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EXACT *D*-OPTIMAL DESIGNS FOR POLYNOMIAL REGRESSION

BY

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Abstract. For polynomial regression of degree $k-1$ on $[a, b]$, the approximate D-optimal design is well know. Salaevskii (1966) conjectures that an exact D-optimal design ξ^* distributes observations as evenly as possible among the k support points of the approximate D-optimal design. Constantine and Studden (1981) have a simplified proof of Salaevskii's result that the conjecture holds for sufficiently large n. Gaffke and Krafft (1982) have proved Salaevskii's conjecture for quadratic regression for all n using a new and quite simple proof. In this work following the new approach, we are able to prove Salaevskii's result for large sample case quite simply. Also for polynomial of degree ≤ 9, Salaevskii's conjecture is proved except for a few cases.

1. **Introduction.** Consider the regression design problem where for each $x \in [a, b]$ an experiment can be performed. The outcome is a random variable $y(x)$, with mean value $\theta' f(x)$, where $\theta = (\theta_0, \cdots, \theta_{k-1})'$, $f(x) = (1, x, \cdots, x^{k-1})'$, and a common variance σ^2.

Suppose that n uncorrelated observations on the response $y(x)$ are to be obtained at levels $x_1, \cdots, x_n$. Let $Y = [y(x_1), \cdots, y(x_n)]'$, $X = (x_{ij})$, where $x_{ij} = (x_i)^j$, $1 \leq i \leq n$, $0 \leq j \leq k-1$. The unknown parameter vector $\theta = (\theta_0, \cdots, \theta_{k-1})'$ is estimated by the classical least squares estimator $\hat{\theta} = (X'X)^{-1}X'Y$. Then $E\hat{\theta} = \theta$ and $\text{cov}(\hat{\theta}) = \sigma^2(X'X)^{-1}$.

An exact design specifies a probability measure ξ on $[a, b]$ which concentrates mass p_i at x_i, $i = 1, \cdots, r$, where $p_i n = m_i$, $i = 1, \cdots, r$, are integers. The design problem now is to choose

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the design in some optimal way.

For a given design ξ, the matrix $M(\xi)=n^{-1}(X'X) = \int f(x)f'(x)\,d\xi(x)$ is commonly called the "information matrix" of the design ξ. If the matrix $M^{-1}(\xi)$ is "small" or $M(\xi)$ is "large", then roughly speaking $\hat{\theta}$ is close to θ. A simple measure of the magnitude of the information matrix $M(\xi)$ is its determinant. Thus an exact design ξ^* is said to be D-optimal if ξ^* maximizes $|M(\xi)|$ among all the exact designs ξ on $[a, b]$.

The emphasis above is on restricting the m_i to be integers. An approach which is often taken in optimal design work is to extend consideration to the class of all "approximate designs", i.e. arbitrary probability measures ξ on X. This approach has the advantage of permitting a complete characterization of certain optimal designs. Hoel (1958) has obtained the result that an approximate design is D-optimal for polynomial regression of degree $k-1$ on $[a,b]=[-1,1]$ if and only if it concentrates equal mass at the roots of $(1-x^2)P'_{k-1}(x)$, where $P_{k-1}(x)$ is the $(k-1)$th Legendre polynomial. Karlin and Studden (1966a) have extended the result to some special choices of $f(x)=(f_0,f_1,\cdots,f_{k-1})'$, where $f_i(x)$ is x^i multiplied by some weight functions. Its limitation is that, in practice, only an exact design may be implemented. It is usually the case that an optimal approximation design is not exact for many choices of n. For more discussions and further references about the approximate design, see Fedorov (1972), Karlin and Studden (1966b). Therefore it is our purpose to find the exact D-optimal design in this paper for polynomial regression.

Salaevskii (1966) conjectures that an exact D-optimal design ξ^* distributes observations as evenly as possible among the k support points of the approximate D-optimal design. Constantine and Studden (1981) have provided a simpler proof of Salaevskii's result that the conjecture holds for sufficiently large n. Both of their proofs are based on the Taylor series expansion of the determinant of the information matrix with respect to the unknown exact D-optimal design points. Gaffke and Krafft (1982) have proved Salaevskii's conjecture for quadratic regression for all $n\geq 3$ quite simply. Their proof is based on the geometric-arithmetic means inequality of the information matrix.

