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DENSE ORBITS AND DENSE PERIODICITY ON THE INTERVAL

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Abstract. We give descriptions of those continuous functions on the interval for which the set of periodic points or the orbit of some point is dense.

1. Introduction. Throughout this note, let I denote a compact real interval and let f be a continuous function from I into itself. If n is a positive integer, let f^n denote the nth iterate of f : $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n \geq 2$. For $x_0 \in I$, call the set $\{f^n(x_0) \mid n \geq 0\}$ the orbit of x_0 under f and denote it by $O_f(x_0)$. If, for some $x_0 \in I$, the orbit of x_0 under f is dense in I, then we say that f has a dense orbit in I. If $f^m(x_0) = x_0$ for some positive integer m, then we call x_0 a periodic point of f and call the smallest such integer m the minimal period of x_0 under f.

Associate with f is the inverse limit space

$$(I, f) = \{(x_0, x_1, \cdots) \mid f(x_{i+1}) = x_i, \quad i \geq 0\}$$

with metric

$$d((x_0, x_1, \cdots), (y_0, y_1, \cdots)) = \sum_{i=0}^{\infty} |x_i - y_i|/2^i.$$

In [2, 3, 4], Barge and Martin study the dynamics of continuous functions f on the interval by analyzing (I, f) which is a compact, connected metric space and is an example of what Bing [5] has called a snakelike continuum. In particular, they study those continuous functions from I onto itself which have a dense orbit in I or whose periodic points are dense in I.

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In this note, we study the same functions. However, our method is different from theirs and in some cases, we obtain stronger results. We state main results in Section 2 and give the proofs in Sections 3, 4, and 5.

2. **Statement of main results.** We state main results in this section. Recall that a subset S of I is residual if it is the intersection of countably many open dense sets in I.

THEOREM 1. *Let $f \in C^0(I, I)$. Then the following statements are equivalent*:

(1) (a) *The set of all periodic points of f is dense in I.*

(b) *There is no proper closed subinterval of I which is invariant under f^2.*

(2) *For every closed subinterval J of I and every closed subinterval $[c, d]$ of* int(I), *there is a positive integer M such that $f^m(J) \supset [c, d]$ for every integer $m \geq M$.*

(3) *For every $x_0 \in \text{int}(I)$ and every increasing sequence $n_1 < n_2 < \cdots$ of positive integers, the set $\bigcup_{i=1}^{\infty} f^{-n_i}(x_0)$ is dense in I.*

(4) *For any countable dense subset W of* int(I), *the set $\bigcup_{n=1}^{\infty} f^{-2n}(x_0)$ is dense in I for every $x_0 \in W$.*

(5) *f^2 has a dense orbit in I.*

(6) *For any infinite subset A of the positive integers, there is a residual set S in I such that if $x_0 \in S$, then the set $\{f^n(x_0) \mid n \in A\}$ is dense in I.*

(7) *For every positive integer k, f^{2k} has a dense orbit in I.*

(8) *For some positive integer k, f^{2k} has a dense orbit in I.*

(9) (a) *For every open subinterval J of I, there is a prime number p such that if $M = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$, where $\{p_1, p_2, \cdots, p_r\}$ is the set of all prime numbers $< p$ and each t_i, $1 \leq i \leq r$, is any fixed positive integer such that $p_i^{t_i} > p$ and if, for every positive integer k, n_k is the number of periodic points of the function $x \to 1 - 2x^2$ on $[-1, 1]$ with minimal period k, then for every integer $m \geq M$, J contains at least N_m periodic points of f with minimal period m, where*

$$N_m \geq \begin{cases} 2, & \text{if } m \text{ is a prime number,} \\ n_{(m/q)}, & \text{if } m \text{ has a prime factor } q \geq p, \\ n_{(m/p_j^{t_j})}, & \text{if } m = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r} \text{ with each } s_i \geq 0 \text{ and } s_j > t_j \\ & \text{for some } j. \end{cases}$$

(b) *There is no proper closed subinterval of I which is invariant under f^2.*

(10) (a) *f has a dense orbit in I.*

(b) *f has a periodic point of minimal period m for some odd integer $m > 1$.*

The following known result (see [2], [6]) is an easy consequence of the above theorem. Recall that for any set B in I, $\bar{B}$ denotes the closure of B in I.

