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DERIVATIONS CENTRALIZING SYMMETRIC OR SKEW ELEMENTS

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Abstract. Let R be a prime ring with involution $*$ and center Z. If d is a nonzero derivation on R such that $d(x)x - xd(x) \in Z$ for all symmetric x or for all skew x, then we show that R must be a commutative integral domain or an order in a 4-dimensional simple algebra. Similar results are also obtained where the condition $d(x)x - xd(x) \in Z$ is replaced by $d(x)x + xd(x) \in Z$. As a by-product we prove a theorem generalizing Chacron's theorem: if $x^n \in Z$ for all symmetric x, where n is a fixed integer, then R satisfies the standard identity in 4 variables.

In an early paper [7] Posner proved the following theorem: If d is a nonzero derivation on a prime ring R such that, for all elements x in R, $[x, d(x)] = xd(x) - d(x)x$ is in the center Z of R, then the ring R must be commutative. In this paper we shall consider similar problems when the ring R is equipped with an involution $*$. What can we say about the structure of R if $[x, d(x)] \in Z$ merely for all symmetric elements $x = x^*$ or for all skew elements $x = -x^*$? In this case one cannot expect to conclude the commutativity of R even if R is assumed to be a division ring. For instance, in the ring of real quaternions, if $*$ is the usual conjugation $(\alpha + \beta i + \gamma j + \delta k)^* = \alpha - \beta i - \gamma j - \delta k$, all symmetric elements are central and hence the property $[x, d(x)] \in Z$ holds trivially for all symmetric elements x. On the other hand, if $*$ is defined by $(\alpha + \beta i + \gamma j + \delta k)^* = \alpha - \beta i + \gamma j + \delta k$, all skew elements commute with one another, so the property $[x, d(x)] \in Z$ holds for all skew elements x when d is an inner derivation induced by some nonzero skew element. Also, one can easily produce counter-examples by suitably defining an involution $*$ and a derivation d on the ring of

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2×2 matrices over a field. As we shall see in the present paper, the quaternions and the 2×2 matrices are the only objects of which one can make noncommutative examples. Explicitly speaking, any such a prime ring must be either a commutative domain or an order in a 4-dimensional simple algebra. Or equivalently, the ring must satisfy the standard identity

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}.$$

In what follows R will always denote a prime ring with involution $*$ and center Z. S is its set of symmetric elements and K its set of skew elements. For a subset A of R, $\bar{A}$ means the subring of R generated by A. And, for subsets A, B, $[A, B]$ will be the additive subgroup of R generated by elements of the form $[a, b] = ab - ba$ with $a \in A$ and $b \in B$.

We start with a symmetric version of Posner's theorem. For the time being we are concerned first with the case when R is not of characteristic 2.

THEOREM 1. *If d is a nonzero derivation on R such that $[d(s), s] \in Z$ for all $s \in S$, then R satisfies s_4 provided* char $R \neq 2$.

Proof. First, we show that the hypothesis actually assumes a stronger form, namely, $[d(s), s] = 0$ for all $s \in S$. By linearizing on s in $[d(s), s] \in Z$ we have $[d(s), t] + [d(t), s] \in Z$ for all $s, t \in S$. In particular, $[d(s), s^2] + [d(s^2), s] \in Z$ for all $s \in S$. Expanding this and using $[d(s), s] \in Z$ we get $4s[d(s), s] \in Z$. The fact that $[d(s), s] \in Z$ forces $s \in Z$ or $[d(s), s] = 0$ because char $R \neq 2$. Hence $[d(s), s] = 0$ for all $s \in S$.