In this paper, following the new approach, we are able to prove Salaevskii's result for large sample case quite simply. Also for polynomials of degree ≤ 9, Salaevskii's conjecture is proved except for a few cases. Although recently Gaffke (1987) has dealt with the same topic as in this paper, and it is also shown that Salaevskii's conjecture is wrong for certain values of n and N. However, it does not cover our investigations for cubic regression for $n=9$, 10, 11. Constantine, Lim and Studden (1985) has also proved the cases for cubic regression for all n where the method is somewhat different. Since we shall follow the approach of Gaffke and Krafft (1982) for general case, their clever idea is briefly described in the following. We assume throughout that $[a, b]=[-1, 1]$.

If Salaevskii's conjecture holds true for quadratic regression for all $n \geq 3$, then there are three different solutions ξ_1, ξ_2, ξ_3. If $n=3p+1$, ξ_i puts $p+1$ points on one of x_i^*, where $x_1^*=-1$, $x_2^*=0$, $x_3^*=1$, and puts p points on the other two points. If $n=3p+2$, ξ_i puts p points on one of x_i^*, where $x_1^*=-1$, $x_2^*=0$, $x_3^*=1$, and puts $p+1$ points on the other two points.

By the geometric-arithmetic means inequality, we have for any design ξ and for $1 \leq \nu \leq 3$,

$$\det M(\xi) \leq \det M(\xi_\nu)(3^{-1} \operatorname{tr} M(\xi) M^{-1}(\xi_\nu))^3 . \tag{1.1}$$

Note that $\det M(\xi_\nu)=\Pi p_i |F|^2$ is independent of ν where F^2 will be defined later. So if we let $c_n=\det M(\xi_\nu)$, then

$$\det M(\xi) \leq c_n \min_{1 \leq \nu \leq 3} (3^{-1} \operatorname{tr} M(\xi) M^{-1}(\xi_\nu))^3 .$$

To establish the theorem one has to show that for all ξ,

$$\min_{1 \leq \nu \leq 3} ((\operatorname{tr} M(\xi) M^{-1}(\xi_\nu)) \leq 3 . \tag{1.2}$$

Let

$$d(x, \xi)=f'(x) M_f^{-1}(\xi) f(x) . \tag{1.3}$$

since $\operatorname{tr} M(\xi) M^{-1}(\xi_\nu)=\Sigma d(x_i, \xi_\nu)$, (1.2) was proved by showing that

$$\min_{1\le\nu\le 3}\sum_{i=1}^{n} d(x_i,\ \xi_\nu)\le 3\,,$$

or

$$\min_{1\le\nu\le 3}\sum_{i=1}^{n} p(p+1)\,d(x_i,\ \xi_\nu)\le 3p(p+1)\,.$$

Since ξ_ν has support on $-1, 0, 1$ (the approximate D-optimal design points for quadratic polynomial), we know that $p\cdot d(x_i, \xi_\nu)\le 1$ for all x_i.

Let r_ν be the number of x_i such that $(p+1)\,d(x_i, \xi_\nu)\le 1$. Then

$$\begin{aligned}\sum_{i=1}^{n} & p(p+1)\,d(x_i,\ \xi_\nu)\\ &\le p\,r_\nu+(n-r_\nu)(p+1)\\ &\le n(p+1)-r_\nu\,.\end{aligned}$$

If $n=3p+1$ then it turns out for each x_i, $(p+1)\,d(x_i, \xi_\nu)\le 1$ for some ν, which will imply $\max r_\nu\ge p+1$. Then

$$\min_{1\le\nu\le 3}\{n(p+1)-r_\nu\}\le 3p(p+1)\,, \tag{1.4}$$

and the result follows. If $n=3p+2$, then for each x_i, $(p+1)\,d(x_i, \xi_\nu)\le 1$ for at least 2 curves and $\max r_\nu\ge 2p+2$. Then (1.2) will hold.