THEOREM 2. *Assume that x_0 is a point of I whose orbit under f is dense in I. For integers $s \geq 2$ and $0 \leq k \leq s-1$, let $A_{s,k}(x_0) = \{f^{sn+k}(x_0) \mid n \geq 0\}$. Then exactly one of the following holds.*

(i) *If f has a periodic point of minimal period m for some odd integer $m > 1$, then $A_{s,k}(x_0)$ is dense in I for all integers $s \geq 2$ and $0 \leq k \leq s-1$.*

(ii) *If f has no periodic points of minimal period m for any odd integer $m > 1$, then both $\overline{A_{2,0}(x_0)}$ and $\overline{A_{2,1}(x_0)}$ are closed subintervals of I and there is a unique fixed point z of f in I such that $I = \overline{A_{2,0}(x_0)} \cup \overline{A_{2,1}(x_0)}$, $\overline{A_{2,0}(x_0)} \cap \overline{A_{2,1}(x_0)} = \{z\}$, $f(\overline{A_{2,0}(x_0)}) = \overline{A_{2,1}(x_0)}$, $f(\overline{A_{2,1}(x_0)}) = \overline{A_{2,0}(x_0)}$, f^2 has a periodic point of odd period > 1 in $\overline{A_{2,0}(x_0)}$ and in $\overline{A_{2,1}(x_0)}$. Consequently, if f has a dense orbit in I, then f has a periodic point of minimal period $2m$ for some odd integer $m > 1$.*

From Theorem 1, we also obtain a characterization of those continuous functions from I onto itself whose periodic points are dense in I (see [3] also). In the following, if $J = [c, d]$ is a closed subinterval of I, we define min J and max J by letting min $J = c$ and max $J = d$.

THEOREM 3. *Assume that the set of all periodic points of f is dense in I. Then exactly one of the following holds.*

(a) *There exists an at most countable (possibly finite or empty) collection of mutually disjoint open subintervals* $I_1, I_2, \cdots$, *of* I *such that*

(i) *for each* $i \geq 1$, $f(\bar{I}_i) = \bar{I}_i$ *and* f^2 *has a dense orbit in* I_i.

(ii) *if* $x \in I - \bigcup_{i=1}^{\infty} I_i$ *and if* x *is not an endpoint of* I, *then* $f(x) = x$.

(b) *There exist a unique (possibly degenerate) closed subinterval* I_0 *of* I *and an at most countable (possibly finite or empty) collection of mutually disjoint open subintervals* $\cdots, I_{-n}, \cdots, I_{-1}, I_1, \cdots, I_n, \cdots$ *of* $I - I_0$ *such that*

(i) $f(I_0) = I_0$ *and if* I_0 *is nondegenerate, then* f^2 *has a dense orbit in* I_0 *and* f *exchanges both endpoints of* I_0,

(ii) *for each* $i \geq 1$, $f(\bar{I}_i) = \bar{I}_{-i}$, $f(\bar{I}_{-i}) = \bar{I}_i$, *and* f^4 *has a dense orbit in* I_i *and in* $\bar{I}_{-i}$,

(iii) *for each* $j = \pm 1, \pm 2, \cdots$, *if* $\bar{I}_j$ *contains no endpoints of* I, *then* $f(\min \bar{I}_j) = \max \bar{I}_{-j}$ *and* $f(\max \bar{I}_j) = \min \bar{I}_{-j}$,

(iv) *if* $\bar{I}_{-n}$ *contains the left-endpoint of* I, *then* $\bar{I}_n$ *contains the right endpoint of* I *and vice versa. In this case,* $f(\max \bar{I}_{-n}) = \min \bar{I}_n$,

(v) *if* $x \in I - (I_0 \cup \bigcup_{i=1}^{\infty}(I_{-i} \cup I_i))$ *and if* x *is not an endpoint of* I, *then* $f^2(x) = x$ *and* $f(x) \neq x$.

From the above theorem, we easily obtain the following result. We remark that this result also follows from Proposition 2.2 in [8].

COROLLARY 4. *If the set of all periodic points of* f *is dense in* I, *then exactly one of the following holds.*

(i) $f^2(x) = x$ *for all* $x \in I$.

(ii) f *has a periodic point of minimal period* $2m$ *for some odd integer* $m > 1$.

3. Proof of Theorem 1.

We shall prove Theorem 1 by showing $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$, $(2) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (1)$, $(2) \Rightarrow (9) \Rightarrow (1)$, $(2) \Rightarrow (10) \Rightarrow (1)$. We first show three lemmas which will be needed later.

