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POINTWISE CONVERGENCE TO N-WAVES FOR SOLUTIONS OF HYPERBOLIC CONSERVATION LAWS

BY

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Abstract. In this paper we consider solutions with compact support of hyperbolic conservation laws and show that nonlinear waves tend to N-waves. We obtain a pointwise description of the asymptotic state. For each nonlinear field the waves consist of two relatively strong shock waves and between them rarefaction waves dominate. As an immediate consequence of the pointwise estimates, a L_1-convergence to N-waves is obtained, improving earlier results of DiPerna and of the author.

1. **Introduction.** Consider a system of conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0, \qquad t\geq 0, \quad -\infty<x<\infty \tag{1.1}$$

where $u=u(x, t)$ is an n-vector and f is a smooth n-vector-valued function of u. Our purpose is to study the asymptotic behavior of the solutions with compact support:

$$\begin{aligned} u(x, 0)=u_0(x)&=\text{given}, \qquad -\infty<x<\infty, \\ |u_0(x)|&\equiv 0 \qquad \text{for} \quad |x|>K \end{aligned} \tag{1.2}$$

for some positive constant K. System (1.1) is assumed to be strictly hyperbolic, that is, $\partial f(u)/\partial u$ has real and distinct eigenvalues for each u under consideration:

$$\begin{aligned} \frac{\partial f(u)}{\partial u} r_i(u)=\lambda_i(u) r_i(u), &\qquad l_i(u)\frac{\partial f(u)}{\partial u}=\lambda_i(u) l_i(u), \\ l_j\cdot r_i(u)=\delta_{ij}, &\qquad i, j=1, 2,\cdots, n, \\ \lambda_1(u)<\lambda_2(u)&<\cdots<\lambda_n(u). \end{aligned} \tag{1.3}$$

With physical models for the compressible fluid in mind we assume

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that each i-field, $i=1, 2, \cdots, n$ is either genuinely nonlinear,

$$\text{(g. n. l.)} \qquad \nabla \lambda_i(u) \cdot r_i(u) \neq 0 \quad \text{for all} \quad u$$

or linearly degenerate

$$\text{(l. d. g.)} \qquad \nabla \lambda_i(u) \cdot r_i(u) \equiv 0 \quad \text{for all} \quad u,$$

(Lax [4]). It was conjectured by Lax [4] that solutions with compact support for a genuinely nonlinear system decay to N-waves as $t \to \infty$. Each N-wave possesses two time-invariants and there are altogether $2n$ time-invariants. The conjecture was proved by Liu [5]. When the system (1.1) possesses linearly degenerate fields it was shown (Liu [6]) that nonlinear waves tend to N-waves and linear waves to travelling waves. The decay was studied using the random choice method (Glimm [2]) and a careful analysis of interactions of waves of the same family and of different families. Convergence to N-waves is based on the observation that the characteristic field of nonlinear waves satisfies the inviscid Burgers equation time-asymptotically. The approach was motivated in part by the work of Glimm-Lax [3] and DiPerna [1] on two conservation laws.

The aforementioned convergence to N-waves is in the L_1 sense. The present article studies the pointwise convergence. This is achieved through the analysis of the distribution of nonlinear waves in each primary region. One important technique for qualitative study of shock waves is the notion of generalized characteristics. The notion was first introduced by Glimm-Lax [3]. The stochastic feature of the random choice method can be removed through the technique of wave tracing (Liu [7]). The technique can be used to construct various types of generalized characteristics and is useful to study the qualitative properties of systems not necessarily genuinely nonlinear or linearly degenerate (Liu [8]). This more refined description of generalized characteristics is essential for the present approach. The existence of two time-invariants for each nonlinear field allows us to show that two relatively strong shock waves eventually emerge for each nonlinear field. The emergence of these strong shock waves stabilizes the wave pattern and yields a satisfactory pointwise description of the asymptotic state.

The rate of L_1 convergence to N-waves obtained in Liu [5] is $t^{-1/6}$. The rate is achieved using an elegant argument of DiPerna [1] for two conservation laws. As an immediate consequence of our pointwise result we obtain a rate of $t^{-1/4}$ for the L_1-convergence. For scalar conservation laws, the rate is $t^{-1/2}$. Nevertheless, for general systems the optimal rate seems to be $t^{-1/4}$. This is remarked toward the end of Section 5. In the next two sections basic notions are introduced and previous results recalled. Section 4 studies the distribution of waves and the emergence of two relatively strong shock waves for each nonlinear field. Finally a stronger result on the behavior of nonlinear waves in the primary region between the two relatively strong shock waves is studied. The present article is focused on nonlinear waves; linear waves have been satisfactorily studied in Liu [6]. The reason for the present study of pointwise convergence is that the L_1-convergence does not give a precise description of the asymptotic state. In particular, the physically observed relatively strong shock waves were not predicted by the previous results. The strength of these shock waves is explicitly related to the time-invariants.