For general k and $n=kp+t$, it will be shown that $\max r_\nu \ge tp+t$. Then (1.2) will hold with 3 replaced by k on the right hand side of the inequality, and ν is from 1 to m, where m is the usual binomial coefficient with value $k!/(t!(k-t)!)$.

In Section 2, following the new approach Salaevskii's result for large sample case is proved. For polynomials of degree ≤ 9, by the method we use to prove Salaevskii's conjecture for large sample case, we are able to give the value of N such that for $n\ge N$, Salaevskii's conjecture is true. We give a list of the value of N for polynomial of degree ≤ 9 in the following:

degree $(k-1)$	3	4	5	6	7	8	9
N	12	15	24	35	48	63	80

For cubic regression we already know that for $n=4$, $n=8$ the exact D-optimal design coincides with the approximate D-

optimal design. Moreover for $n = 9, 10, 11$, Salaevskii's conjecture is proved by using a modification of the new approach, which is presented in Section 3.

2. Another look at Salaevskii's result for large sample case.

Again recall that the approximate D-optimal design for polynomial regression of degree $k-1$ on $[-1, 1]$ concentrates equal mass at the roots of $(1-x^2)P'_{k-1}(x)$, where $P_{k-1}(x)$ is the $(k-1)$th Legendre polynomial. At this point, we introduce the following notation:

(i) Let $\{x_1^*, \cdots, x_k^*\}$ be the set of the approximate D-optimal design points for polynomial regression of degree $k-1$ on $[-1, 1]$.

(ii) Let $g_i(x)$, $i = 1, \cdots, k$, be the fundamental Lagrange interpolation polynomials induced by the points $\{x_1^*, \cdots, x_k^*\}$.

Then it is clear that

$$\sum_{i=1}^{k} g_i^2(x) \leq 1, \qquad x \in [-1, 1]. \tag{2.1}$$

Suppose that we have n observations, where $n = kp + t$, for some $1 \leq t \leq k-1$.

(iii) Let $m = \binom{k}{t}$, $q = \binom{k-1}{t-1}$, where the brackets denote the usual binomial coefficient.

Then there are m designs which distribute the n observations as evenly as possible among $\{x_1^*, \cdots, x_k^*\}$. Thus let ξ_ν be one of such designs, say ξ_ν puts $p+1$ points on each of $x^*_{\nu_1}, x^*_{\nu_2}, \cdots, x^*_{\nu_t}$, where $x^*_{\nu_1} < \cdots < x^*_{\nu_t}$, and $\{\nu_i, i = 1, \cdots, t\} \in \{1, \cdots, k\}$, and puts p points on each of the points in $S - \{x^*_{\nu_i}, 1 \leq i \leq t\}$.

(iv) Let $d(x, \xi_\nu)$ be the variance function of design ξ_ν, $1 \leq \nu \leq m$. Then $d(x, \xi_\nu)$ can be written as

$$\begin{aligned} d(x, \xi_\nu) &= f'(x) M_f^{-1}(\xi_\nu) f(x) = g'(x) M_g^{-1}(\xi_\nu) g(x) \\ &= \sum_{i=1}^{k} \frac{g_i^2(x)}{p_{i,\nu}} \end{aligned} \tag{2.2}$$

where

$$\begin{cases} p_{i,\nu} = p+1, & \text{if } i \in \{\nu_1, \cdots, \nu_t\}, \\ p_{i,\nu} = p, & \text{otherwise}. \end{cases}$$

Now by the geometric-arithmetic means inequality for the general case where the polynomial regression function is of degree $k-1$, we have for any design ξ and for all ξ_ν, $1 \leq \nu \leq m$,

$$\det M(\xi) \leq \det M(\xi_\nu)(k^{-1} \operatorname{tr} M(\xi) M^{-1}(\xi_\nu))^k .$$

where

$$F^2 = \prod_{1 \leq i < j \leq k} (x_j^* - x_i^*)^2 \tag{2.3}$$

is the square of the Vandermonde determinant corresponding to the points $x_1^*, \cdots, x_k^*$. This in turn implies that for any ξ

$$\det M(\xi) \leq c_n \min_{1 \leq \nu \leq m} (k^{-1} \operatorname{tr} M(\xi) M^{-1}(\xi_\nu))^k . \tag{2.4}$$

As in the quadratic case, we need to show that for all ξ

$$\min_{1 \leq \nu \leq m} (\operatorname{tr} M(\xi) M^{-1}(\xi_\nu)) \leq k . \tag{2.5}$$

In the following, two lemmas which are useful for proving (2.5) are proved. The first lemma is a generalization of the inequality used in Gaffke and Krafft (1982). More notation are needed and introduced below.