LEMMA 1. *Let $I=[a, b]$ and $J=[c, d] \subset I$. Assume that the set of all periodic points of f is dense in I. Then the following hold.*

(i) *f cannot map any open interval in I into a point.*

(ii) *If $f^k(J) \subset J$ for some positive integer k, then $f^k(J)=J$ and $f^k(\overline{I-J})=\overline{I-J}$.*

(iii) *If $f^m(J) \subset J$ for some integer $m \geq 2$, then $f^2(J)=J$, $f^2([a, c])=[a, c]$ if $a \neq c$, and $f^2([d, b])=[d, b]$ if $d \neq b$.*

Proof. (i) and (ii) are obvious. So, we prove (iii). If $f^2(J) \subset J$, then it is rather easy to establish the desired result. So, we assume $m>2$ and, without loss of generality, we may assume that $J \subset \operatorname{int}(I)$.

Let $S=\bigcup_{k=0}^{m-1} f^k(J)$ and let p and q be, respectively, the smallest and largest elements in S. If $[p, q]-S=\varnothing$, then $f([p, q]) \subset [p, q]$. If $[p, q]-S \neq \varnothing$, let L be a closed subinterval of $[p, q]$ whose interior is an open component of $[p, q]-S$. Since the periodic points of f are dense in I, we must have $f(L) \subset [p, q]$ Therefore, in any case, we have $f([p, q]) \subset [p, q]$. Let z be a fixed point of f in $[p, q]$.

Recall that $J=[c, d]$. Now, we have five cases to consider:

Case 1. z lies in the interior of J.

If $f(J) \subset J$, we are done. So, assume that $f(J) \not\subset J$ and, without loss of generality, assume that $f(y)>d$ for some $c<y<d$. If $f(c')>d$ for some $z<c'<d$, then since $f(J) \supset [z, d]$, we have $f(c') \in f^m(J) \subset J$ which is a contradiction. So, $y<z$ and $f(x) \leq d$ for all $z \leq x \leq d$. If $f(x_0)>z$ for some $z<x_0<d$, let $A=f([z, d]) \cap [z, d]$. Then $f^m(A) \subset A$. So, f^{2m} fixes both endpoints of A. This is impossible since z is a fixed point of f. Thus, $f(x) \leq z$ for all $z \leq x \leq d$. Similarly, $f(x) \geq z$ for all $c \leq x \leq z$.

If $f(J) \supsetneqq J$, then $J \supsetneqq f^m(J) \subset J$. A contradiction. So, there is a point $d' \in (c, z)$ such that $f([z, d])=[d', z]$. If $f(x_0)>d$ for some $d'<x_0<z$, then $d<f(x_0) \in f^m(J) \subset J$. A contradiction. So, $z \leq f(x) \leq d$ for all $d' \leq x \leq z$. In particular, $f([d', d])$

$\subset [d', d]$. By a similar argument, we have $f^2([c, d']) \subset [c, d']$. Consequently, $f^2(J) \subset J$. By part (ii), $f^2(J) = J$.

Case 2. z lies in the interior of $f^i(J)$ for some i.

In this case, the proof is similar to that of Case 1. So $f^2(J) = J$.

Case 3. $z = c$ or $z = d$.

Without loss of generality, we may assume that $z = d$. If $f(J) \subset J$, we are done. So, assume that $f(J) \not\subset J$. Since $f^m(J) \subset J$, we must have $f(x) \geq c$ for all $c \leq x \leq d$. Hence $f(x_0) > d$ for some $c < x_0 < d$. As in the proof of Case 1, we have $f(x) \geq d$ for all $c \leq x \leq d$.

Let $f(J) = [d, d']$. If $f(x_0) > d'$ for some $d < x_0 < d'$, then $d < d' < f(x_0) \in f^m(J) \subset J$. A contradiction. If $f([d, d']) \subset [d, d']$, then since for some $\varepsilon > 0$ with $(d - \varepsilon, d) \subset J$, we have $f([d - \varepsilon, d]) \subset [d, d']$, we see that f cannot have periodic points in $(d - \varepsilon, d)$ which contradicts the assumption that periodic points of f are dense in I. Consequently, as in the proof of Case 1, we have $f(x) \leq d$ for all $d \leq x \leq d'$. Since $f^m(J) \subset J$, we have $f^m(f(J)) \subset f(J)$. Hence $f^m(J \cup f(J)) \subset J \cup f(J)$. That is, $f^m([c, d']) \subset [c, d']$. If $f(x_0) < c$ for some $d < x_0 < d'$, then $f(x_0) \in f^m([c, d']) \subset [c, d']$. A contradiction. Therefore, $f^2(J) \subset J$. By part (ii), we have $f^2(J) = J$.