2. **Preliminaries.** There are two kinds of nonlinear waves for (1.1), the shock waves and the rarefaction waves. For each given state u_0 we denote by $R_i(u_0)$ the integral curve through u_0 of $r_i(u)$ in the u-space, $i = 1, 2, \cdots, n$. For each genuinely nonlinear i-field, $R_i(u_0)$ is divided into $R_i^+(u_0)$ and $R_i^-(u_0)$ so that $\lambda_i(u) > \lambda_i(u_0)$ (or $\lambda_i(u) < \lambda_i(u_0)$) for $u \in R_i^+(u_0)$ (or $u \in R_i^-(u_0)$). For $u \in R_i^+(u_0)$, an i-rarefaction wave (u_0, u) exists which connects u_0 on the left and u on the right and propagates with speed λ_i. For a shock wave (u_-, u_+) with speed σ, the following jump (Rankine-Hugoniot) condition is satisfied:

$$\text{(R-H)} \qquad \sigma(u_+ - u_-) = f(u_+) - f(u_-).$$

It follows from weakly nonlinear arguments that for a fixed initial state u_-, there exist n curves $S_1(u_-), \cdots, S_n(u_-)$, the shock curves, through u_-. For $u_+ \in S_i(u_-)$, (u_-, u_+) satisfies (R-H) and $\sigma \equiv \sigma(u_-, u_+)$ is close to $\lambda_i(u_-)$ when u_+ is close to u_-. More precisely, for $u_+ \in S_i(u_-)$ there exists u_* on $R_i(u_-)$ such that

$$(2.1)\qquad \begin{aligned} &|u_* - u_+| = O(1)|u_+ - u_-|^3, \\ &\sigma(u_-, u_+) = (\lambda_i(u_-) + \lambda_i(u_+))/2 + O(1)|u_+ - u_-|^2. \end{aligned}$$

A shock wave (u_-, u_+) pertaining a g. n. l. i-field is required to satisfy the stability condition

$$(2.2)\qquad \lambda_i(u_+) < \sigma(u_-, u_+) < \lambda_i(u_-).$$

When an i-field is l.d. g., $S_i(u_0) \equiv R_i(u_0)$ and an i-discontinuity wave (u_-, u_+) is a contact discontinuity, $\sigma(u_-, u_+) = \lambda_i(u_-) = \lambda_i(u_+)$, $u_+ \in R_i(u_-) = S_i(u_-)$.

A special initial value problem for (1.1) with data

$$(2.3)\qquad u(x, 0) = \begin{cases} u_l, & \text{for } x < 0 \\ u_r, & \text{for } x > 0, \end{cases}$$

is called the Riemann problem, where u_l and u_r are constant states. Since both the system (1.1) and the data (2.3) are invariant under $(x, t) \to c(x, t)$ for any $c > 0$, solutions to the Riemann problem are functions of x/t. Thus solving the Riemann problem is reduced to solving ordinary differential equations along R_i and the algebraic equations (R-H). For u_l close to u_r, (1.1) and (2.3) can be solved by the implicit function theorem in the class of elementary waves just described. The above results are due to Lax [4].

To solve the general initial-value problem, Glimm [3] introduced a random choice method. The method depends on a random sequence $\{a_k\}_{k=1}^{\infty}$ in the interval (0, 1). First, approximate the inital data $u(x, 0)$ by a step function with respect to mesh size Δx. Choose a mesh size Δt which satisfies the (C-F-L) condition. An approximate solution $u_{\Delta x}$ consists of elementary waves issued from the mesh points $(h\Delta x, k\Delta t)$, $k = 0, 1, 2, \cdots$, $h = 0, \pm 1, \pm 2, \cdots$, defined as follows: Resolve the discontinuity at time $t = 0$ by solving translated Riemann problems. At time $t = \Delta t$ approximate these elementary waves by a step function:

$$u_{\Delta x}(x, \Delta t + 0) = u_{\Delta x}((h + a_1)\Delta x, \Delta t - 0),$$

$$h\Delta x < x < (h+1)\Delta x, \qquad h = 0, \pm 1, \pm 2, \cdots.$$

The process repeats itself so that the random number a_k is used to approximate the elementary waves by a step function at time level $t = k\Delta t$. Thus a general solution is approximated by elementary waves. Between each time level $k\Delta t$ and $(k+1)\Delta t$ these elementary waves do not interact; the interactions are discretized to concentrate at mesh points. At each mesh point $(h\Delta x, k\Delta t)$ the incoming waves are two sets of waves, each is a part of a solution of a Riemann problem. These two sets of waves interact and produce the outgoing waves issued from $(h\Delta x, k\delta t)$. Let the strength of the incoming i-waves be α_i and β_i and that of the outgoing i-wave γ_i, then Glimm [2] shows

