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NOTE ON CONSTRUCTIONS OF HIRANO'S SYSTEM OF CENTER CIRCLES

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Abstract. Given a line l_0 and a system of center circles of Steiner-Kantor-Morley, K. Hirano gave a construction of a system of center circles wtih reference to the given line l_0 [1][2]. In this note a general construction, which includes Hirano's construction as a particular case, of a similar system of center circles with reference to the line l_0 is given. A general construction of Hirano's another system of circles [1][2] is also given.

1. **Introduction.** Starting from the system of center circles of Steiner-Kantor-Morley, K. Hirano constructed a system of center circles with reference to a given line l_0 (Hirano [1]).

Some alternative constructions of Hirano's system or similar system of center circles are also known (Hsu [2], [3]). In this short note, we intend to show that a more general construction gives similar system of center circles.

More precisely, let $\{12\cdots n\}$ be an n-line which consists of the n lines: line $1,\cdots$, line n. The center circles of a 3-line $\{123\}$, a 4-line $\{1234\}$, a 5-line $\{12345\}$, $\cdots$, an n-line $\{12\cdots n\}$ are denoted respectively as $A(123)$ (which is the circumcircle of the triangle $\{123\}$ formed by three lines 1, 2 and 3), $A(1234)$ (called Steiner circle), $A(12345)$ (called Kantor circle),$\cdots$, and $A(12\cdots n)$. Then the respective centers are denoted as (123), (1234), $(12345),\cdots$, $(12\cdots n)$, and are respectively the centric point of the 3-line $\{123\}$, the 4-line $\{1234\}$, the five line $\{12345\},\cdots$, and the n-line $\{12\cdots n\}$. The point of intersection of the line i and line j is denoted by

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(ij) and is called the centric point of the 2-line $\{ij\}$.

Now the typical construction of Hirano's system of center circles is as follows: Let l_0 be a given line. Let k be an integer such that $3 \leq k \leq n$, and let $\{12\cdots k\}$ be the given k-line, The k centric points $(12\cdots(k-1))$, $(12\cdots(k-2)k),\cdots,(23\cdots k)$ are on the center circle $A(12\cdots k)$. Now, 1° let $(12\cdots(k-1))\|_{l_0}$ be the second point of intersection with the circle $A(12\cdots k)$ of the line through the point $(12\cdots(k-1))$ and parallel to the given line l_0. Then, 2° construct k perpendicular lines, one through $(12\cdots(k-1))\|_{l_0}$ and perpendicular to the line k, one through $(12\cdots(k-2)k)\|_{l_0}$ and perpendicular to the line $(k-1),\cdots$, and finally one through $(23\cdots k)\|_{l_0}$ and perpendicular to the line 1. These k perpendiculars all meet at a same point of $A(12\cdots k)$. This point is denoted by $R_{l_0}(12\cdots k)$. For a $(k+1)$-line we have $(k+1)$ such points: $R_{l_0}(12\cdots k)$, $R_{l_0}(12\cdots(k-1)(k+1)),\cdots, R_{l_0}(2\cdots k(k+1))$. These $(k+1)$ points are on a circle denoted as $C_{l_0}(12\cdots k(k+1))$, the center of which is the point $R_{l_0}(12\cdots k(k+1))$. Thus for an n-line $\{12\cdots n\}$ we have Hirano's system of center circles with reference to the line l_0 : $C_{l_0}(1234)$, $C_{l_0}(12345),\cdots, C_{l_0}(12\cdots n)$.

The known modifications so far of the abve construction of Hirano's system which also give rise to a system of center circles are either by changing 1° to get the second point of intersection $(12\cdots(k-1))_{\perp l_0}$ with $A(12\cdots k)$ of the line through $(12\cdots(k-1))$ and *parpendicular* to the given line l_0, or by changing 2° to construct k *parallel* lines, one through $(12\cdots(k-1)\|_{l_0}$ [or $(12\cdots(k-1))_{\perp l_0}$] and parallel to the line $k,\cdots$.

It is now natural to raise a question whether we can still get a system of center circles with reference to the line l_0 if we replace 1° by taking the line through $(12\cdots(k-1))$ and making a fixed angle θ with the given line l_0, and replace 2° by constructing lines which are making a fixed angle ϕ with the remaining line.