Case 4. z equals one of the endpoints of $f^i(J)$ for some i.

In this case, the proof is similar to that of Case 3. So, $f^2(J) = J$.

Case 5. $z \in [p, q] - S$.

In this case, let L be the closed interval in $[p, q]$ whose interior is the open component of $[p, q] - S$ which contains z. Since the periodic points of f are dense in I and since $f(S) \subset S$, we have $f(L) = L$. Let $L = [r, s]$. Since $L \cap S = \{r, s\}$ and since $f(S) \subset S$, we must have $f(r) = s$ and $f(s) = r$. Consequently, we have $f^2(J) = J$.

Therefore, iu any case, we have $f^2(J) = J$. The rest is easy and omitted. This proves Lemma 1.

The following result can be found in [7].

LEMMA 2. *If J is an open subinterval of I which contains no periodic point of f, then, for every $x_0 \in I$, the points of the orbit $O_f(x_0)$ which lie in J form a strictly monotonic, finite or infinite, sequence.*

Proof. Since J contains no periodic point of f, we have, for every positive integer k, either $f^k(x) > x$ for all $x \in J$ or $f^k(x) < x$ for all $x \in J$. Now, assume that $y_0 < f^m(y_0) < f^n(y_0)$ (*other cases can be similarly proved*) for some $0 < n < m$ with $\{y_0, f^m(y_0), f^n(y_0)\} \subset J$. Then, by induction, we obtain that $f^{hm+kn}(x) > x$ and $f^{j(m-n)}(x) < x$ for all $x \in J$ and all integers $h,k \geq 0$ and $j \geq 1$ with $h^2 + k^2 \neq 0$. But if we take $h = k = m - n$ and $j = m + n$, then we have a contradiction. This proves Lemma 2.

Before we state Lemma 3, we first introduce some notations. Let Σ_2 denote the collection of all infinite one-sided sequences of 0's and 1's. A metric for Σ_2 is given by

$$d((a_0, a_1, a_2, \cdots), (b_0, b_1, b_2, \cdots)) = \sum_{k=0}^{\infty} |a_k - b_k| / 2^k.$$

Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift transformation $\sigma((a_0, a_1, a_2, \cdots)) = (a_1, a_2, \cdots)$. Then (Σ_2, d) is a compact metric space and σ is a continuous, onto, two to one map. The pair (Σ_2, σ) is called the one-sided shift on the symbols 0 and 1. In the following lemma, we omit parentheses and commas in writing the elements of Σ_2. (See [1] also).

LEMMA 3. *Assume that L and R are two disjoint closed subintervals of I and g is a continuous function from I into itself with $g(L) \cap g(R) \supset L \cup R$. Let $Z = L \cup R$. If $Z^* = \bigcap_{n=0}^{\infty} g^{-n}(Z)$ and if h is a function from Z^* into Σ_2 defined by $h(z) = a = a_0 a_1 a_2 \cdots$, where $a_k = 0$ or 1 according as $g^k(z) \in L$ or R, then the following hold.*

(i) *Z^* is a nonempty closed subset of I with $g(Z^*) = Z^*$.*

(ii) *If $z \in Z$ satisfies $g(z) \in Z^*$, then $z \in Z^*$.*

(iii) *Z^* contains all periodic orbits of g which lie in Z.*

(iv) *h is a continuous function from Z^* onto Σ_2.*

(v) *g is semiconjugate to σ through h, i.e., $h \circ g = \sigma \circ h$, on Z^*.*

(vi) *If $a \in \Sigma_2$ is a periodic point of σ with minimal period m, then $h^{-1}(a)$ contains a periodic point of g in Z^* with minimal period m. Consequently, if, for every positive integer k, n_k is the number of periodic points of σ, or, equivalently, of the function $x \to 1-2x^2$ on $[-1, 1]$, then, for every positive integer m, g has at least n_m distinct periodic points of minimal period m in $Z = L \cup R$.*

Proof. We only give a proof of (vi). All others are easy and omitted.