$$(2.4_1) \qquad \gamma_i = \alpha_i + \beta_i + O(1)\, Q_1(\alpha, \beta)$$

where $Q_1(\alpha, \beta)$ is quadratic and measures the nonlinear interaction. Here the strength of i-shock waves and i-rarefaction waves is conveniently measured by the jump of λ_i so that shock waves have negative strength and rarefaction waves have positive strength. The interaction of waves of different families is measured by

$$(2.5) \qquad Q^d(\alpha, \beta) = \sum_{i>j} |\alpha_i \beta_j|$$

where α lies to the left of β so that α_i and β_j are approaching for $i > j$. Two i-waves α_i and β_i approach when at least one of them is a shock wave. Thus the interaction of waves of the same family is measured by

$$(2.6_1) \qquad \begin{aligned} Q_1'(\alpha, \beta) &= \sum \{|\alpha_i \beta_i| : i\text{-field g. n. l. and} \\ &\qquad \alpha_i \le 0 \text{ and/or } \beta_i \le 0\}, \\ Q_1(\alpha, \beta) &\equiv Q^d(\alpha, \beta) + Q_1'(\alpha, \beta). \end{aligned}$$

For the study of the large-time behavior one needs a sharper estimate than (2.6), Liu [5]:

$$(2.6_2) \qquad \begin{aligned} Q_2'(\alpha, \beta) &= \sum \{|\alpha_i \beta_i|(|\alpha_i| + |\beta_i|) : i\text{-field g. n. l. and} \\ &\qquad \alpha_i \le 0 \text{ and or } \beta_i \le 0\} \\ Q_2(\alpha, \beta) &= Q^d(\alpha, \beta) + Q_2'(\alpha, \beta), \end{aligned}$$

so that (2.4_1) still holds with Q_1 replaced by Q_2

$$\gamma_i = \alpha_i + \beta_i + O(1)\, Q_2(\alpha, \beta). \tag{2.4$_1$}$$

The total amount of i-shock (or i-rarefaction) wave at time t is denoted by $X_i^-(t)$ (or $X_i^+(t)$). The amount of i-waves crossing a horizontal interval I is accordingly denoted by $X_i^\pm(I)$. For a given region A in the (x, t)-plane, the total amount of interaction is denoted by $Q(A)$ as the sum of Q_2 at each mesh point in A. The existence of weak solutions is based on the estimate of the total amount of waves through a nonlinear functional. It is shown that when the total amount of waves at $t = 0$ is small, then the amount of waves at a later time is also small and the approximate solutions converge to a weak solution for almost all choices of $\{a_k\}$ ([2]). In particular, it is shown that the total amount of interactions after time t can be estimated

$$Q_2(\text{time} \geq t) \leq 2D(t) \tag{2.8}$$

where $D(t)$ measures the potential wave interactions and is defined as follows: When an i-wave α_i lies to the left of a j-wave β_j at time t, then $|\alpha_i \beta_j|$ is included in the sum defining $D(t)$. Included is also $|\alpha_i \beta_i|(|\alpha_i| + |\beta_i|)$ where α_i and β_i are two i-waves at time t and at least one of them is a shock wave. A crude estimate yields

$$D(t) \leq (X(t))^2$$

where $X(t)$ is the total amount of waves at time t.

3. **Generalized characteristics and decay.** Although our results hold for systems with linearly degenerate fields, for the simplicity of presentation we will assume from now on that all characteristic fields are genuinely nonlinear. No new idea beyond those of [6] is needed to generalize our arguments to the general case. As mentioned earlier the notion of generalized characteristics is basic in the study of the behavior, in particular the decay, of the solutions. Suppose that $u(x, t)$ is a piecewise smooth solution of (1.1). A generalized i-characteristic χ travels along an i-characteristic curve. However, an i-characteristic may impinge on an i-shock wave ((2.2)).

When this happens, χ travels with the shock wave. The importance of such a notion is that i-waves do not cross an i-characteristic and thereby one has conservation laws of nonlinear waves. Of course a general solution is not necessarily piecewise smooth. The generalized characteristics may be defined nevertheless for approximate solutions ([3]). Using the same notations as in Section 2, suppose that a generalized characteristic χ is contained in an incoming wave α_i or β_i, then χ is contained in the outgoing wave γ_i. When γ_i is an i-shock wave, there is only one choice. When γ_i is an i-rarefaction wave, χ is located so that the aforementioned conservation law of i-waves holds for the interaction of α and β. More precisely, the amount of i-waves to the left (and right) of χ before and after the interaction do not change except for the amount $O(1)\ Q(\alpha, \beta)$ due to nonlinear interaction. Here the amount of i-waves includes shock waves with negative strength and rarefaction waves with positive strength so that wave cancellation is acounted for.