Next we show that we may assume d to be inner. From $[d(s), s] = 0$ we have $[d(s), t] + [d(t), s] = 0$ for all $s, t \in S$. Set $t = [s, k]$ where $k \in K$; then

$$\begin{aligned} 0 &= [d(s), [s, k]] + [d([s, k]), s] \\ &= [d(s),[s, k]] + [[d(s), k], s] + [[s, d(k)], s] \\ &= [[d(s), s], k] + [[s, d(k)], s] \\ &= [[s, d(k)], s]. \end{aligned}$$

Thus $[\delta_k(s), s]=0$ for all $s \in S$ if we denote by δ_k the inner derivation induced by $d(k)$. Suppose that this theorem has been proved for nonzero inner derivations, then we can conclude that $d(k) \in Z$ for all $k \in K$ or R satisfies s_4. Thus we are done because $d(K) \subseteq Z$ implies R satisfying s_4 by [6; Lemma 1.6].

Now let $d(x)=[a, x]$ for all $x \in R$, where a is a fixed noncentral element. Applying $*$ to $[d(s), s]=[[a, s], s]=0$ we have $[[a^*, s], s]=0$. Thus $[[a+a^*, s], s]=0=[[a-a^*, s], s]$ for $s \in S$. Since $a \notin Z$, $a+a^*$ and $a-a^*$ cannot be both in Z. Hence we may, if necessary, replace a by $a+a^*$ or $a-a^*$ and assume that a is either symmetric or skew.

Assume first that $a \in S$. For $s \in S$ we have $[d(a+s), a+s]=0$. Thus $0=[[a, a+s], a+s]=[[a, s], a+s]=[[a, s], a]$ for all $s \in S$. That is, $d^2(S)=0$ from which it follows that R satifies s_4 [6; Theorem 2.2].

It remains to check the case when $a \in K$. For $s \in S$ we have $[d(a^2+s), a^2+s]=0$ and hence $[[a, s], a^2]=0$. Thus $[[a^2, s], a^2]=0$ for all $s \in S$. Now $a^2 \in S$ so we are done unless $a^2 \in Z$. Hence assume that $a^2 \in Z$. Then $ad(s)+d(s)a=d(as+sa)=0$. Commuting this with s we have $2d(s)^2=0$, whence $d(s)^2=0$ and $d(s)d(t)+d(t)d(s)=0$ for all $s, t \in S$. Replace t by $st+ts$;

$$\begin{aligned} 0 &= d(s)\,d(st+ts)+d(st+ts)\,d(s) \\ &= d(s)\,sd(t)+d(s)\,d(t)\,s \\ &\quad +2d(s)\,td(s)+sd(t)\,d(s)+d(t)\,sd(s). \end{aligned}$$

Since

$$d(s)\,sd(t)+sd(t)\,d(s)=s[d(s)\,d(t)+d(t)\,d(s)]=0$$

and similarly $d(s)\,d(t)s+d(t)\,sd(s)=0$, we end up with $d(s)\,Sd(s)=0$ for all $s \in S$. Note that $d(s) \in S$ so we conclude that $d(S)=0$ and hence R satisfies s_4 [4; Lemma 5]. This completes the proof.

Next we trun to a corresponding result in the skew case.

THEOREM 2. *If d is a nonzero derivation on R such that $[d(k), k] \in Z$ for all $k \in K$, then R satisfies s_4 provided* char $R \neq 2$.

Proof. From $[d(k), k] \in Z$ we have $[d(k), h] + [d(h), k] \in Z$ for all $h, k \in K$. Expanding $[d(k), [h, k]] + [d([h, k]), k]$ and using $[d(k), [h, k]] + [[h, d(k)], k] = [h, [d(k), k]] = 0$ we get $[[d(h), k], k] \in Z$ for all $h, k \in K$. If this theorem holds for nonzero inner derivations, then either $d(K) \subseteq Z$ or R satisfies s_4. But if $d(K) \subseteq Z$ we still have that R satisfies s_4 by [6; Lemma 1.6]. So we suppose that $d(x) = [a, x]$ for all $x \in R$, where $a \notin Z$. As in the proof of the preceding theorem, we may assume further that a is in S or K.

