\left(\int_{\mathcal{R}_3^+}(\det Y)^{k-18-s}\det(Y+iH/2)^{-k}\,dY\right)$$
$$=\tilde{\eta}(k,s)\cdot\cos\pi(1+s)/2\cdot\xi_3^-(9+s),$$

with $H=\mathrm{diag}[1,1,-1]$. This proves our assertion.

To get functional equations for ξ_3^+ and ξ_3^-, we need a more detailed description similar to those given in section 4 of [9]. Define $\mathcal{R}_3=\{X\in\mathfrak{J}_R\mid\det X\neq 0\}$ and let $\mathcal{R}_3^-$ be the set of matrices in $\mathcal{R}_3$ having 2 positive and 1 negative eigenalues. Then

$$\mathcal{R}_3=\mathcal{R}_3^+\cup(-\mathcal{R}_3^+)\cup(\mathcal{R}_3^-)\cup(-\mathcal{R}_3^-)$$

is a decompositions of $\mathscr{R}_3$ into disjoint subsets. Let $A_1 = \text{diag}[-1, 1, 1]$, $A_2 = \text{diag}[1, -1, 1]$, $A_3 = \text{diag}[1, 1, -1]$ and $\mathscr{G}$ be the set of upper triangular matrices T in $M_3(\mathscr{C}_R)$ with positive numbers on the main diagonal. Define mappings ϕ_j $(j = 1, 2, 3)$ of $\mathscr{G}$ into $\mathscr{R}_3^-$ by

$$\phi_j : T \longrightarrow {}^t\bar{T} A_j T.$$

Then ϕ_j is one to one and onto a connected component $\mathscr{R}_{3j}^-$ of $\mathscr{R}_3^-$. Obviously, we have

$$\mathscr{R}_3^- = \bigcup_{j=1}^{3} \mathscr{R}_{3j}^-.$$

For any

$$T = \begin{bmatrix} \tau_1 & t_{12} & t_{13} \\ 0 & \tau_2 & t_{23} \\ 0 & 0 & \tau_3 \end{bmatrix}$$

in $\mathscr{G}$, we let

$$dT = d\tau_1\, d\tau_2\, d\tau_3\, dt_{12}\, dt_{13}\, dt_{23}$$

be the euclidean measure on $\mathscr{G}$. Then

$$dX = 2^3\, \tau_1^{17}\tau_2^{9}\, \tau_3\, dT$$

if $X = {}^t\bar{T}A_j T$ or $X = {}^t\bar{T}T$.

Denote by $\mathscr{S}(\mathscr{R}_3)$ be the functional space of rapidily decreasing functions on $\mathscr{R}_3$. For any f in $\mathscr{S}(\mathscr{R}_3)$, we define

$$\Phi^+(f, s) = \int_{\mathscr{R}_3^+} f(X)\,|\det(X)|^{-s}\, dX,$$

$$\Phi^-(f, s) = \int_{\mathscr{R}_3^-} f(X)\,|\det(X)|^{-s}\, dX$$

and

$$\hat{f}(X) = \int_{\mathscr{G}_R} f(Z) \cdot \varepsilon((Z, X))\, dZ.$$

PROPOSITION 4. *Let $\Phi^+(f, s)$ and $\Phi^-(f, s)$ be defined as above. Then we have for any $f \in \mathscr{S}(\mathscr{R}_3)$ satisfying $f(X) = f(-X)$,*

(1) *$\Phi^+(f, s)$ and $\Phi^-(f, s)$ are holomorphic functions of s in the*

half plane $\operatorname{Re} s < 0$ *and have analytic continuations which are memermorphic functions in the whole complex plane,*

(2)

$$\begin{bmatrix}\Phi^+(\hat{f}, 9-s)\\ \Phi^-(\hat{f}, 9-s)\end{bmatrix} = 2\alpha(s)\begin{bmatrix}\cos 3\pi s/2 & \cos \pi s/2\\ 3\cos \pi s/2 & \cos 3\pi s/2 + 2\cos \pi s/2\end{bmatrix}\begin{bmatrix}\Phi^+(f, s)\\ \Phi^-(f, s)\end{bmatrix}.$$

Here $\alpha(s) = \pi^{12}(2\pi)^{-3s}\Gamma(s)\Gamma(s-4)\Gamma(s-8)$.