In [8] a more refined notion of generalized characteristics based on the wave tracing technique of [7] was introduced. Generalized characteristics are classified into three types. Type I is always contained in a rarefaction wave and thereby travels with characteristic speed. It may change its speed only due to the crossing of waves of other families. Type II is a more comon one, it travels with characteristic speed until it impinges on a shock wave. It may change its speed also due to combination with other shock waves and cancellation with rarefaction waves. Type III is less common and exists in general only backward in time. Through a point (x_0, t_0) we may construct several type III curves backward in time, $t \leq t_0$. When a shock is present at (x_0, t_0), we take either side of the characteristic line of the shock wave as the starting direction for a type III curve. As it progresses backward in time, it never travels with shock waves and may be contained in rarefaction waves. Such a type III curve has the aforementioned property of conservation of waves. It changes its speed only due to nonlinear interactions, controlled by Q, and the crossing of waves of other families. Another important fact is that through any given point (x, t) there always exist type III curves backward in time. On the other hand, in the forward time direction a characteristic curve may hit a shock wave and become a type II curve.

Through $(-K, 0)$ and $(K, 0)$ draw generalized 1-characteristic and n-characteristic curves χ_1 and χ_n, respectively. From (1.2), $u(x, t) \equiv 0$ outside χ_1 and χ_n. Between χ_1 and χ_n it has been shown that the total variation and amount of interactions decay in time, [3], [5], [6],

$$\begin{aligned} X_i(t) &= O(1)\,\varepsilon\, t^{-1/2}, \qquad i = 1, 2, \cdots, n, \\ D(t) &= O(1)\,\varepsilon^2\, t^{-3/2}, \end{aligned} \tag{3.1}$$

where ε is the total variation of the initial data which is assumed to be small. To gain more precise information we need to divide the (x, t)-plane into subregions:

$$\begin{aligned} &\Omega_0 \equiv \{x \leq \mu_0 t\}, \qquad \Omega_{n+1} \equiv \{x \geq \mu_n t\}, \\ &\Omega_i \equiv \{\mu_{i-1} t \leq x \leq \mu_i t\}, \qquad i = 1, 2, \cdots, n, \\ &\Gamma_i \equiv \Omega_i \cap \Omega_{i-1} = \{x = \mu_i t\}, \qquad i = 0, 1, \cdots, n. \end{aligned} \tag{3.2}$$

where the constants $\mu_0 < \mu_1 < \cdots < \mu_n$ are chosen to separate the characteristics:

$$\mu_0 < \lambda_1(u) < \mu_1 < \lambda_2(u) < \cdots < \mu_{n-1} < \lambda_n(u) < \mu_n \tag{3.3}$$

for all u under consideration. Through Γ_0 and Γ_n at time t, draw generalized i-characteristics which, by (3.3), enter Ω_i before time Ct, C some constant independent of t. The i-waves outside these characteristics are produced by interactions and are therefore $O(1)\,D(t) = O(1)\,\varepsilon^2 t^{-3/2}$ by (3.1). Thus the total amount of i-waves not in Ω_i at time Ct is $O(1)\varepsilon^2 t^{-3/2}$. Since C is independent of t, we conclude:

$$\bar{X}_i(t) = O(1)\,\varepsilon^2\, t^{-3/2} \tag{3.4}$$

where $\bar{X}_i(t)$ denotes the amount of i-waves outside Ω_i at time t. Since shock curves S_i and characteristic curves R_i are tangent of second order, (2.1), and i-waves dominate Ω_i, (3.4), we have from the decay (3.1) that for any (x_1, t) and (x_2, t) in Ω_i, $u(x_1, t) \in R_i(u(x_2, t)) + O(1)\,\varepsilon^2 t^{-3/2}$. This holds for all $i \in \{1, 2, \cdots, n\}$, and it follows from the implicit function theorem that

$$(3.5)\qquad \begin{aligned} &u(\mu_i t, t) = O(1)\,\varepsilon^2 t^{-3/2}, \\ &u(x, t)|_{\Omega_i} \in R_i(0) + O(1)\,\varepsilon^2 t^{-3/2}, \qquad i = 1, 2, \cdots, n. \end{aligned}$$

That is, on each Γ_i, $u(x, t)$ is close to the state generated by $0 = u(\pm\infty, t)$. In Ω_i, we have from the second estimate in (2.1) and the decay (3.1) and (3.4) that

$$(3.6)\qquad \frac{\partial \lambda_i}{\partial t} + \frac{\partial}{\partial x}((\lambda_i^2)/2)\,|\,\Omega_i = \nu_i\,|\,\Omega_i$$

for some measure ν_i which decays at the rate $t^{-3/2}$:

$$(3.7)\qquad \int_t^\infty \int_{-\infty}^\infty |\nu_i(x, r)|\,dx\,dr = O(1)\,\varepsilon^2 t^{-1/2}.$$

It follows from geometric measure theory, [9], that λ_i is close to a solution of the inviscid Burgers equation

$$(3.8)\qquad \frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial x}(\lambda^2/2) = 0$$

for which an extensive theory is available. In particular there are two time-invariants

$$\min_x \int_{-\infty}^x \lambda(y, t)\,dy \quad \text{and} \quad \max_x \int_x^{+\infty} \lambda(y, t)\,dy, \qquad t \geq 0.$$