Let $a \in \Sigma_2$ be periodic with minimal period m and write $a = a_0 a_1 \cdots a_{m-1} a_0 a_1 \cdots a_{m-1} \cdots$. Let $J_i = L$ or R according as $a_i = 0$ or 1, $0 \le i \le m-1$. Then $g(J_i) \supset J_{i+1}$ for all $0 \le i \le m-2$ and $g(J_{m-1}) \supset J_0$. So, there exist, for all $0 \le i \le m-1$, closed intervals $I_i \subset J_i$ such that $g(I_k) = I_{k+1}$ for all $0 \le k \le m-2$ and $g(I_{m-1}) = J_0$. Therefore, there exists a point $z_0 \in J_0 \subset Z$ such that $g^k(z_0) \in J_k$ for all $0 \le k \le m-1$ and $g^m(z_0) = z_0$. It is obvious that z_0 is a periodic point of g in Z^* with minimal period m such that $h(z_0) = a$. This proves Lemma 3.

We can now show $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

$(1) \Rightarrow (2)$. Let p be a fixed periodic point of f in J. For any periodic point q of f with $O_f(q) \subset \text{int}(I)$, let q_1 and q_2 be, respectively, the smallest and largest elements in $O_f(q)$ and let k be any fixed positive integer such that $f^k(p) = p$ and $f^k(q) = q$. Let $L = \bigcup_{i=0}^{\infty}(f^k)^i(J)$. Then L is an interval and $f^k(\bar{L}) = \bar{L}$. By Lemma 1, $f^2(\bar{L}) = \bar{L}$. So, by assumption, $\bar{L} = I \supset [q_1, q_2]$. Consequently, there exist two nonnegative integers u and v such that $q_1 \in (f^k)^u(J)$ and $q_2 \in (f^k)^v(J)$. So, $q_1 \in f^{kuv}(J)$ and $q_2 \in f^{kuv}(J)$ and hence $O_f(q) \subset f^{kuv}(J)$. This shows that, for any periodic point q of f with $O_f(q) \subset \text{int}(I)$, there exists a nonnegative integer M such that $O_f(q) \subset f^M(J)$.

Now for any closed interval $[c, d] \subset \text{int}(I)$, choose two periodic points y and z such that $y < c < d < z$ and $O_f(y) \cup O_f(z) \subset \text{int}(I)$.

Then there are positive integers M_1 and M_2 such that $O_f(y) \subset f^{M_1}(J)$ and $O_f(z) \subset f^{M_2}(J)$. Thus, if $m \geq \max\{M_1, M_2\}$, then $[c, d] \subset [y, z] \subset f^m(J)$. This shows that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Let $n_1 < n_2 < \cdots$ be an increasing sequence of positive integers and let $x_0 \in \text{int}(I)$. Choose c, d so that $x_0 \in [c, d] \subset \text{int}(I)$. Then, by (2), for every open interval J in I, there is a positive integer M such that $f^m(J) \supset [c, d]$ for all $m \geq M$. So, $f^{n_i}(J) \supset [c, d]$ for all sufficiently large n_i. But then $x_0 = f^{n_i}(y)$ for some $y \in J$. Thus, $y \in f^{-n_i}(x_0)$. Therefore, $\bigcup_{i=1}^{\infty} f^{-n_i}(x_0)$ is dense in I. This shows that (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4). Trivial.

(4) $\Rightarrow$ (5). Let $\{I_k\}$ be an enumeration of all oepn intervals in I with rational endpoints. By (4), the set $\bigcup_{n=1}^{\infty} f^{-2n}(I_k)$, for each $k \geq 1$, is open and dense in I. Let $S = \bigcap_{k=1}^{\infty} (\bigcup_{n=1}^{\infty} f^{-2n}(I_k))$. Then it is easy to see that, for every $x_0 \in S$, the orbit of x_0 under f^2 is dense in I. Hence (4) $\Rightarrow$ (5).

(5) $\Rightarrow$ (1). If f^2 has a dense orbit in I, then it is clear that there is no proper closed subinterval of I which is invariant under f^2. Therefore, the implication (5) $\Rightarrow$ (1) is an easy consequence of Lemma 2.

In the following, we show (2) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (1).