Proof. The first assertion follows from the general theory of [9]. Here we only prove the functional equation for $\Phi^-(\hat{f}, 9-s)$. The functional equation for $\Phi^+(\hat{f}, 9-s)$ can be proved in the same way.

With the measure induced from ϕ_j, we then heve

$$\begin{aligned}\Phi(\hat{f}, 9-s) &= \sum_{j=1}^{3}\int_{\mathfrak{G}} \hat{f}({}^t\bar{T}A_j T)\, 2^3 \tau_1^{2s-1}\tau_2^{2s-9}\tau_3^{2s-17}\, dT\\ &= \lim_{\delta\to 0}\sum_{j=1}^{3}\int_{\mathfrak{G}} \hat{f}({}^t\bar{T}A_j T)\, 2^3\tau_1^{2s-1}\tau_2^{2s-9}\tau_3^{2s-17}\,\varepsilon(({}^t\bar{T}T, i\delta E))\, dT\\ &= \lim_{\delta\to 0}\int_{\mathfrak{J}_R} f(X)\, dX\cdot\sum_{j=1}^{3}\int_{\mathfrak{G}} 2^3\tau_1^{2s-1}\tau^{22s-9}\tau_3^{2s-17}\,\varepsilon(({}^t\bar{T}T, i\delta E)\\ &\qquad + ({}^t\bar{T}A_j\, T, X))\, dT.\end{aligned}$$

Set

$$g_j(X) = \lim_{\delta\to 0}\int_{\mathfrak{J}} 2^3\,\tau_1^{2s-1}\tau_2^{2s-9}\tau_3^{2s-17}\,\varepsilon(({}^t\bar{T}T, i\delta E) + ({}^t\bar{T}A_j\,T, X))\, dT.$$

By appropriate changes of variables, we have

$$g_j(X) = \det|X|^{-s}\begin{cases} g(E) & \text{if } X\in\mathscr{R}_3^+,\\ g(-E) & \text{if } X\in(-\mathscr{R}_3^+),\\ g(A_k) & \text{if } X\in\mathscr{R}_{3k}^-,\\ g(-A_k) & \text{if } X\in(-\mathscr{R}_{3k}^-).\end{cases}$$

Thus it suffices to evaluate $g_j(E)$, $g_j(-E)$, $g_j(A_k)$ and $g_j(-A_k)$. An elementary calculation using only the definition of the gamma function shows that

$$\begin{cases} g_j(E) = \alpha(s)\exp(\pi i s/2),\\ g_j(-E) = \alpha(s)\exp(-\pi i s/2),\end{cases}$$

and

$$g_j(A_k) = \alpha(s) \begin{cases} \exp(3\pi is/2), & j = k, \\ \exp(-\pi is/2), & j \neq k, \end{cases}$$

$$g_j(-A_k) = \alpha(s) \begin{cases} \exp(-3\pi is/2), & j = k, \\ \exp(\pi is/2), & j \neq k. \end{cases}$$

By our assumption $f(-X) = f(X)$, it follows that

$$\Phi^-(\hat{f}, 9-s) = 2\alpha(s)\{3\cos \pi s/2 \cdot \Phi^+(f, s) + (\cos 3\pi s/2 + 2\cos s/2)\, \Phi^-(f, s)\}.$$

With Proposition 4 and a general argument as in p. 153 of [9] (Indeed, function equation for ξ_3^+ and ξ_3^- is the same the function equation for Φ^+ and Φ^- was shown.) We then

THEOREM 2. *Let* $\xi_3^+(s)$ *and* $\xi_3^-(s)$ *be the zeta functions as defined in* (A) *and* (B) *respectively. Then we have*