For (3.6) satisfying (3.7) there exist constants p_i, q_i, $p_i \leq 0 \leq q_i$, such that

$$(3.9)\qquad \begin{aligned} p_i(t) &\equiv \min_x \int_{-\infty}^x \lambda_i(u(y, t))|_{\Omega_i}\,dy = p_i + O(1)\,\varepsilon^2 t^{-1/2}, \\ q_i(t) &\equiv \max_x \int_x^{+\infty} \lambda_i(u(y, t))|_{\Omega_i}\,dy \\ &= q_i + O(1)\,\varepsilon^2 t^{-1/2}, \qquad i = 1, 2, \cdots, n, \end{aligned}$$

as $t \to \infty$.

4. **Distribution of waves.** Throughout this section and the next we assume that the total variation of the initial data is small (except for the special system [6]) so that a solution exists by [2]. Given a time $r \gg 1$, consider an earlier time r^α. Here α is a

constant whose optimal choice turns out to be 2/5. For the moment α is assumed to lie between 0 and 1. For convenience and without loss of generality, we assume that $\lambda_i(0) = 0$ and Γ_i and Γ_{i-1} are taken to be symmetric:

$$\Gamma_i = \{x = \mu t\}, \qquad \Gamma_{i-1} = \{x = -\mu t\}$$

for some positive constant μ. Through $(-\mu\tau^\alpha, \tau^\alpha)$ and $(\mu\tau^\alpha, \tau^\alpha)$ on Γ_{i-1} and Γ_i we draw i-characteristics χ_l and χ_r, respectively, which intersect time τ at $x = \chi_l \equiv \chi_l(\tau)$ and $x = \chi_r \equiv \chi_r(\tau)$, Figure 1. The amount of j-waves, $j \neq i$, between $(-\mu\tau, \tau)$ and $(\mu\tau, \tau)$ is $O(1)\,\varepsilon^2\tau^{-3/2}$, (3.1). Our purpose in this section is to show that except at (χ_l, τ) and (χ_r, τ), the amount of i-shock waves elsewhere decays faster than $\tau^{-1/2}$ and has proper distribution. A crude estimate follows from (3.4). Since i-waves between $(-\mu\tau, \tau)$ and (χ_l, τ) and also between (χ_r, τ) and $(\mu\tau, \tau)$ are produced by interaction after time $O(1)\tau^\alpha$, we have from (3.4) that they are $O(1)\,D(\tau^\alpha) = O(1)\,\varepsilon^2\tau^{-3\alpha/2}$. With the choice of $\alpha = 2/5$, this is $O(1)\varepsilon^2\tau^{-3/5}$, which is smaller than $O(1)\,\varepsilon^2\tau^{-1/2}$. The first lemma improves upon this.

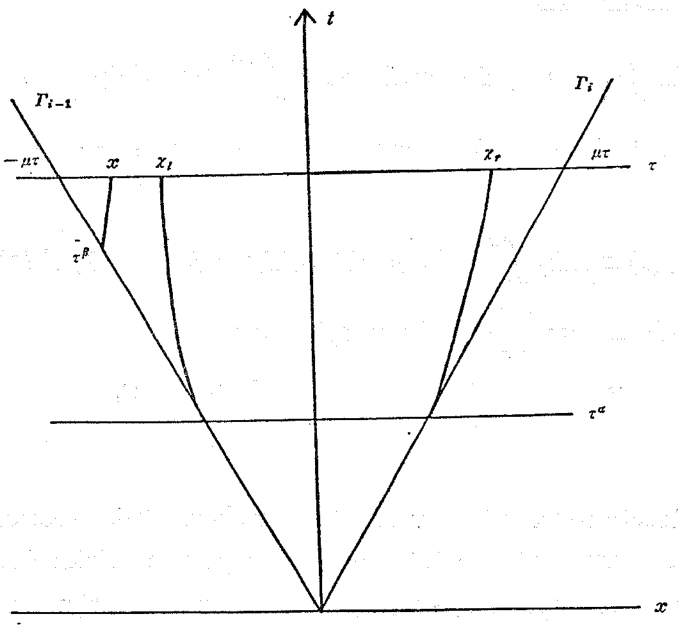

FIG. 1.

LEMMA 4.1. *For* $x \in (-\mu r, \chi_l)$ *(or* $(\chi_r, \mu r)$*), the amount of* i*-waves between* $(-\mu r, r)$ *and* (x, r) *(or between* (x, r) *and* $(\mu r, r)$*) is* $O(1)\varepsilon^2|x|^{-3/2}$ *provided that* $\alpha \geq 2/5$.