The case when $a \in K$ is much easier. If $Z \cap K \neq 0$, let $\alpha \in Z$ such that $\alpha^* = -\alpha \neq 0$. For $s \in S$, $\alpha s \in K$ so

$$\alpha^2[[a, s], s] = [[a, \alpha s], \alpha s] = [d(\alpha s), \alpha s] \in Z.$$

Thus $[[a, s], s] \in Z$ for all $s \in S$ and hence R satisfies s_4 by Theorem 1. However, if $Z \cap K = 0$ we have that $[[a, k], k] = 0$ for all $k \in K$. Then $0 = [[a, a+k], a+k] = [[a, k], a]$ for all $k \in K$. That is $d^2(K) = 0$ and hence R satisfies s_4 by [6; Theorem 2.6].

Finally, we assume that $a \in S$. For $k \in K$, set $h = ak + ka$. Expanding $[d(k), h] + [d(h), k]$ and using $[d(k), k] \in Z$ and $[d^2(k), k] = 0$, we obtain that $-d^2(k)k + 2[d(k), k]a + d(k)^2 \in Z$. Commuting this with k we have $d(k)[d(k), k] = 0$ and so $[d(k), k] = 0$. Hence $[d(k), h] + [d(h), k] = 0$ for all $h, k \in K$. Replacing h by $ah + ha$ and expanding, we have $d(k)d(h) + d(h)d(k) = d^2(k)h + hd^2(k)$. In particular, $2d(k)^2 = d^2(k)k + kd^2(k) = 2d^2(k)k$. Thus $d(k)^2 = d^2(k)k$ and hence $d(k)d(h) + d(h)d(k) = d^2(k)h + d^2(h)k$ for all $h, k \in K$. Comparing this with the previous expression for $d(k)d(h) + d(h)d(k)$ we have $hd^2(k) = d^2(h)k$ for all $h, k \in K$. As a result, $kxd^2(k) = d^2(k)xk$ for all $x \in \bar{K}$. If $K^2 \subseteq Z$ then R satisfies s_4 [5; Lemma 2]. Otherwise, $\bar{K}^2$ contains a nonzero ideal I of R. Thus $kxd^2(k) = d^2(k)xk$ for all $x \in I$ and hence for all $x \in R$, whence $d^2(k) = \lambda_k k$ for some $\lambda_k \in C$, the extended centroid of R [2; p. 23]. Since $hd^2(k) = d^2(h)k$, we have $\lambda_h = \lambda_k$ whenever $hk \neq 0$. Fix two elements $a, b \in K$ such that $ab \neq 0$ and let $\mu \in C$ such that $d^2(a) = \mu a$. Then $d^2(k) = \mu k$ for all $k \in K$ with $ak \neq 0$. But if $ak = 0$, then $a(k+b) \neq 0$ and so $\mu(k+b) = d^2(k+b) = d^2(k) + d^2(b) = \lambda_k k + \mu b$. Thus we have $\mu k = \lambda_k k$ and hence $\lambda_k = \mu$ if $k \neq 0$. In other words, $d^2(k) = \mu k$ for all $k \in K$. If $\mu \neq 0$ then $\mu k^3 = d^2(k^3) = d[3k^2 d(k)] = 6kd(k)^2 + 3k^2 d^2(k) = 9k^2 d^2(k) = 9\mu k^3$ and so $k^3 = 0$ for all $k \in K$. Then $(k^2x - x^* k^2)^3 = 0$ for all $k \in K$ and $x \in R$. Post-multiplying by k^2 we have $(k^2x)^4=0$. Thus k^2R is a right ideal of R in which the fourth power of every element is 0; by a result of Levitzki [2; Lemma 2.1.1] this cannot happen in a semiprime ring unless $k^2 = 0$. Hence $k^2 = 0$ for all $k \in K$. Again, from $(kx + x^* k)^2 = 0$ for all $k \in K$ and $x \in R$, we can conclude $K = 0$ via the same argument and so R is commutative. However, if $\mu = 0$ then $d(k)^2 = d^2(k)\,k = 0$ for all $k \in K$ and so R satisfies s_4 by [6; Theorem 2.17]. This proves the theorem.