(1) $\xi_3^+(s)$ *and* $\xi_3^-(s)$ *are holomorphic functions on the half plane* $\operatorname{Re} s > 9$ *and have analytic continuations which are holomorphic functions in the whole complex plane except possible poles at* $s = 9, 5$ *or* 1,

(2)

$$\begin{bmatrix} \xi_3^+(9-s) \\ \xi_3^-(9-s) \end{bmatrix} = 2\pi^{12}(2\pi)^{-3s}\,\Gamma(s)\,\Gamma(s-4)\,\Gamma(s-8) \cdot \begin{bmatrix} \cos 3\pi s/2 & \cos \pi s/2 \\ 3\cos \pi s/2 & \cos 3\pi s/2 + 2\cos \pi s/2 \end{bmatrix} \begin{bmatrix} \xi_3^+(s) \\ \xi_3^-(s) \end{bmatrix}.$$

COROLLARY. *Let* $\xi_3^+(s)$ *and* $\xi_3^-(s)$ *be the zeta functions as defined in* (A) *and* (B), *respectively. Then we have*

$$\xi_3^+(0) = -\xi_3^-(0) = 2^{-26}\pi^{-14}\,\Gamma(9)\,\Gamma(5)\int_{\mathcal{G}_0\backslash\mathcal{R}_3^+} (\det Y)^{-9}\, dY.$$

Proof. From our functional equation, we have

$$\begin{aligned} \xi_3^+(0) &= \lim_{s\to 9} \xi_3^+(9-s) \\ &= \lim_{s\to 9} 2\pi^{12}(2\pi)^{-35}\Gamma(s)\,\Gamma(s-4) \cdot \Gamma(s-8)(\xi_+^3(s)\cos 3\pi s/2 + \xi_3^-(s)\cos \pi s/2) \\ &= 2^{-26}\pi^{-15}\,\Gamma(9)\,\Gamma(5)\lim_{s\to 9}(\xi_3^+(s)\cos 3\pi s/2 + \xi_3^-(s)\cos \pi s/2). \end{aligned}$$

On the other hand, by Lemma 7 of [9], $\xi_3^+(s)$ and $\xi_3^-(s)$ have the same residue

$$\int_{\mathscr{G}_0\backslash\mathscr{R}_3^+} (\det Y)^{-9}\, dY$$

at $s=9$. Note that

$$\lim_{s\to 9} \frac{1}{s-9} (\cos 2\pi s/2 + \cos \pi s/2) = \pi.$$

Thus

$$\xi_3^+(0) = 2^{-26}\pi^{-14}\, \Gamma(9)\, \Gamma(5) \int_{\mathscr{G}_0\backslash\mathscr{R}_3^+} (\det Y)^{-9}\, dY.$$

In the same way, we prove $\xi_3^-(0) = -\xi_3^+(0)$.

4. **Zeta functions associated with Hermitian forms of rank 2.** Arguments in this section can be proved in the same way as those in previous section, so we omit most of the proofs.

Let $\mathfrak{J}'$ be the 10-dimensional vector space over $\boldsymbol{R}$ defined by

$$\mathfrak{J}' = \left\{\begin{bmatrix} \xi_1 & x \\ \bar{x} & \xi_2 \end{bmatrix} \middle| \xi_1, \xi_2 \in \boldsymbol{R} \text{ and } x \in \mathscr{C}_R\right\}.$$

For an element $X = \begin{bmatrix} \xi_1 & x \\ \bar{x} & \xi_2 \end{bmatrix}$ in $\mathfrak{J}'$, we let

$$\begin{cases} \det X = \xi_1 \xi_2 - N(x), \\ \text{trace } X = \xi_1 + \xi_2. \end{cases}$$