Proof. Through (x, r) draw a type III characteristic χ which intersects Γ_{i-1} or Γ_i at time r^β, $0 < \beta < 1$, Figure 1. Since χ is of type III, it changes its speed by at most $O(1)\, D(r^\beta) = O(1)\, \varepsilon^2 r^{-3\beta/2}$. From (3.5) and $\lambda_i(0) = 0$, the speed of χ at time r^β is $O(1)\, \varepsilon^2 r^{-3\beta/2}$. Thus for $-\mu r < x < \chi_l$,

$$\begin{aligned} x &= -\mu r^\beta + (r - r^\beta)\, O(1)\, \varepsilon^2 r^{-3\beta/2} \\ &= -\mu r^\beta + O(1)\, \varepsilon^2 r^{1-3\beta/2}. \end{aligned}$$

Since $\beta \geq \alpha \geq 2/5$, $1 - 3\beta/2 \leq 1 - 3/5 = 2/5$, we have from the above estimate that

$$x = O(1)\, r^\beta,$$

$$r^\beta = O(1)\,|x|.$$

The amount of i-waves between $(-\mu r, r)$ and (x, r) is produced interactions after time $O(1)\, r^\beta$ and is

$$O(1)\, D(r^\beta) = O(1)\varepsilon^2 r^{-3\beta/2} = O(1)\, \varepsilon^2 |x|^{-3/2}.$$

This proves the lemma.

COROLLARY 4.2. *We have*

$$\begin{aligned} &\int_{-\mu r}^{\chi_l} |\lambda_i(y, r)|\, dy + \int_{\chi_r}^{\mu r} |\lambda_i(y, r)|\, dy \\ &\qquad = O(1)\, r^{-1/5} \qquad \textit{for} \quad \alpha = 2/5. \end{aligned}$$

Proof. From Lemma 4.1, for any positive C

$$\begin{aligned} &\int_{-\mu r}^{-C r^{1/2}} |\lambda_i(y, r)|\, dy + \int_{C r^{1/2}}^{\mu r} |\lambda_i(y, r)|\, dy \\ &\qquad = O(\varepsilon^2) \int_{C r^{1/2}}^{\mu r} |y|^{-3/2}\, dy = O(\varepsilon^2) r^{-1/4}. \end{aligned} \tag{4.1}$$

The position of χ_l and χ_r is expected to be $\chi_l \sim -r^{1/2}$ and $\chi_r \sim r^{1/2}$, Lemma 4.5. For the time being we don't have this. From the crude estimate mentioned just before Lemma 4.1, we have (choosing $\alpha = 2/5$)

$$\int_{-C_{r^{1/2}}}^{\chi_l} |\lambda_i(y, r)|\,dy$$

$$= O(1)\int_{-C_{r^{1/2}}}^{-r^\beta} |y|^{-3/2}\,dy + O(1)\int_{-r^\beta}^{0} |r|^{-3/5}\,dy$$

$$= O(\varepsilon^2)\, r^{-\beta/2} + O(\varepsilon^2)\, r^{\beta-3/5}.$$

With the optimal choice $\beta = 2/5$ we have

$$\int_{-C_{r^{1/2}}}^{\chi_l} |\lambda_i(y, r)|\,dy = O(\varepsilon^2)\, r^{-1/5}.$$

This proves the Corollary.

LEMMA 4.3. *$|\chi_l - \chi_r| = O(\varepsilon^3)\, r^{1/2}$ provided that $\alpha \le 1/2$.*

Proof. From (3.1) the speeds of χ_l and χ_r at time t are $O(\varepsilon^3)\, t^{-3/2}$. At time r^α, the distance between χ_l and χ_r is $2\mu r^\alpha$, which implies the lemma immediately.

LEMMA 4.4. *Between (χ_l, r) and (χ_r, r), $u(x, r)$ is within the distance $O(1)\, r^{-3\alpha/2}$ from the i-centered rarefaction wave through the state zero provided that $\alpha \le 2/5$.*

Proof. By (3.5) one needs only to show that

$$(4.2) \qquad |\lambda_i(u(x, r)) - x/r| = O(1)\, r^{-3\alpha/2}, \qquad \chi_l < x < \chi_r.$$

Given any x_0 between χ_l and χ_r, let χ be a type III characteristic through (x_0, r) and hitting time r^α between $x = -\mu r^\alpha$ and $x = \mu r^\alpha$. Let $\sigma(t)$ be the speed of χ at time t. Since χ is of type III, and $D(r^\alpha) = O(1)\, r^{3\alpha/2}$, we have

$$|\sigma(t) - \sigma(r)| = O(1)\, r^{-3\alpha/2}, \qquad r^\alpha \le t \le r.$$

At time r, χ is located at x_0 with speed $\sigma(r) = \lambda_i(u(x_0, r))$ and between $-\mu r^\alpha$ and μr^α. Thus

$$x = O(1)\, r^\alpha + (r - r^\alpha)(\lambda_i(u(x_0, r)) + O(1)\, r^{-3\alpha/2})$$
$$= r\lambda_i(u(x_0, r)) + O(1)\, r^\alpha + O(1)\, r^{-3\alpha/2+1}$$

This implies (4.2) since $\alpha \le 2/5$.