(2) $\Rightarrow$ (6). For every open interval J in I, $\bigcup_{n \in A} f^{-n}(J)$ is open and, by (2), dense in I. Let $\{I_k\}$ be an enumeration of all open intervals in I with rational endpoints and let $S = \bigcap_{k=1}^{\infty}(\bigcup_{n \in A} f^{-n}(I_k))$. If $x_0 \in S$ and L is an open interval in I, then $L \supset I_k$ for some $k \geq 1$. Since $x_0 \in f^{-n}(I_k)$ for some $n \in A$, $f^n(x_0) \in I_k \subset L$. This shows that (2) $\Rightarrow$ (6).

(6) $\Rightarrow$ (7) $\Rightarrow$ (8). Trivial.

(8) $\Rightarrow$ (1). The proof is similar to that of (5) $\Rightarrow$ (1).

We now show (2) $\Rightarrow$ (9) $\Rightarrow$ (1).

(2) $\Rightarrow$ (9). Let J be any open interval in I and let L and R be two disjoint closed subintervals of J such that $f(L) \cap (L \cup R) = \emptyset$. By (2), there is a positive integer M_0 such that $f^n(L) \cap f^n(R) \supset L \cup R$ for all $n \geq M_0$. Fix any prime number $p \geq M_0$ and let $M = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$, where $\{p_1, p_2, \cdots, p_r\}$ is the set of all prime numbers $< p$ and each t_i, $1 \leq i \leq r$, is any fixed

positive integer such that $p_i^{t_i} > p$. Let m be a positive integer $\geq M$. If m is a prime number, then, by Lemma 3, f^m has a fixed point in L and in R. Since $f(L) \cap (L \cup R) = \emptyset$, these two fixed points of f^m are periodic points of f with minimal period m. If m has a prime factor $q \geq p$, then, by Lemma 3, f^q has $n_{(m/q)}$ distinct periodic points of minimal period m/q in $L \cup R \subset J$. Since $f(L) \cap (L \cup R) = \emptyset$, these periodic points of f^q are all periodic points of f with minimal period m.. If all prime factors of m are $< p$, write $m = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$ with each $s_i \geq 0$ and $s_j > t_j$ for some $1 \leq j \leq r$ and let $u = p_j^{t_j}$. Then, by Lemma 3, f^u has $n_{(m/u)}$ periodic points of minimal period m/u in $L \cup R \subset J$. Since p_j divides both u and m/u, these periodic points of f^u are all periodic points of f with minimal period m, This shows that (2) $\Rightarrow$ (9).

(9) $\Rightarrow$ (1). Trivial.

Finally, we show (2) $\Rightarrow$ (10) $\Rightarrow$ (1).

(2) $\Rightarrow$ (10). This follows from (2) $\Rightarrow$ (5) and (2) $\Rightarrow$ (9).

(10) $\Rightarrow$ (1). Since f has a dense orbit in I, it follows from Lemma 2 that the set of all periodic points of f is dense in I. Now if there were a proper closed subinterval J of I such that $f^2(J) \subset J$, then, by Lemma 1, we may assume that $J = [a, c]$ with $I = [a, b]$ and $a < c < b$. We then also have $f^2([c, b]) \subset [c, b]$. Since f has a dense orbit in I, $f(J) \cap J$ can contain at most one point. So, $f(J) \subset [c, b]$ and $f([c, b]) \subset J$. Consequently, f cannot have periodic points of odd periods > 1. This is a contradiction. Therefore, there is no proper closed subinterval of I which is invariant under f^2. This shows (10) $\Rightarrow$ (1).

The proof of Theorem 1 is now complete.

4. Proof of Theorem 2.

To show (i), let $s \geq 2$ be any fixed integer and let $q > s$ be any prime number. If y is a periodic point of f with minimal period q, then, since $I = \overline{\bigcup_{k=0}^{s-1} A_{s,k}(x_0)}$, there exist an integer $0 \leq j \leq s-1$ and an increasing sequence $\{n_i\}$ of positive integers such that $f^{sn_i+j}(x_0) \to y$ as $i \to \infty$. For every integer $0 \leq k \leq s-1$, since s and q are coprime, there exist positive integers u and v such that $uq - sv = s + k - j$. But then $f^{s(n_i+v+1)+k}(x_0) = f^{(uq-sv)+s(n_i+v)+j}(x_0)$

$= f^{uq+sn_i+j}(x_0) \to f^{uq}(y) = y$ as $i \to \infty$. Thus, $y \in \overline{A_{s,k}(x_0)}$ for every $0 \leq k \leq s-1$. From the equivalence of (9) and (10) in Theorem 1, we see that the set of all periodic points of f with minimal period $q > s$ and q prime is dense in I. Therefore, $\overline{A_{s,k}(x_0)} = I$ for all integers $s \geq 2$ and $0 \leq k \leq s-1$.