Define two subgroups GL($\mathfrak{J}'$) as follows:

$$\mathscr{G}'_R = \{g \mid g \in \mathrm{GL}(\mathfrak{J}'),\ \det(g\cdot X) = \det X\},$$

$$\mathscr{G}'_0 = \{g \mid g\cdot \mathfrak{J}'_0 \subset \mathfrak{J}'_0\}$$

where $\mathfrak{J}'_D = \left\{\begin{bmatrix} \xi_1 & x \\ \bar{x} & \xi_2 \end{bmatrix} \middle| \xi_1, \xi_2 \in \boldsymbol{Z},\ x \in \mathfrak{o}\right\}$ Let M^+ be the set of positively definite elements of $\mathfrak{J}'_0$ and M^- be the set of elements with signature $+, -$ in $\mathfrak{J}'_0$. Define an equivalent relation $\sim$ on $\mathfrak{J}'_0$ as

$$S_1 \sim S_2 \text{ iff there exists } g \in \mathscr{G}'_0 \text{ such that } g\cdot S_1 = S_2.$$

Now our zeta functions $\xi_2^+(t)$ and $\xi_2^-(t)$ are defined as

(C) $$\xi_2^+(t) = \sum_{M^+/\sim} \frac{1}{\omega(S)(\det S)^t}$$

and

(D) $$\xi_2^-(t) = \sum_{M^-/\sim} \frac{\mu(S)}{|\det S|^t}.$$

where $\omega(S)$ and $\mu(S)$ are defined as in the previous section.

Consider the series defined by

$$L_2(s) = \int_{\mathcal{G}'_0\backslash\mathcal{G}'^+} \sum_{S\in\mathfrak{Z}_0,\ \det S\neq 0} (\det Y)^{k-18-s} \det(Y + iS/2)^{-k}\, dY.$$

By the integral test, $L_2(s)$ is absolutely convergent for $k > 18$ and $\operatorname{Re} s > -8$. Furthermore, we have

$$L_2(s) = \xi_2^+(13+s) \cdot 2\operatorname{Re}\Big(\int_{\mathfrak{Z}'_+} (\det Y)^{k-18-s} \det(Y + iE/2)^{-k}\, dY\Big)$$
$$+ \xi_2^-(13+s) \cdot 2\operatorname{Re}\Big(\int_{\mathfrak{Z}'_+} (\det Y)^{k-18-s} \det(Y + iU/2)^{-k}\, dY\Big)$$

with $U = \operatorname{diag}[1, -1]$. An elementary calculation shows

$$L_2(s) = 2^{27+2s}\pi^4\,\Gamma(k-s-17)\,\Gamma(k-s-13)$$
$$\cdot\,\Gamma(13+s)\,\Gamma(9+s)/\Gamma(k)\,\Gamma(k-4)$$
$$\cdot\,\{\cos\pi(1+s)\cdot\xi_2^+(13+s) + \xi_2^-(13+s)\}.$$

This proves the following:

PROPOSITION 5. *Suppose $L_2(s)$ is the zeta function as defined above, then we hve for $k > 18$ and $s > -8$,*

$$L_2(s) = 2^{27+2s}\pi^4\,\Gamma(k-s-17)\,\Gamma(k-s-13)$$
$$\cdot\,\Gamma(13+s)\,\Gamma(9+s)/\Gamma(k)\,\Gamma(k-4)$$
$$\cdot\,\{\cos\pi(1+s)\cdot\xi_2^+(13+s) + \xi_2^-(13+s)\}.$$

By the general argument [9], $\xi_2^+(s)$ and $\xi_2^-(s)$ are holomorphic functions in the whole complex plane except for possible simple poles at $s = 5$ and $s = 1$. A similar consideration as that in Proposition 4 yields the following.

THEOREM 3. *Let $\xi_2^+(s)$ and $\xi_2^-(s)$ be zeta functions as defined in* (C) *and* (D), *respectively. Then we have*

$$\begin{bmatrix}\xi_2^+(5-s)\\ \xi_2^-(5-s)\end{bmatrix} = 2\pi^4(2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s-4)\begin{bmatrix}\cos\pi s & 1\\ 1 & \cos\pi s\end{bmatrix}\begin{bmatrix}\xi_2^+(s)\\ \xi_2^-(s)\end{bmatrix}.$$

REMARK. The functional equation for ξ_2^+ and ξ_2^- are consistent with the functional equation given in p. 156 of [9] or [12] with $m=10$, $n=1$.