It can easily be checked that $g_i^2(x)$ and $g_{i+1}^2(x)$ intersect only once in $[x_i^*, x_{i+1}^*]$, for $i = 1, \cdots, k-1$. Therefore,

(v) Let $\{x_0', \cdots, x_k'\}$ be the set of points where x_i' is the unique intersection point of $g_i^2(x)$ and $g_{i+1}^2(x)$ in $[x_i^*, x_{i+1}^*]$ for $i = 1, \cdots, k-1$; and $x_0' = x_1^*$, $x_k' = x_k^*$.

From (i) it is clear that $x_0' = x_1^* = -1$, $x_k' = x_k^* = 1$.

(vi) Let $R^2(x) = g_i^2(x)$, for all $x \in [x_{i-1}', x_i']$, $1 \leq i \leq k$.

LEMMA 2.1. *There exists p_0 such that*

$$\begin{aligned} \sum_{i=1}^k g_i^2(x) &\leq \frac{p_0}{p_0+1} + \frac{1}{p_0+1} R^2(x) \\ &\leq \frac{p}{p+1} + \frac{1}{p+1} R^2(x) \end{aligned} \tag{2.6}$$

for $p \geq p_0$, and for every $x \in [-1, 1]$.

Proof. By the fact that S is the set of the approximate D-optimal design points, it is known that $x_2^*, \cdots, x_{k-1}^*$ are the local maxima of $\sum g_i^2(x)$ in $[-1, 1]$. Also

$$\sum_{i=1}^{k} g_i^2(x_j^*) = 1, \qquad j = 1, \cdots, k.$$

this in turn implies that

$$\sum_{i=1}^{k} g_i^2(x_j') < 1, \qquad j = 1, \cdots, k-1; \tag{2.7}$$

and for $2 \leq j \leq k-1$

$$\frac{d}{dx} \sum_{i=1}^{k} g_i^2(x) \Big|_{x=x_j^*} = 0, \qquad \frac{d^2}{dx^2} \sum_{i=1}^{k} g_i^2(x) \Big|_{x=x_j^*} < 0. \tag{2.8}$$

Let

$$R_{j,p}(x) = \sum_{i=1}^{k} g_i^2(x) - \frac{1}{p+1} g_j^2(x),$$

for $j = 1, \cdots, k$; and for all positive integer p. Then

$$R_{j,p}(x_j^*) = p/(p+1),$$

and for $i = 2, \cdots, k-1$, $j = 1, \cdots, k$,

$$d/dx(R_{j,p}(x))|_{x=x_i^*} = 0.$$

From (2.8), for fixed i, j, there exists a $p_{i,j,1}$ such that for $p \geq p_{i,j,1}$,

$$\frac{d^2}{dx^2}(R_{j,p}(x)) \Big|_{x=x_i^*} < 0,$$

i.e. x_i^* is a local maximum of $R_{j,p}(x)$.

Let $p_{j,1} = \max_{2 \leq i \leq k-1} p_{i,j,1}$, then for $p \geq p_{j,1}$, it is easy to see that $x_1^*, \cdots, x_{k-1}^*$ are the only local maxima for function $R_{j,p}(x)$. In order to find the absolute maxima of $R_{j,p}(x)$ for $p \geq p_{j,1}$ at the interval $[x_j', x_{j+1}']$, we need to check the values of $R_{j,p}(x)$ at the boundary points x_j' and x_{j+1}'.