With Lemmas 4.1 and 4.4 we make the optimal choice $\alpha = 2/5$ from here on. With the same technique of investigating the backward type III characteristics as in the above proof, we have the following estimate of shock waves; its proof is omitted.

LEMMA 4.5. *Between* χ_l *and* χ_r *each i-shock wave decays at the rate* $r^{-3\alpha/2}=r^{-3/5}$.

The following lemma shows that the dominating i-shock waves are those on χ_l and χ_r.

LEMMA 4.6. *Suppose that* $q_i\neq 0$ (*or* $p_i\neq 0$). *Then for sufficiently large* r *an i-shock wave of strength*

$$-(2q_i)^{1/2}r^{-1/2}(1+O(1)\,r^{-1/10})$$
$$(or\ -(-2p_i)^{1/2}r^{-1/2}(1+O(1)\,r^{-1/10})),$$

is located at (χ_r, r) (*or* (χ_l, r)). *Moreover, the location of* χ_r (*or* χ_l) *is*

$$\chi_r=(2q_i\,r)^{1/2}(1+O(1)\,r^{-1/10}),$$
$$(or\ -(-2p_i\,r)^{1/2}(1+O(1)\,r^{-1/10})),$$

and to the right of χ_r, *i-waves are* $O(1)r^{-3/4}$ *and*

$$\int_{\chi_r}^{\mu r}|\lambda_i(y,r)|\,dr=O(1)\,r^{-1/4}$$
$$\left(or\ \int_{-\mu r}^{\chi_l}|\lambda_i(y,r)|\,dr=O(1)\,r^{-1/4}\right).$$

Proof. Suppose that $q_i\neq 0$ and that

$$q_i(r)=\int_{x_0}^{\mu r}\lambda_i(u(y,r))\,dy.$$

From Corollary 4.2, Lemmas 4.3, 4.4 and (3.8) we know that x_0 is close to zero and that

$$\begin{aligned}q_i&=O(1)\,r^{-1/2}+\int_0^{\chi_r}\frac{y}{r}\,dy\\&\quad+\int_0^{\chi_r}O(1)\,r^{-3/5}\,dy+\int_{\chi_r}^{\mu r}\lambda_i(u(y,r))\,dy\\&=\frac{\chi_r^2}{2r}+O(1)\,r^{-3/5}\cdot r^{1/2}+O(1)\,r^{-1/5}+O(1)\,r^{-1/2}\\&=\frac{\chi_r^2}{2r}+O(1)\,r^{-1/5}.\end{aligned}\tag{4.3}$$

This gives the estimate on χ_r. From Lemmas 4.1 and 4.4 and the above estimate

$$
\begin{aligned}
\lambda_i(u(\chi_r - 0, r)) & \\
&= \chi_r r^{-1} + O(1) r^{-3/5} \\
&= ((2q_i r)^{1/2}(1 + O(1) r^{-1/10})) r^{-1} + O(1) r^{-3/5} \qquad (4.3)\\
&= (2q_i)^{1/2} r^{-1/2}(1 + O(1) r^{-1/10}), \\
\lambda_i(u(\chi_r + 0, r)) &= O(1) \chi_r^{-3/5} = O(1) r^{-1/10}.
\end{aligned}
$$

This yields the strength of the shock wave on χ_r. Finally the last estimate follows immediately from Lemma 4.1 and the above estimate of χ_r. The proof concerning χ_l is analogous.

5. **Asymptotic behavior.** The following theorem deals with the generic case where $p_i < 0 < q_i$. The estimate is somewhat stronger in this case. In particular it is stronger than those obtained in the last section. This is due to the stabilizing effect of the relatively stronger shock waves predicted in Lemma 4.6. Recall that we have made the assumption $\lambda_i(0) = 0$ for convenience.

THEOREM 5.1. *Suppose that $p_i < 0 < q_i$. Then there exist two Lipschitz continuous curves $x = \chi_l(t)$ and $x = \chi_r(t)$ with the properties that*

(1)
$$
\begin{aligned}
\chi_l(t) &= -(2p_i t)^{1/2}(1 + O(1) t^{-1/4}), \\
\chi_r(t) &= (2q_i t)^{1/2}(1 + O(1) t^{-1/4}),
\end{aligned}
$$

(2) *for t sufficiently large there are i-shock waves at $(\chi_l(t), t)$ and $(\chi_r(t), t)$ with strength (measured by a jump in λ_i)*

$$-(-2p_i)^{1/2} t^{-1/2}(1 + O(1) t^{-1/4})$$

and

$$-(2q_i)^{1/2} t^{-1/2}(1 + O(1) t^{-1/4}),$$

respectively,

(3) *in Ω_i and not between $x = \chi_l(t)$ and $x = \chi_r(t)$, the total amount of waves is $O(1)t^{-3/4}$, in $L_\infty(x)$ u decays at the rate $O(1) t^{-3/4}$ and in $L_1(x)$ u decays at the rate $O(1) t^{-1/4}$,*