5. **Contributions to the dimension formula from conjugacy classes represented by translations.** Let $\Pi=\{g\cdot P_B\cdot g^{-1}\,|\,g\in\Gamma,\ P_B: Z\to Z+B,\ B\in\mathfrak{J}_0\}$ be the set all conjugate elements of translation $P_B: Z\to Z+B$. We decompose Π into disjoint union of following sets:

$$\Pi_0=\{\mathrm{id}\},$$

$$\Pi_i=\{g\cdot P_B\cdot g^{-1}\,|\,g\in\Gamma,\ \mathrm{rank}\ B=i\},\qquad i=1,2,3.$$

LEMMA 5. *Let $B\in\mathfrak{J}_0$ with $B=i$ $(i=1,2)$. Then there exists $g\in\mathfrak{J}_0$ such that*

$$g\cdot B=\mathrm{diag}[s,0,0],\quad s\in\boldsymbol{Z}-\{0\}\qquad(i=1),$$

$$g\cdot B=\begin{bmatrix}s_1 & x & 0\\ \bar{x} & s_2 & 0\\ 0 & 0 & 0\end{bmatrix},\quad s_1s_2-N(x)\neq 0\qquad(i=2).$$

Proof. See Lemma in 3.2 of [1].

Now we begin to compute the contributions from Π_i $(i=0,1,2,3)$. Once and for all, we assume that k is an even integer and $k\geq 36$.

PROPOSITION 6. *The contribution from identity of Γ to the dimension formula is*

$$N_0=\Gamma(k)\,\Gamma(k-4)\,\Gamma(k-8)/\Gamma(k-9)\,\Gamma(k-13)\,\Gamma(k-17)\cdot 2^{-30}\pi^{-27}\,\mathrm{vol}\,(\Gamma\backslash\mathcal{H}).$$

Proof. The identity forms a single conjugacy class and $j(\mathrm{id},Z)=1$. It follows

$$N_0=c(k)\int_{\Gamma\backslash\mathcal{H}}(\det Y)^{-18}\,dX\,dY$$
$$=c(k)\cdot\mathrm{vol}(\Gamma\backslash\mathcal{H})$$

with

$$c(k)=\frac{2^{-30}\pi^{-27}\Gamma(k)\Gamma(k-4)\Gamma(k-8)}{\Gamma(k-9)\Gamma(k-13)\Gamma(k-17)}.$$

PROPOSITION 7. *The contribution from Π_3 to the dimension formula is*

$$N_3=2^{24}\xi_3^+(0)=2^{-2}\pi^{-14}\Gamma(9)\Gamma(5)\operatorname{vol}(\mathcal{G}_0\backslash\mathcal{R}_3^+).$$

Proof. Let $\tilde{\Lambda}=\{P_B\mid B\in\mathfrak{J}_0,\ \det B\neq 0\}$. Then the normalizer of $\tilde{\Lambda}$ in Γ is $\Gamma\cap N_0$ (in the notation of [1]) which is the semidirect product of $\mathfrak{J}_0\cdot\{\pm\,\mathrm{id}\}$ and $\{P_B|B\in\mathfrak{J}_0\}$. Note that the series

$$c(k)L_3(s)=c(k)\int_{\mathcal{G}_0\backslash\mathcal{R}_3^+}\sum_{S\in\Lambda}(\det Y)^{k-18-s}\det(Y+iS/2)^{-k}\,dY$$

is absolutely convergent for any positive number s. Thus we have

$$c(k)L_3(s)=c(k)\int_{\Gamma\backslash\mathcal{H}}\sum_{\gamma\in\Pi_3}(\det Y)^{k-18-s}$$
$$\cdot\det\left[\frac{1}{2i}(Z-{}^t\overline{\gamma(Z)})\right]^{-k}\overline{j(\gamma,Z)}^{-k}\,dZ.$$

Hence the contribution is given by

$$N_3=\lim_{s\to 0}c(k)L_3(s)=2^{24}\xi_3^+(0)$$
$$=2^{-5}\pi^{-14}\Gamma(9)\Gamma(5)\operatorname{vol}(\mathcal{G}_0\backslash\mathcal{R}_3^+).$$

The above equalities are follow from Proposition 3 and Corollary to Theorem 2.

