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ON DERIVATIONS OF PRIME RINGS WITH INVOLUTION (II)

BY

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Abstract. Let R be a prime ring with involution $*$, of characteristic not 2. By a $*$-derivation d(resp. skew $*$-derivation) on R we mean that d is a derivation of R satisfying $d(x^*)=d(x)^*$ (resp. $d(x^*)=-d(x)^*$) for all $x\in R$. In this paper we show the following.

THEOREM *Let R be an S_4-free ring and d a nonzero derivation of R. Suppose that $b\in K$ (that is, $b^*=-b$) such that $[b, d(k)]\in Z$, the center of R, for all $k\in K$. If* $\dim R_z\neq 16$ *or if d is a skew $*$-derivation, then $b\in z$.*

Here R_z denotes the localization of R at $Z-\{0\}$ and dim R_z denotes the dimension of R_z over its center.

Let R be an associative ring. Recall that a derivation of R is an additive mapping d from R into itself such that $d(xy)=d(x)y+xd(y)$ for all $x, y\in R$. In the previous paper [5] we showed the following.

THEOREM. *Let R be a prime ring with involution $*$,* char$R\neq 2$ *and d a nonzero derivation of R such that $[a, d(s)]\in Z$, the center of R, $a^*=a$, for all symmetric elements $s\in R$. Then $a\in Z$ unless R satisfies*

$$S_4(x_1, x_2, x_3, x_4)=\sum_{\sigma\in P_4}(-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)},$$

the standard identity of degree 4.

In the present paper we shall handle the skew case.

Let R be a prime ring satisfying a polynomial identity of degree k. Then by [3; Theorem 1.4.2] Z, the center of R, is always nontrivial and it follows from [3; Theorem 1.4.3] that R_z, the localization of R at $Z-\{0\}$, is a central simple algbra of dimension

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at most $[k/2]^2$ over its center. We also not that the center of R_z is exactly the quotient field of Z. We shall denote by dim R_z the dimension of R_z over its center in this case.

We call R to be an S_k-ring if R satisfies the standard identity

$$S_k(x_1, \cdots, x_k) = \sum_{\sigma \in p_k} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}.$$

Otherwise, R is called an S_k-free ring.

Throughout this paper R will always denote a prime ring with involution $*$, of characteristic not 2. Let S and K denote the set of symmetric elements of R and the set of skew elements of R respectively. $Z(R)$ (or Z in brief) will be the center of R. For $a, b \in R$, $[a, b]$ will be the element $ab - ba$ and $a \circ b$ the element $ab + ba$. However, given two subsets A and B of R then $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[a, b]$ with $a \in A$ and $b \in B$; $A \circ B$ is defined similarly.

Finally, a derivation d of R is called a $*$-derivation (resp. skew $*$-derivation) if $d^* = d$ (resp. $d^* = -d$), where d^* is defined by $d^*(x) = d(x^*)^*$ for all $x \in R$. In this paper we shall prove the following.

THEOREM. *Let R be an S_4-free ring and d a nonzero derivation of R. Suppose that $b \in K$ is such that $[b, d(k)] \in Z$ for all $k \in K$. If* dim $R_z \neq 16$ *or if d is a skew $*$-derivation, then $b \in Z$.*

To complete the proof of the theorem above we shall proceed with a series of lemmas. *In what follows we assume, unless otherwise stated, that R is always an S_4-free ring.* Let d be a nonzero derivation of R and let $b \in K$ be such that $[b, d(k)] \in Z$ for all $k \in K$. We begin with.

LEMMA 1. *If either $d(Z) \neq 0$ or $Z \not\subset S$, then $b \in Z$.*

We omit the proof since it is similar to those of [5; Lemma 3 and Lemma 4].

With Lemma 1 we may assume that both $d(Z) = 0$ and $Z \subseteq S$ in all that follows.

LEMMA 2. $b^2 \in Z$.

Proof. We divide into some cases.

Case 1. *Assume that* $d(b) = 0$. For $s \in S$ we have $b \circ s \in K$. So $[b, d(b \circ s)] = [b, b \circ d(s)] + [b, d(b) \circ s] = [b^2, d(s)] \in Z$. Since $b^2 \in S$ and R is S_4-free, it follows from [5, Theorem 2] that $b^2 \in Z$.