In view of (2.7), there exists a $p_{j,2}$ such that for $p \geq p_{j,2}$ and for $i = j$ or $i = j+1$,

$$R_{j,p}(x_i') \leq p/(p+1),$$

Thus for every j, $1 \leq j \leq k$, we have

$$R_{j,p}(x) \leq p/(p+1),$$

where $x \in [x'_{j-1}, x'_j]$ and $p \geq \max(p_{j,1}, p_{j,2})$. Let

(2.9) $$p_0 = \max_{1 \leq j \leq k} (p_{j,1}, p_{j,2}),$$

then for $p \geq p_0$,

$$R_{j,p}(x) \leq p/(p+1)$$

for every $x \in [x'_{j-1}, x'_j]$, and for all $j = 1, \cdots, k$. Therefore the first inequality in (2.6) is proved. The second inequality follows from the fact that $R^2(x) \leq 1$ for $x \in [-1, 1]$.

LEMMA 2.2. *For $p \geq p_0$, p_0 is as defined in* (2.9)

$$(p+1)\, d(x, \xi_\nu) \leq 1, \qquad \textit{for} \quad x \in [x'_{\nu_{j-1}}, x'_{\nu_j}],$$

$1 \leq \nu_j \leq t$ *and for all* $1 \leq \nu \leq m$.

Proof. Divide the interval $[-1, 1]$ into k subintervals by the points x'_i, $i = 0, \cdots, k$, such that

$$[-1, 1] = \bigcup_{i=1}^{k} [x'_{i-1}, x'_i].$$

In view of (2.2), for every ν, $1 \leq \nu \leq m$,

$$p(p+1)\, d(x, \xi_\nu) = (p+1) \sum_{i=1}^{k} g_i^2(x) - \sum_{j=1}^{t} g_{\nu_j}^2(x)$$

where $\{\nu_1, \cdots, \nu_t\}$ is a subset of $\{1, \cdots, k\}$.

If $1 \in \{\nu_1, \cdots, \nu_t\}$, then by Lemma 2.1 it implies that for $x \in [-1, x'_1]$,

$$\begin{aligned} p(p+1)\, d(x, \xi_\nu) & \\ &= (p+1) \sum_{i=1}^{k} g_i^2(x) - g_1^2(x) - \sum_{j=2}^{t} g_{\nu_j}^2(x) \\ &\leq p + R^2(x) - g_1^2(x) - \sum_{j=2}^{t} g_{\nu_j}^2(x) \\ &\leq p. \end{aligned}$$

Similarly, it can be proved that,

$$(p+1)\, d(x, \xi_\nu) \leq 1, \qquad \text{for} \quad x \in [x'_{\nu_{j-1}}, x'_{\nu_j}].$$

thus the lemma is proved.

In view of Lemma 2.2, we see that there are at least q out of m functions of $\{d(x, \xi_\nu)\}$ such that

$$(p+1)\,d(x, \xi_\nu) \leq 1,$$

for every $x \in [x'_{i-1}, x'_i]$ and $1 \leq i \leq k$. In other words, for every $x \in [-1, 1]$, there are at least q indices ν such that $(p+1)\,d(x, \xi_\nu) \leq 1$. The particular indices depend on the interval x is in.

Now we are ready to prove the main theorem.

THEOREM 2.1. *For* $n = kp + t \geq N = kp_0$, *where* $1 \leq t \leq k-1$, *and* p_0 *as defined in* (2.9), *there is an exact D-optimal design* $\xi^* = \xi_\nu$, *for some* ν, $1 \leq \nu \leq m$.

Proof. As discussed previously, once (2.5) holds for all ξ, the theorem is true. From (2.2), we have

$$\begin{aligned}\operatorname{tr} M(\xi)\,M^{-1}(\xi_\nu) &= \sum_{i=1}^{n} f'(x_i)\,M^{-1}(\xi_\nu)\,f(x_i) \\ &= \sum_{i=1}^{n} d(x_i, \xi_\nu)\,,\end{aligned} \tag{2.10}$$

where x_i, $i = 1, \cdots, n$ are the support points of a design ξ.