To show (ii), we see that, from the equivalence of (1) and (10) in Theorem 1, there is a proper closed subinterval J of I such that $f^2(J) \subset J$. By Lemma 1, we may assume that $J = [a, c]$ with $I = [a, b]$ and $a < c < b$. We then also have $f^2(J) = J$ and $f^2([c, b]) = [c, b]$. If J ($[c, b]$ resp.) contains $f^k(x_0)$ for some odd $k > 0$, and some even $k > 0$, then, since $f^2(J) = J$ ($f^2([c, b]) = [c, b]$ resp.), J ($[c, b]$ resp.) contains $f^k(x_0)$ for all sufficiently large k. But this would imply that $J = I$ ($[c, b] = I$ resp.). So, J ($[c, b]$ resp.) contains $f^k(x_0)$ for all, say, even $k \geq 0$ (odd $k > 0$ resp.). Since the orbit of x_0 under f is dense in I, $f(J) \cap J$ can contain at most one point and since f has a fixed point z in I, we must have $z = c$. Hence $f(J) = [c, b]$ and $f([c, b]) = J$. Note that the above argument also shows that J and $[c, b]$ are the only proper closed subintervals of I which are invariant under f^2. So, by (1) in Theorem 1 and Lemma 1, we obtain that f^2 has a periodic point of odd period > 1 in J and in $[c, b]$.

The proof of Theorem 2 is now complete.

5. Proof of Theorem 3.

For the proof of Theorem 3, we need three lemmas.

LEMMA 4. *Assume that the set of all periodic points of f is dense in I and f fixes an endpoint of I. If J is a closed subinterval of I such that $f^2(J) \subset J$, then $f(J) = J$ and if y is an endpoint of J which is not an endpoint of I, then $f(y) = y$.*

Proof. Let $I = [a, b]$ and $J = [c, d]$. Without loss of generality, we may assume that $f(b) = b$. If $d = b$, then $f(J) \subset J$ and hence $f(J) = J$. For otherwise, $J \supsetneqq f(J) \subset f^2(J) \subset J$. This is a contradiction. So, we now assume that $d < b$. If for some $w \in J$, $f(w) > d$, then, since $f(f(J)) \subset J$, $f(x) < b$ for all $x \in J$. Thus, there is a nondegenerate closed interval L in $[\max f(J), b]$

such that $f(L) = f(J) \cap [d, b]$ and hence no point of the interior of L can be a periodic point of f. This is a contradiction. Consequently, $f(x) \leq d$ for all $x \in J$.

If $f(d) < d$, then there is a nondegenerate closed interval L in $[d, b]$ such that $f(L) = [f(d), d]$. So, no point of the interior of L is a periodic point of f. This is again a contradiction. So, $f(d) = d$. The same argument as above then shows that $f(J) = J$. If $c \neq a$, then it is clear that $f(c) = c$ since, in this case, we have $f([d, b]) = [d, b]$, $f(J) = J$, and $f([a, c]) = [a, c]$. This proves Lemma 4.

LEMMA 5. *Let $I = [a, b]$. Assume that the set of all periodic points of f is dense in I and $L = [p, q] \subset (a, b)$ satisfies that $p < q$, $f(L) = L$, $f(p) = q$ and $f(q) = p$. If $J = [c, d] \subset [a, p] \cup [q, b]$ satisfies $c < d$ and $f^2(J) = J$, then the following hold.*

(i) *If $d \leq p$, then $q \leq f(d) \leq f(c)$, $f(d) = \min f(J)$, and if c is not an endpoint of I, then $f(c) = \max f(J)$.*

(ii) *If $q \leq c$, then $f(d) \leq f(c) \leq p$, $f(c) = \max f(J)$, and if d is not an endpoint of I, then $f(d) = \min f(J)$.*

Proof. First we note that, since the periodic points of f are dense in I, the set $f(I - L) \cap L$ can contain at most two points. If $J \subset [a, p]$ and $f(w) < q$ for some $w \in J$, then there exists a closed interval $K \subset [w, p]$ such that $f(K) = L \cap [f(w), q]$. This contradicts the above observation. So, $f(x) \geq q$ for all $x \in J$. On the other hand, if $f(c) < f(d)$, then there exists a closed interval $K \subset [d, p]$ such that $f(K) = [f(c), f(d)]$. Thus, no point of the interior of K can be a periodic point of f. This is again a contradiction, So, $f(c) \geq f(d)$. The proof of the other statement in (i) is similar and omitted. If $J \subset [q, b]$, the proof is similar. This proves Lemma 5.