Case 2. *Assume that* d *is a* $*$*-derivation.* By $d(K) \subseteq K$ and $Z \cap K = 0$ we have $[b, d(K)] = 0$. In particular, $[b, d(b)] = 0$. For $k \in K$ we have $[b, d([b, k])] = 0$. Thus $[b, d([b, k])] = [b, [d(b), k]] + [b, [b, d(k)]] = [b, [d(b), k]] = 0$. Since $[b, d(b)] = 0$, it follows from Cases 1 that either $b^2 \in Z$ or $d(b) \in Z$. If $d(b) \in Z$, then $d(b) \in Z \cap K = 0$ and hence $b^2 \in Z$ by Case 1 again.

Case 3. *Assume that* d *is a skew* $*$*-derivation.* For $k \in K$ we have

$$[b, [d(b), k]] \in Z. \tag{1}$$

Since $b^2 \circ k \in K$, replacing k by $b^2 \circ k$ in (1) we obtain that $[b, [d(b), b^2 \circ k]] \in Z$. Expanding it and using $[b, d(b)] \in Z$, we yield

$$b^2[d(b), [b, k]] + (b[d(b), b]) \circ [b, k] \in Z. \tag{2}$$

If $[b, d(b)] \neq 0$, then commuting b with (2) we get $[b, (b[d(b), b]) \circ [b, k]] = 0$. So $[b, [b^2, k]] = 0$ for all $k \in K$. From Case 1 we get $b^2 \in Z$. So we assume $[b, d(b)] = 0$. By (1) and Case 1 either $b^2 \in Z$ or $d(b) \in Z$. Hence we may assume that $d(b) \neq 0$ in Z. By $d(Z) = 0$ we have that $d([b, d(k)]) = [b, d^2(k)] = 0$ for all $k \in K$. For $s \in S$, by $d(s) \in K$ we get $[b, d^2(s)] \in Z$. Thus $[b, d^2(R)] \subseteq Z$. Expanding $[b, d^2(bx)] \in Z$ where $x \in R$ and noting $d^2(b) = 0$, we get that $b[b, d^2(x)] + 2d(b)[b, d(x)] \in Z$. Commuting this with b, we yield $d(b)[b, [b, d(x)]] = 0$. But $d(b) \neq 0$ in Z, so $[b, [b, d(x)]] = 0$. Replacing x by bx and expanding $[b, [b, d(bx)]] = 0$, we get $[b, [b, x]] = 0$. Then, by char $R \neq 2$, this implies $b \in Z$.

Now for the general case since $2d$ is the sum of a $*$-derivation and a skew $*$-derivation, Lemma 2.2 follows immediately from Case 2 and Case 3. This completes the proof.

For further discussion we must quote two easy results. We record them here for covenience of references. Their proofs are similar to that of [6; Lemma 1.11].

REMARK 1. Let R be a prime ring of characteristic not 2 and let $a, b \in R$ be such that $axb + bxa = 0$ for all $x \in R$. Then either $a = 0$ or $b = 0$.

REMARK 2. Let R be a prime ring with involution $*$, of characteristic not 2. Suppose that $a \in K$ and $b \in R$ are such that $akb + bka = 0$ for all $k \in K$. Then either $a = 0$ or $b = 0$.

LEMMA 3. $d(b) = 0$.

Proof. Since $d(Z) = 0$, by Lemma 2 we have $0 = d(b^2) = bd(b) + d(b)b$. However, by hypothesis, $[b, d(b)] \in Z$; thus $bd(b) \in Z$. If $b^2 = 0$, then $bd(b) = 0$ and hence $d(b)b = 0$. Expanding $[b, [d(b), k]] \in Z$ where $k \in K$, we get $bkd(b) + d(b)kb \in Z$. But $b(bkd(b) + d(b)kb) = 0$, this implies that $bkd(b) + d(b)kb = 0$ for all $k \in K$. Thus $d(b) = 0$ by Remark 2. So we assume $b^2 \in Z - \{0\}$. Since $[b, bd(b)] = 0$, we have $b[b, d(b)] = 0$ and hence $[b, d(b)] = 0$. Combining with fact that $d(b^2) = bd(b) + d(b)b = 0$, we obtain $bd(b) = 0$ and hence $d(b) = 0$. This completes the proof.

Now we come to the first result.