Now let r_ν be the number of i's for which

$$(p+1)\,d(x_i, \xi_\nu) \leq 1, \qquad 1 \leq \nu \leq m.$$

In the following, we shall prove that for $p \geq p_0$

$$\max_{1 \leq \nu \leq m} r_\nu \geq tp + t\,, \tag{2.11}$$

then

$$\begin{aligned} p(p+1) \min_{1\leq\nu\leq m} \left(\sum_{i=1}^{n} d(x_i, \xi_\nu)\right) &\leq \min_{1\leq\nu\leq m} \left(pr_\nu + (n - r_\nu)(p+1)\right) \\ &= (kp+t)(p+1) - \max_{1\leq\nu\leq m} r_\nu \\ &\leq kp(p+1)\,. \end{aligned}$$

Together with (2.10), (2.5) is obtained for all ξ.

Now we prove (2.11) for $p \geq p_0$. Let λ_i be the number of observations, in the interval $[x'_{i-1}, x'_i]$, for $1 \leq i \leq k$, and let $\{\lambda_{[i]}\}$ be the ordered number of $\{\lambda_i\}$ such that

$$\lambda_{[1]} \geq \lambda_{[2]} \geq \cdots \geq \lambda_{[k]},$$

therefore,

$$n = kp + t = \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \lambda_{[i]}.$$

From Lemma 2.2, we claim that for $p \geq p_0$, there exists an s_ν, where $s_\nu = \sum_{i=1}^{t} \lambda_{[i]}$, such that $r_\nu \geq s_\nu \geq tp + t$.

For $p \geq p_0$, from Lemma 2.2, we know $r_\nu \geq s_\nu$. The second inequality can be obtained as follows:

If $s_\nu < tp + t$, it means that

$$\sum_{i=1}^{t} \lambda_{[i]} < tp + t \Longrightarrow \lambda_{[t]} < p + 1.$$

and

$$\begin{aligned}\sum_{i=t+1}^{k} \lambda_{[i]} &= n - \sum_{i=1}^{t} \lambda_{[i]} \\ &= (kp + t) - \sum_{i=1}^{t} \lambda_{[i]} \\ &> (k - t)\, p \\ \Longrightarrow \lambda_{[t+1]} &> p + 1,\end{aligned}$$

which contradicts that $\lambda_{[t]} > \lambda_{[t+1]}$. Therefore the theorem is proved.

It is natural to ask that how many observations are sufficient for the theorem to be true. In the following, for polynomial of degree from 3 to 9, we give a list of the smallest p_0 and N values such that Lemma 2.1 holds. Therefore Theorem 2.1 is true for $n \geq N = k\, p_0$.

degree $(k-1)$	3	4	5	6	7	8	9
p_0	3	3	4	5	6	7	8
N	12	15	24	35	48	63	80

3. **Cubic regression for** $N = 9, 10, 11$. In the case of cubic regression, for $p = 2$, it can be checked that inequality (2.5) holds

for $x \in [-1, -\varepsilon] \cup [\varepsilon, 1]$, for a constant $\varepsilon \leq 0.03$. Also the left side of the inequality is greater than the right side by only a small amount for $x \in [-\varepsilon, \varepsilon]$. Therefore after making some modifications of the proof in the previous section, we are able to prove the conjecture for $n = 9, 10, 11$.

Again by the geometric-arithmetic means inequality, we have for any design ξ', ξ'' with 4 or more design points,

$$\det M(\xi') \leq \det M(\xi'')(k^{-1} \operatorname{tr} M^{-1}(\xi'') M(\xi'))^k ,$$

which can be written as

$$\det M(\xi') \leq \det M(\xi_\nu) \left\{ \frac{\det M(\xi'')}{\det M(\xi_\nu)} \left(\frac{1}{k} \operatorname{tr} M^{-1}(\xi'') M(\xi') \right)^k \right\}. \tag{3.1}$$

For convenience, we divide the interval $[-1, 1]$ into 5 subintervals. Let

$$I_1 = [-1, x_1'] , \qquad I_2 = [x_1', -0.03] ,$$
$$I_3 = [0.03, x_3'] , \qquad I_4 = [x_3', 1] ,$$
$$I_5 = [-0.03, 0.03] .$$

Case (i) $n = 9$. Then there are 4 designs ξ_ν, $1 \leq \nu \leq 4$, where ξ_ν puts 3 points on x_ν^* and 2 points on each of $S - \{x_\nu^*\}$.