Before removing the restriction on char R in the statements of the previous theorems we need a result on power-central symmetric elements. The following theorem was proved by Chacron [1] under the additional condition that R has no nonzero nil ideals.

THEOREM 3. *Let n be a fixed natural number such that $s^n \in Z$ for all $s \in S$. Then R satisfies s_4.*

Proof. If $Z \cap S = 0$, then $s^n = 0$ for all $s \in S$. An argument similar to that in the proof of Theorem 2 reduces n successively to yield $S = 0$ and so R satisfies s_4. If $Z \cap S \neq 0$, we can localize R at $Z^+ = Z \cap S$ to obtain a simple ring R_{Z^+} with an involution defined by $(x\alpha^{-1})^* = x^* \alpha^{-1}$ for $x \in R$ and $\alpha \in Z^+\backslash 0$. Thus R_{Z^+} satisfies the same power-central hypothesis on symmetric elements. In light of [1; Theorem 4] R_{Z^+} satisfies s_4 and, a fortiori, R satisfies s_4 too.

In addition to Theorem 3 we need one more lemma concerning the centralizer of $d(S)$.

LEMMA 4. *Assume that* char $R = 2$. *Let $a \in S$ and d a nonzero derivation on R such that $[a, d(S)] = 0$. Then $a^8 \in Z$.*

Proof. Since $a \in S$, $d(a^2) = [a, d(a)] = 0$ by hypothesis. For $x \in R$ we have

$$
\begin{aligned}
0 &= a^2 d(a^2 x + x^* a^2) + d(a^2 x + x^* a^2) a^2 \\
&= a^4 d(x) + a^2 d(x^* + x) a^2 + d(x^*) a^4 \\
&= a^4 d(x) + d(x^* + x) a^4 + d(x^*) a^4 \\
&= a^4 d(x) + d(x) a^4.
\end{aligned}
$$

That is, $[a^4, d(R)] = 0$ and hence $a^8 \in Z$ by a theorem due to Herstein [3].

Now we dispose of the case of characteristic 2. Note that K coincides with S and $[x, y]$ assumes the form $xy + yx$ if char $R = 2$. Therefore, our hypothesis reads $d(x)x + xd(x) \in Z$ for all $x \in S = K$ in this case.

THEOREM 5. *If d is a nonzero derivation on R such that $d(s)s + sd(s) \in Z$ for all $s \in S$, then R satisfies s_4 provided* char $R = 2$.

Proof. For $s \in S$, $d(s^2) = d(s)s + sd(s) \in Z$ by assumption. Then, $d(st + ts) \in Z$ for all $s, t \in S$. Expanding $d(s^2 t + ts^2)$, we get $d(s^2)t + s^2 d(t) + d(t)s^2 + td(s^2) = s^2 d(t) + d(t)s^2$ since $d(s^2)t = td(s^2)$. This tells us that $s^2 d(t) + d(t)s^2 \in Z$ and so $s^4 d(t) = d(t)s^4$ for all $s, t \in S$. By Lemma 4 we obtain that $s^{32} \in Z$ for all $s \in S$. With this the theorem is proved by Theorem 3.

In view of the preceding theorem one might ask whether the conclusion remains true if $d(s)s + sd(s) \in Z$ for all $s \in S$ in case char $R \neq 2$. The answer is affirmative indeed as we see in the next

THEOREM 6. *If d is a nonzero derivation on R such that $d(s)s + sd(s) \in Z$ for all $s \in S$, then R satisfies s_4.*

Proof. Because of Theorem 5, it suffices to prove the theorem in the situation when char $R \neq 2$.