PROPOSITION 8. *The contribution from Π_1 to the dimension formula is $N_1=0$.*

Proof. Let $M(s)=P_B$ with $B=\operatorname{diag}[s,0,0]$ and $s\in Z'=Z-\{0\}$. Then by Lemma 5, we have

$$\Pi_1=\bigcup_{g\in\Gamma}\bigcup_{s\in Z'}g\cdot M(s)\cdot g^{-1}.$$

The centralizer of $M(s)$ is the stabilizer of the boundary component $\mathcal{H}_2^\infty$ define by

$$\mathcal{H}_2^\infty=\lim_{\lambda\to\infty}\left\{\begin{bmatrix}i\lambda & 0\\ 0 & Z\end{bmatrix}\in\mathcal{H}\right\}.$$

By the argument of 6–3 of [1], we get that the set

$$F : Z = \mathrm{diag}[iy, 0, 0], \qquad y > 0$$

is a fundamental domain for the contralizer of $M(s)$ and

$$N_1 = c(k) \sum_{s \in Z'} \int_0^\infty \frac{y^{k-18}\, dy}{(y + is/2)^k} \cdot c$$

where c is the volume of a fundamental domain on $\mathcal{H}_2^\infty$ for the centralizer of $M(s)$ restricted on $\mathcal{H}_2^\infty$. Since

$$\sum_{s \in Z'} \int_0^\infty \frac{y^{k-18}\, dy}{(y + is/2)^k} = \frac{\Gamma(k-17)\,\Gamma(17)}{\Gamma(k)} \sum_{s \in Z'} \left(\frac{2}{is}\right)^{17}$$
$$= 0,$$

it follows that $N_1 = 0$.

With the same argument as in Proposition 8 and by using Theorem 3, $\mathrm{vol}(\mathrm{SL}(Z)\backslash \mathrm{SL}_2(R)) = \pi^2/3$, we get

PROPOSITION 9. *The contribution from Π_2 to the dimension formula is given by*

$$N_2 = 2^{21}\, 3^{-1}(k-9)\xi_2^+(-8).$$

Combine Proposition 6 to 9, we have

THEOREM 4. *The total contribution from conjugacy classes represented by translations $P_B : Z \to Z + B$, $B \in \mathfrak{J}_0$, to the dimension formula is*

$$\begin{aligned} &\Gamma(k)\,\Gamma(k-4)\,\Gamma(k-8)/\Gamma(k-9)\Gamma(k-13)\,\Gamma(k-17) \\ &\qquad \cdot 2^{-30}\,\pi^{-27}\,\mathrm{vol}(\Gamma\backslash\mathcal{H}) \\ &\quad + 2^{21}\,3^{-1}(k-9) \cdot \xi_2^+(-8) \\ &\quad + 2^{-2}\,\pi^{-14}\,\Gamma(9)\,\Gamma(5)\,\mathrm{vol}(\mathcal{G}_0\backslash\mathcal{R}_3^+). \end{aligned}$$

REMARK. For the case of Siegel's cusp forms on the upper-half plane of degree n, the contributions from conjugacy classes represented by $P_{NB} : Z \to Z + NB$, N is a positive integer and B is a symmetric $n \times n$ matrix over integer, to the dimension formulae with respect to the principal congruence subgroup $\Gamma_n(N)$ of $\mathrm{Sp}(n, Z)$ is computed in [10]. Especially, when $n = 1, 2, 3$ and $N \geq 3$, these contributions give the dimensions of cusp forms with respect to $\Gamma_n(N)$. We hope that Theorem 4 can be applied to obtain explicit dimension formulae for cusp forms with respect to certain subgroups of Γ of finite index.

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