From the proof of Lemma 2.2 it is easy to see that for $x \in I_\nu$,

$$3 \cdot d(x, \xi_\nu) \leq 1 , \qquad 1 \leq \nu \leq 4 .$$

Therefore for any design ξ with 3 points or more in any one of the intervals I_ν, $1 \leq \nu \leq 4$, from the proof of Theorem 2.1, we have

$$\det M(\xi) \leq \det M(\xi_\nu) .$$

Since in the set $[-1, 1] - I_5$ there can be at most 8 support points of design ξ, there is at least one point in I_5. Now consider designs with one or more design points in I_5.

Let ξ'' be the exact design with 10 observations, which has support points x_1^*, x_2^*, x_3^*, x_4^* with corresponding weight 2/10, 3/10, 3/10, 2/10. Then

$$\begin{aligned}
\operatorname{tr} M^{-1}(\xi'')\, M(\xi) &= \frac{1}{9}\sum_{i=1}^{9}\sum_{\nu=1}^{4}\frac{g_\nu^2(x_i)}{p_\nu} \\
&= \frac{1}{9}\sum_{i=1}^{9}\left\{\frac{10}{2}g_1^2(x_i)+\frac{10}{3}g_2^2(x_i)+\frac{10}{3}g_3^2(x_i)+\frac{10}{2}g_4^2(x_i)\right\} \\
&= \frac{10}{9}\sum_{i=1}^{9}\left\{\frac{g_1^2(x_i)}{2}+\frac{g_2^2(x_i)}{3}+\frac{g_3^2(x_i)}{3}+\frac{g_4^2(x_i)}{2}\right\} \\
&= \frac{10}{9}\sum_{i=1}^{9} d(x_i, \xi'')
\end{aligned}$$

where $d(x, \xi'')$ is the variance function of design ξ''.

It is clear that $3\cdot d(x, \xi'') \le 3/2$ for every $x \in [-1, 1]$. Also it can be checked that

$$3\cdot d(x, \xi'') \le 1\,, \qquad \text{for} \quad x \in I_2 \cup I_3\,,$$

and since

$$\begin{aligned}
3\cdot d(x, \xi'') \\
&= (25/64)\{2(x^2-1)^2(5x^2+1)+3(x^2-1/5)^2(x^2+1)\}\,,
\end{aligned}$$

we have

$$3\cdot d(x, \xi'') \le 3\cdot d(.04, \xi'') < .840167\,, \qquad \text{for} \quad x \in I_5\,.$$

Therefore,

$$\begin{aligned}
\sum 3d(x_i, \xi'') &\le (.840167+4+6) \\
&= (10.840167)\,.
\end{aligned}$$

Together with the fact that for $1 \le \nu \le 4$

$$\det M(\xi) = 9^{-4}\cdot 2^3\cdot 3\cdot F^2\,,$$

$$\det M(\xi'') = 10^{-4}\cdot 3^2\cdot 2^2\cdot F^2$$

where F^2 is as defined in (2.3), it implies that

$$\det M(\xi) \le \det M(\xi_\nu)\,.$$

Therefore ξ_ν, $1 \le \nu \le 4$, are the exact D-optimal designs.

For cases (ii) and (iii) with $n = 11, 10$, the conjecture can be proved along the lines of that of case (i). We outline the proof as follows: Let $I = [-1, 1]$.

Case (ii) $n=11$. First eliminate those designs with more than 9 points in any of the following four intervals $I-I_1$, $I-I_4$, $I-I_2-I_5$, $I-I_3-I_5$ by the property of the variance function of design ξ_ν with 11 design points. Then choose ξ'' to be the one with support on x_1^*, x_2^*, x_3^*, x_4^* and with corresponding weights 3/10, 2/10, 2/10, 3/10.

Case (iii) $n=10$. Similarly, first eliminate those designs with more than 6 points in any one of the following four intervals, $I_1 \cup I_2$, $I_3 \cup I_4$, $I-I_1-I_4$, $I-I_2-I_3$. Then choose ξ'' to be the one with 11 design points on x_1^*, x_2^*, x_3^*, x_4^* and with corresponding weights either 3/11, 3/11, 3/11, 2/11 or 2/11, 3/11, 3/11, 3/11.

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