For $s \in S$, we have $d(s^2) = d(s)s + sd(s) \in Z$ and $2s^2 d(s^2) = d(s^2)s^2 + s^2 d(s^2) \in Z$. Hence, either $s^2 \in Z$ or $d(s^2) = 0$. Assume first that $d(Z \cap S) = 0$. Then $d(s^2) = 0$ holds always for all $s \in S$. Thus, for $s, t \in S$, we have $d(st + ts) = 0$ and so

$0 = d(s^2 t + ts^2) = s^2 d(t) + d(t) s^2$. Hence $[s^4, d(S)] = 0$ for all $s \in S$. On the other hand, if $s \in S$ and $k \in K$, then $[s, k] \in S$ and so $0 = d(s[s, k] + [s, k]s) = d([s^2, k]) = [s^2, d(k)]$. Thus, $[s^2, d(K)] = 0$ and, a fortiori, $[s^4, d(K)] = 0$ for all $s \in S$. Consequently, $[s^4, d(R)] = 0$ and so $s^4 \in Z$ for all $s \in S$ [3]. Therefore, R satisfies s_4 by Theorem 3. Now to the case when $d(Z \cap S) \neq 0$. Let $\alpha \in Z \cap S$ such that $d(\alpha) \neq 0$; then $d(\alpha^2) = 2\alpha d(\alpha) \neq 0$. For $s \in S$ we have $d(\alpha^2 s^2) \in Z$, that is, $d(\alpha^2) s^2 + \alpha^2 d(s^2) \in Z$. Hence $s^2 \in Z$ for all $s \in S$ and with this we have the theorem.

In [2; Theorem 2.1.11] Herstein generalized a result of Baxter on $K \circ K$, the additive subgroup of R generated by elements of the form $hk + kh$ with $h, k \in K$. An inspection of his proof reveals that $2^{n-1} K^n \subseteq K + K \circ K$ for each natural number n. As a consequence, for any $x \in \bar{K}$, there exists some n such that $2^n x \in K + K \circ K$ and, in particular, $2^n x \in K \circ K$ in case $x \in S \cap \bar{K}$. With this in hand, we are ready to prove a skew analogue to Theorem 6 and conclude this paper.

THEOREM 7. *If d is a nonzero derivation on R such that $d(k) k + kd(k) \in Z$ for all $k \in K$, then R satisfies s_4.*

Proof. As before we need only consider the case char $R \neq 2$. If $K^2 \subseteq Z$, there is nothing to prove. So we assume that $\bar{K}^2$ contains a nonzero $*$—ideal I of R. By hypothesis, $d(k^2) = d(k) k + kd(k) \in Z$ for all $k \in K$. For $h, k \in K$, we have $hk + kh = (h + k)^2 - h^2 - k^2$, so $d(K \circ K) \subseteq Z$ follows. For $s \in S \cap I \subseteq S \cap \bar{K}$, we have $2^n s \in K \circ K$ for some natural number n. Then $2^n d(s) = d(2^n s) \in Z$ and hence $d(s) \in Z$. That is, $d(S \cap I) \subseteq Z$. For $s \in S \cap I$ we have both $d(s) \in Z$ and $2sd(s) = d(s^2) \in Z$; then either $d(s) = 0$ or $s \in Z$. Thus $S \cap I$ is the union of two additive subgroups, namely, $S \cap I \cap \operatorname{Ker} d$ and $S \cap I \cap Z$, so either $d(S \cap I) = 0$ or $S \cap I \subseteq Z$. If $S \cap I \subseteq Z$, then I satisfies s_4 and so does R. But if $d(S \cap I) = 0$ then $d(\overline{S \cap I}) = 0$. Being a Lie ideal of the prime ring I, $\overline{S \cap I}$ contains a nonzero ideal of I unless I satisfies s_4. Then, $\overline{S \cap I}$ contains a nonzero ideal J of R as well and so $d(J) = 0$, a contradiction. This completes the proof.

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