LEMMA 6. *Let $I = [a, b]$ and $J = [c, d]$. Assume that the set of all periodic points of f is dense in I. If z is a fixed point of f in (a, b) and if the interior of J is a nonempty open subset of $I - \{z\}$ such that $f(J) \neq J$ and $f^2(J) = J$, then Conclusions* (i) *and* (ii) *of Lemma* 5 *(assuming $p = q = z$) hold.*

Proof. First assume $d=z$. If $J\subset f(J)$, then $J\subset f(J)\subset f^2(J)\subset J$. So, $f(J)=J$, contradicting the assumption that $f(J)\neq J$. Thus, $J\not\subset f(J)$. In particular, $f(x)\geq c$ for all $x\in J$. Let $L=J\cap f(J)$. Then $f(L)\subset L$. Since $f(J)\not\subset J$, $L\subsetneqq J$. Hence there is a point $w\in J$ such that $f(w)>z$. Let $y=\max f(J)$. Then since $f(f(J))\subset J$ and $f(L)\subset L\subsetneqq J$, $f([z, y])=J$. So, there is a closed subinterval K of $[z, y]$ such that $f(K)=L$. If L contains more than one point, then, since $f(L)\subset L$, no point of the interior of K can be a periodic point of f. This is a contradiction. So, $L=J\cap f(J)=\{z\}$. Consequently, $x\leq z\leq f(x)$ for all $x\in J$.

Now assume that $d<z$. If $f(w)<z$ for some $w\in J$, then, since $z\notin J$ and $f(f(J))\subset J$, $f(x)<z$ for all $x\in J$. So, $J\cup f(J)\subset[a, z]$. If $f(w)>d$, then there is a closed interval K in $[\max f(J), z]$ such that $f(K)=[d, z]\cap f(J)$. Thus, no point of the interior of K can be a periodic point of f. This is a contradiction. So, $f(w)\leq d$. That is, $f(x)\leq d$ for all $x\in J$.

If $f(d)=d$, the first part of this proof shows a contradiction. So, assume $f(d)<d$. But then there is a closed interval K in $[d, z]$ such that $f(K)=[f(d), d]\cap J$. This is again a contradiction. Therefore, $x\leq z\leq f(x)$ for all $x\in J$. The proof of the rest is easy and omitted. If $J\subset[z, b]$, the proof is similar. This proves Lemma 6.

Let K be a closed subinterval of I which is invariant under f^2. If K contains no proper closed subinterval which is invariant under f^2, then we call K a minimal invariant closed interval of f^2 in I. We can now prove Theorem 3.

Let $S=\{x\in I\,|\,f^2(x)=x\}$. Then S is a closed subset of I. If $S=I$, then the conclusion is clear. So, we assume that $S\neq I$. By part (iii) of Lemma 1, every open component of $I-S$ is contained in a minimal invariant closed interval of f^2. Let $\{J_1, J_2,\cdots\}$ be the (possibly finite but nonempty) collection of all minimal invariant closed intervals of f^2 with nonempty interior in $I-\text{int}(S)$. If $I-\bigcup_{i=1}^{\infty}J_i\neq\emptyset$, then every $x\in I-\bigcup_{i=1}^{\infty}J_i$ satisfies $f^2(x)=x$. If, for some positive integer k, $f(J_k)\subset J_k$, then $f(J_k)=J_k$ and we have two cases to consider:

Case 1. $J_k = I$ or, $J_k \neq I$ and f fixes at least one endpoint of J_k. In this case, it follows from Theorem 1 and Lemma 4 that part (a) holds.

Case 2. $J_k \neq I$ and f exchanges both endpoints of J_k. In this case, it follows from Theorem 1 and Lemma 5 that part (b) with a non-degenerate I_0 (which equals J_k) holds.

If, for all positive integers i, $f(J_i) \not\subset J_i$, then it follows from Theorem 1 and Lemma 6 that part (b) with a degenerate I_0 (which equals the set consisting of the unique fixed point of f) holds.

The proof of Theorem 3 is now complete.

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