LEMMA 4. *If d is a skew $*$-derivation, then $b = 0$.*

Proof. *Claim* 1 ; $b^2 = 0$. Assume on the contrary that $b^2 \neq 0$. For $k \in K$, $[b, d(k)] \in Z$. By Lemma 2, $b^2 \in Z$. It follows from $[b, [b, d(k)]] = 0$ that $2b[b, d(k)] = 0$. Thus $[b, d(k)] = 0$ for all $k \in K$. For $s \in S$ we have that $d(s) \in K$ since d is a skew $*$-derivation. Thus $[b, d^2(s)] = 0$. By Lemma 3 we note that $d(b) = 0$. Thus $0 = d([b, d(k)]) = [d(b), d(k)] + [b, d^2(k)] = [b, d^2(k)]$. So $[b, d^2(R)] = 0$. For $x \in R$, $k \in K$, expanding $[b, d^2([b, k]x)] = 0$ and using $d([b, k]) = 0$, $b^2 \in Z$, we get $b[b, k]\, d^2(x) = 0$. Thus $[b, k]\, d^2(x) = 0$. So $[b, K]\, d^2(R) = 0$. But $b \in K$, the inner derivation induced by b is a $*$-derivation.

Suppose that $b \notin Z$. Then by [6; Theorem 2.8] $d^2(R)=0$. Hence $d=0$, which is absurd. This implies $b \in Z$. However $Z \subseteq S$; hence $b=0$. This proves Claim 1.

Claim 2. $bd(s)+d(s)b=0$ for all $s \in S$.

Let $s \in S$ and $k \in K$; then $d(s) \in K$ and $d(k) \in S$. Thus, by hypothesis, $[b, d^2(s)] \in Z$. Since $d(b)=0$ and $d(Z)=0$, we get $[b, d^2(k)]=0$. Thus $[b, d^2(x)] \in Z$ for all $x \in R$. Replacing x by bx and expanding $[b, d^2(bx)]$, we get $b[b, d^2(x)] \in Z$. Thus $[b, d^2(x)]=0$ for all $x \in R$. Replacing x by $[b, k]x$ where $k \in K$ and expanding $[b, d^2([b, k]x)]=0$, we obtain

$$(1) \qquad [b, d(k)][b, d(x)] - bkbd^2(x) = 0$$

for all $x \in R$, $k \in K$, by using $d(b)=0=b^2$ and $d^2([b, k])=0$. Now suppose that $[b, d(k)] \neq 0$ for some $k \in K$. Replacing x by k in (1), we get $[b, d(k)]^2 = bkbd^2(k)$. But $b^2=0$ and $[b, d(k)] \in Z$, so $[b, d(k)]=0$, which is absurd. Hence

$$(2) \qquad [b, d(k)] = 0 \quad \text{for all } k \in K.$$

From this (1) is reduced to $bKbd^2(R)=0$. By [6; Lemma 1.8] $bd^2(x)=0$ for all $x \in R$. Replacing x by xby where $y \in R$ and expanding $bd^2(xby)=0$, we get $bd(x)bd(y)=0$ by using $d(b)=bd^2(x)=bd^2(y)=0$. Thus $bd(x)bd(R)=0$ for all $x \in R$. But the annihilator of $d(R)$ is zero, thus

$$(3) \qquad bd(x)b = 0 \quad \text{for all } x \in R.$$

Let $l \in K$ and $s \in S$; then $sbl - lbs \in K$. Expanding $[b, d(sbl-lbs)]=0$ and using $d(b)=0$, (2) and (3), we get

$$(4) \qquad bsbd(l) - blbd(s) - d(s)blb + d(l)bsb = 0.$$

From (2) and (3) we have that $bsbd(l) = bsd(l)b = b(d(sl)-d(s)l)b = -bd(s)lb$ and similarly $d(l)bsb = -bld(s)b$. Thus (4) implies that $(b \circ d(s))lb + bl(b \circ d(s)) = 0$ for all $s \in S$ and $l \in K$. By Remark 2 again we obtain that $b \circ d(s) = 0$ for all $s \in S$ and hence Claim 2 holds.

Claim 3. *$bRb \subseteq Cb$, where C is the extended centroid of R* [3; pp 20–23].