(4) *between $x = \chi_l(t)$ and $x = \chi_r(t)$, $u(x, t)$ and the i-rarefaction wave through the state zero differ pointwise by $O(1) t^{-1}$.*

Proof. The theorem is contained in Lemmas 4.1, 4.4 and 4.5 except for the improved rates of decay. We start with the proof of (4). The proof is similar to that of Lemma 4.4. We construct a type III characteristic χ through a point (x, t) between $x=\chi_l(t)$ and $x=\chi_r(t)$ and, instead of stopping at time t^α we let it continue until time T, where the two relatively strong shock waves have already emerged. Thus χ lies between $x=\chi_l(t)$ and $x=\chi_r(t)$ between time t and T. At time t the speed $\sigma(t)$ of χ is $\lambda_i(u(x, t))$. At an earlier time s, the speed $\sigma(s)$ differs from $\sigma(t)$ by $O(1)\,D(s)=O(1)\,s^{-3/2}$:

$$|\sigma(s)-\sigma(t)|=|\sigma(s)-\lambda_i(u(x, t))|=O(1)\,s^{-3/2}, \qquad T\le s\le t.$$

At time T the location of χ is $O(1)\,T=O(1)$ since T is independent of t, and so

$$\begin{aligned} x &= O(1)+\int_T^t (\lambda_i(u(x, t))+O(1)\,s^{-3/2})\,ds \\ &= \lambda_i(u(x, t))\,t+O(1)\,t^{-1/2}+O(1), \end{aligned}$$

$$x/t-\lambda_i(u(x, t))=O(1)\,t^{-1} \qquad t\to\infty.$$

This and (3.5) yield (4). With (4) and the last estimate in Lemma 4.6 we may improve (4.3) to

$$\begin{aligned} q_i &= O(1)\,r^{-1/2}+\int_0^{\chi_r}\frac{y}{r}\,dy+\int_0^{\chi_r}O(1)\,r^{-1}\,dy+O(1)\,r^{-1/4} \\ &= \frac{\chi_r^2}{2r}+O(1)\,r^{-1/4} \end{aligned}$$

whence the estimate (1). (2) and (3) follow from (1) and Lemma 4.1. This completes the proof of the theorem.

A N-wave for the inviscid Burger equation (3.8) with time invariants p and q is a rarefaction wave sandwiched by two shock waves of strength $-(2pt^{-1})^{1/2}$ and $-(2qt^{-1})^{1/2}$:

$$N_{p,q}(x, t)=\begin{cases} x/t & \text{for } \ -(-2pt)^{1/2}<x<(2qt)^{1/2} \\ 0 & \text{otherwise.} \end{cases}$$

For the system (1.1), a N-wave $u(x, t)=N^i_{p,q}(x, t)$ for the i-field with zero base state is defined as

$$N^i_{p,q}(x, t) \in R_i(0),$$

$$\lambda_i(N^i_{p,q}(x, t)) = N_{p,q}(x, t).$$

Here for convenience we have assumed as before that $\lambda_i(0) = 0$. The following corollary on L_1-convergence follows from the above theorem.

COROLLARY 5.2. *Suppose that $p_i < 0 < q_i$. Then*

$$\int_{-\infty}^{\infty} \left| \, |u(x, t)|_{\Omega_i} - N^i_{p,q}(x, t) \right| dx = O(1)\, t^{-1/4}.$$

REMARKS:

(1) The rate $t^{-1/4}$ of L_1-convergence seems to be optimal. Since the shock waves on $x = \chi_l(t)$ and $x = \chi_r(t)$ decay at the rate $t^{-1/2}$, their cancellation with rarefaction waves produces waves of other families with strength $t^{-3/2}$. These waves of other families fill up the region outside the primary regions. In other words, the estimate in Lemma 4.1 should be optimal. Our rate of $t^{-1/4}$ essentially comes from the estimate in Lemma 4.1 and the location of $x = \chi_l(t)$ and $x = \chi_r(t)$ in Lemma 4.6. Thus the rate should be optimal.

(2) When both p_i and q_i are zero, the solution decays to zero at a faster rate. It decays in $L_1(x)$ at the rate $t^{-1/5}$, Lemma 4.1. When one of p_i and q_i is zero, we get a one-sided N-wave. The rate $t^{-1/5}$ may not be optimal though.

(3) The convergence results in Theorem 5.1 and Corollary 5.2 remain the same for nonlinear waves when (1.1) has linearly degenerate fields. In this case, the linear waves tend to traveling waves at the rate $t^{-1/2}$ in $L_1(x)$, [6], which is an optimal rate.

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