From (2) and Claim 2,

(5) $$bd(x)+d(x^*)b=0 \quad \text{for all } x\in R.$$

Since d is a skew $*$-derivation, $d(x^*)=-d(x)^*$ for all $x\in R$. Thus, by (5), $(bd(x))^*=-bd(x)$, that is

(6) $$bd(x)\in K \quad \text{for all } x\in R.$$

Let $x, y\in R$. From $d(b)=0$, (3) and (6) we have that $bd(xby)=bxbd(y)\in K$. In other words, $bxbd(y)=-(bxbd(y))^*=-(bd(y))^*x^*b^*=-bd(y)x^*b$. Thus

(7) $$bxbd(y)+bd(y)x^*b=0.$$

Let $z\in R$ and set $a=bd(y)$. Then by (7) we obtain $bxbza=-bxaz^*b=ax^*bz^*b=bzbxa$. Hence it follows that $(bxbzb-bzbxb)d(y)=0$ for all $x, y, z\in R$. So $bxbzb=bzbxb$ for all $x, z\in R$. By Martindale's theorem [3; p23] $bRb\subseteq Cb$.

For $s\in S$ we have $sbs\in K$. Thus by (2) $[b, d(sbs)]=0$. Expanding this and using (3) and $d(b)=0$, we get $[bsb, d(s)]=0$. From Claim 3 we have $bsb\in Cb$. Thus, for each $s\in S$, either $bsb=0$ or $[b, d(s)]=0$. Thus either $bSb=0$ or $[b, d(S)]=0$. We claim that $[b, d(S)]=0$ holds always. Suppose that $bSb=0$. For $x\in R$, $s\in S$ we have that $bd(xs)b=0$ by (3). Thus $bxd(s)b=-bd(x)sb=d(x^*)bsb=0$ by (5), that is, $bRd(S)b=0$. So $d(S)b=0$. Since $b\in K$ and d is a skew $*$-derivation, we also have $bd(S)=0$ and hence $[b, d(R)]=0$. Then by a theorem of Herstein [4] $b\in Z$ and hence $b=0$ as claimed.

With Lemma 4 in hand it remains to dispose of the case when d is a $*$-derivation. To arrive at this aim we first handle a technical lemma.

LEMMA 5. *Let R be a prime ring with involution $*$, of characteristic not 2 and let $a\in K$ and $t\in S$ be nonzero elements satisfying $a^2=t^2=at=ta=0$. Suppose that $akt=tka$ for all $k\in K$. If $tKt=0$ then $aKa\subseteq Ca$.*

Proof. Let $k\in K$. By hypothesis, $akt-tka=0$. Note that $ast-tsa\in S$ for all $s\in S$. Thus

(1) $$axt-txa\in S \quad \text{for all } x\in R.$$

Let $x \in R$ and $k \in K$. Then by $ak + ka \in S$ we have that $[axt - txa, ak + ka] \in [S, S] \subseteq K$. On the other hand, $[axt - txa, ak + ka] = (a(xak)t - t(xak)a) - (a(kax)t - t(kax)a) \in S$ by (1). But $K \cap S = (0)$, $[axt - txa, ak + ka] = 0$. Similarly, $[axt - txa, tk - kt] = 0$. Expanding the last two formlas, we get

$$axtka - txaka - akaxt + aktxa = 0. \tag{2}$$

$$txakt + tktxa - axtkt - tkaxt = 0. \tag{3}$$

From (3) and $tKt = 0$, $tx(akt) = (tka)xt$ for all $x \in R$. Applying Martindale's theorem, for each $k \in K$ we can find $\lambda_k \in C$ such that $akt = \lambda_k t$. From (2) this implies that $(\lambda_k a - aka)xt + tx(\lambda_k a - aka) = 0$. By Remark 1 we get $aka = \lambda_k a$, which completes the proof.

For the following discussion we need two theorems. The first is due to Herstein and its proof is almost the same as that of [2; Theorem 2.12]. So we only give its statement. The second is due to Erickson [1; Theorem 3]. By [6] we provide an easy proof.

THEOREM H. *Let R be a simple ring with involution $*$, of characteristic not 2, $\dim_z R > 4$ and let $*$ be of the first kind. Suppose that U is a nonzero Lie ideal of K such that $u^2 \in Z$ for all $u \in U$. Then* $\dim_z R = 16$.

THEOREM E. *Let R be a prime S_4-free ring with involution $*$, of characteristic not 2 and let $*$ be of the first kind. Suppose that U is a nonzero Lie ideal of K such that $u^2 \in Z$ for all $u \in U$. Then* $\dim R_z = 16$.

Proof. Since U is a nonzero Lie ideal of K, it follows from [1; Lemma 2] that $Z \neq 0$. Let F denote the center of R_z. Then $*$ can be canonically extended to R_z and the involution $*$ on R_z is still of the first kind. Let $U_z = FU = \{\sum \alpha_i u_i \mid \alpha_i \in F, u_i \in U\}$ and $K_z = \{k/\alpha \mid k \in K, \alpha \in Z - \{0\}\}$. Then K_z is exactly the set of skew elements of R_z and U_z is also a nonzero Lie ideal of K_z. By hypothesis we still have that $v^2 \in F$ for all $v \in U_z$. Applying Theorem H, to prove $\dim R_z = 16$ it is sufficient to show that R_z is a simple ring. Assume on the contrary that R_z is not a simple ring. Then there exists a nonzero proper $*$-ideal I of R_z. If $I \cap U_z = 0$ then $[U_z, I \cap K_z] = 0$ and hence $U_z \subseteq F$ by [6; Lemma 1.6], since I is also S_4-free. But $U_z \subseteq K_z$ and $*$ is of the first kind, $U_z = 0$ follows. This is absurd. Thus $I \cap U_z \neq 0$. Pick $u \neq 0$ in $I \cap U_z$. Then $[u, k]^2 \in F \cap I$ for all $k \in I \cap U_z$. Since F is a field, we have $F \cap I = 0$. So $[u, k]^2 = 0$ for all $k \in I \cap K_z$. Let δ be the inner derivation of I induced by u. Then $\delta(k)^2 = 0$ for all $k \in I \cap K_z$ Applying [6; Theorem 2.17], we get that I is an S_4-ring unless $u \in Z(I)$. However, if $u \in Z(I)$, then $u \in F$ and hence $u = 0$, which is absurd. Thus I is an S_4-ring and so is R_z. Of course, R is also an S_4-ring, a contradiction. Thus R_z is simple, which completes the proof.

Our next process is to handle the inner case and the general case is easily concluded from the inner one. For a subset $B \subseteq R$ we denote by $C_R(B)$ the centralizer of B in R, that is, $C_R(B) = \{x \in R \mid xb = bx \text{ for all } b \in B\}$.

LEMMA 6. *Let h be a skew element of R such that $h^2 \in Z - \{0\}$ and let $\delta(x) = [h, x]$ for all $x \in R$. Then $C_R(\delta(K)) \cap K = 0$ unless* $\dim R_z = 16$.

Proof. Set $A = C_R(\delta(K)) \cap K$. Let $a \in A$; then $[a, \delta(K)] = 0$. For $k, l \in K$ we have that $[k, \delta(l)] \in K$. By Lemma 3 $\delta(a) = 0$ and hence $[a, \delta^2(l)] = 0$. Thus $0 = [a, \delta([k, \delta(l)])] = [a, [\delta(k), \delta(l)]] + [a, [k, \delta^2(l)]] = [[a, k], \delta^2(l)]$. But $h^2 \in Z - \{0\}$, So $\delta^2(l) = 2h\delta(l)$. We also note that $\delta(a) = 0 = \delta([a, k])$. Thus $[[a, k], \delta^2(l)] = 2h[[a, k], \delta(l)] = 0$. But $h^2 \in Z - \{0\}$, we get that $[[a, k], \delta(l)] = 0$ for all $k, l \in K$. Therefore $[A, K] \subseteq A$, that is, A is a Lie ideal of K. Applying Lemma 2 we have that $a^2 \in Z$ for all $a \in A$. Thus, if $A \neq 0$, then Theorem E implies $\dim R_z = 16$, since R is an S_4-free ring. This completes the proof.

With Lemmas 5, 6 we are in a position to prove the key

LEMMA 7. *Let $e, f \in K$ be such that $[e, [f, k]] = 0$ for all $k \in K$. Then either $e = 0$ or $f = 0$ unless* $\dim R_z = 16$.

Proof. We note first that R may be assumed to be a centrally closed prime algebra with center Z as its extended centroid.

Assume further that $\dim R_z \neq 16$. Set $A = C_R(\delta(K)) \cap K$, where $\delta(x) = [f, x]$ for all $x \in R$. So $e \in A$. Now we assume that e and f are both nonzero. Then, by Lemma 3, $[e, f] = 0$ and Lemma 2 implies that both $e^2 \in Z$ and $f^2 \in Z$. From Lemma 3 we have $[f, A] = 0$. Thus by Lemma 6 we have that $f^2 = 0$ and $x^2 = 0$ for all $x \in A$. But A is an additive subgroup of K, so

(1) $$xy + yx = 0 \quad \text{for all} \quad x, y \in A.$$

In particular, $xy \in K$ for all $x, y \in A$. Let $x, y \in A$; then $[x, [f, k]] = 0$ for all $k \in K$. Expanding this and using $[x, f] = 0$, we get

(2) $$xkf + fkx = xfk + kxf \quad \text{for all} \quad k \in K.$$

Multiplying (2) by x from the left, we get

(3) $$xkxf = xfkx \quad \text{for all} \quad k \in K.$$

From (2) and Remark 2 we have that if $xf = 0$ then $x = 0$. Suppose now that $x \neq 0$; then $xf \neq 0$ and $xKx \neq 0$. But by (3) $xfKxf = 0$, so by $xf \in S$ and Lemma 5 we get $xKx \subseteq Zx$. Thus $yxKxy \subseteq Zyxy = 0$ by (1) and $y^2 = 0$. Since $xy \in K$ and $xyKxy = 0$, we get $xy = 0$. Up to now we have seen that $A^2 = 0$.

Let $\mu(x) = [e, x]$ for all $x \in R$. Note that $\delta\mu = \mu\delta$. Let $s \in S$ and $k \in K$; then $[k, f \circ s] \in K$ and $\delta(f \circ s) = [f^2, s] = 0$. Thus

$$0 = \mu\delta([k, f \circ s]) = \mu([\delta(k), f \circ s]) = [\delta(k), \mu(f \circ s)].$$

Since $f \circ s \in K$, we have that $\mu(f \circ s) \in K$ and hence $\mu(f \circ s) \in A$. By $A^2 = 0$ we get $e\mu(f \circ s) f = 0$. Expand this and use $[e, f] = e^2 = f^2 = 0$; then $(ef)s(ef) = 0$ for all $s \in S$. But $ef \in S$, it follows that $ef = 0$; this implies $e = 0$, which is absurd. This completes the proof.

Now we may dispose of the case when d is a $*$-derivation..

LEMMA 8. *If d is a $*$-derivation of R, then $b = 0$ unless* $\dim R_z = 16$.

Proof. Assume that $\dim R_z \neq 16$. Since $d(K) \subseteq K$ and $K \cap Z = 0$, we get $[b, d(K)] = 0$. For $k, l \in K$ we have $[d(k), l] \in K$. Thus $0 = [b, d([d(k), l])] = [d^2(k), [b, l]]$. Suppose that $b \neq 0$. Since $d^2(k) \in K$, by Lemma 7 we get $d^2(k) = 0$. That is $d^2(K) = 0$. From [6; Theorem 2.6] R must be an S_4-ring. This is a contradiction. So $b = 0$ as claimed.

Proof of Theorem.

Assume that R is S_4-free. By Lemma 1 we may assume that both $d(Z) = 0$ and $Z \subseteq S$. If d is a skew $*$-derivation, then Lemma 4 completes the proof. For the general case, we write $2d = d_1 + d_2$, where d_1 is a skew $*$-derivation and d_2 is a $*$-derivation. Thus $[b, d_1(K)] \subseteq Z$ and $[b, d_2(K)] = 0$. Now suppose that $b \neq 0$ and $\dim R_z \neq 16$. Then Lemma 4 implies that $d_1 = 0$ and Lemma 8 concludes that $d_2 = 0$. This is a contradiction. So $b = 0$, which completes the proof.

We conclude this paper with two remarks.

REMARK 3. In Theorem it is essential to assume that $b \in K$. A counterexample can be found in [6; Example].

REMARK 4. Let $R = C_4$ be the 4×4 matrix ring over the complex number field C and let the involution $*$ be the usual transpose mapping. Let d be the inner derivation of R induced by b, where

$$b = \begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & -i \\ -i & 0 & 0 & 1 \\ 0 & i & -1 & 0 \end{pmatrix}.$$

Then by an elementary calculation we have

$$d(K) = \left\{ \begin{pmatrix} 0 & -ia & a & b \\ ia & 0 & b & -a \\ -a & -b & 0 & -ia \\ -b & a & ia & 0 \end{pmatrix} \middle| a, b \in C \right\},$$

and

$$C_R(d(K)) \cap K = \left\{ \begin{pmatrix} 0 & a & x & y \\ -a & 0 & -y & x \\ -x & y & 0 & -a \\ -y & -x & a & 0 \end{pmatrix} \middle| x, y, a \in C \right\}.$$

Note that $b \in K$ and $b^2 = 0$. The example shows that $\dim R_z = 16$ may occur in this case which is in contrary to the symmetric case.

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