2. **General construction of Hirano's system of center circles.** The following proposition answers the above question:

PROPOSITION 1. *Let an n-line $\{12\cdots n\}$ and a line l_0 be given. Let k be an integer such that $3 \leq k \leq n-1$. Let θ, ϕ, $0<\theta$, $\phi<\pi$*

be two fixed angles. For the k-line $\{12\cdots k\}$, *let* $(12\cdots(k-1))_{l_0\theta}$ *be the second point of intersection with the center circle* $A(12\cdots k)$, *on which the centric point* $(12\cdots(k-1))$ *lies, of the line through* $(12\cdots(k-1))$ *and making the angle* θ *with the given line* l_0. *Next draw k lines, one through the point* $(12\cdots(k-1))_{l_0\theta}$ *and making the angle* ϕ *with the line k, one through the point* $(12\cdots(k-2)k)_{l_0\theta}$ *and making the angle* ϕ *with the line* $(k-1),\cdots$, *and the last one through the point* $(23\cdots k)_{l_0\theta}$ *and making the angle* ϕ *with the line* 1. *Then these k lines meet at one point on the circle* $A(12\cdots k)$, *denoted as* $R_{l_0\theta\phi}(12\cdots k)$. *For the* $(k+1)$*-line* $\{12\cdots k(k+1)\}$, *we have* $k+1$ *such points*: $R_{l_0\theta\phi}(12\cdots k)$, $R_{l_0\theta\phi}(12\cdots(k-1)(k+1)),\cdots$, $R_{l_0\theta\phi}(23\cdots k(k+1))$. *These* $k+1$ *points lie on a circle denoted as* $C_{l_0\theta\phi}(12\cdots k(k+1))$. *In this way, for the n-line* $\{12\cdots n\}$, *we have a system of circles* $C_{l_0\theta\phi}(1234)$, $C_{l_0\theta\phi}(12345),\cdots$, $C_{l_0\theta\phi}(12\cdots n)$. *Moreover,* $R_{l_0\theta\phi}(12\cdots k)$ *is the center of the circle* $C_{l_0\theta\phi}(12\cdots k)$, *and the circle* $C_{l_0\theta\phi}(12\cdots k(k+1))$ *contains all the centers* $R_{l_0\theta\phi}(12\cdots k)$, $R_{l_0\theta\phi}(12\cdots(k-1)(k+1)),\cdots$, $R_{l_0\theta\phi}(23\cdots k(k+1))$ *of the respective circles* $C_{l_0\theta\phi}(12\cdots k)$, $C_{l_0\theta\phi}(12\cdots(k-1)(k+1)),\cdots$, $C_{l_0\theta\phi}(23\cdots k(k+1))$. *Thus the system of circles* $C_{l_0\theta\phi}(1234)$, $C_{l_0\theta\phi}(12345),\cdots$, $C_{l_0\theta\phi}(12\cdots n)$ *is a system of center circles.*

Proof. Let the k lines: line $1,\cdots$, line k of the k-line $\{12\cdots k\}$ be represented by

$$(1)\qquad z t_i + \bar{z} = z_i t_i, \qquad i = 1,\cdots, k,$$

where z_i are points and $t_i = \bar{z}_i/z_i$. Then the characteristic constants of this k-line, defined by F. Morley, are

$$(2)\qquad a_\alpha^{(k)} = \sum \frac{z_1 t_1^{k-\alpha}}{(t_1-t_2)(t_1-t_3)\cdots(t_1-t_k)}, \qquad \alpha = 1,\cdots, k.$$

Then the centric point $(12\cdots(k-1))$ of the $(k-1)$-line $\{12\cdots(k-1)\}$ is given by $a_1^{(k-1)}$, and the center circle $A(12\cdots k)$ of the k-line $\{12\cdots k\}$ is given by

$$(3)\qquad z = a_1^{(k)} - a_2^{(k)} t, \qquad |t| = 1.$$

The line through the point $(12\cdots(k-1))$ and making the angle θ with the given line l_0:

$$zt_0 + \bar{z} = z_0 t_0 \tag{4}$$

is given by the equation:

$$\begin{aligned} e^{2i\theta}t_0 z + \bar{z} &= e^{2i\theta}\, t_0 a_1^{(k-1)} + \bar{a}_1^{(k-1)} \\ &= e^{2i\theta} t_0 a_1^{(k-1)} + (-1)^{k-2}\, s_{k-1}^{(k-1)}\, a_{k-1}^{(k-1)}, \end{aligned} \tag{5}$$

since we have the relation:

$$\bar{a}_\alpha^{(k)} = (-1)^{k-1}\, s_k^{(k)}\, a_{k+1-\alpha}^{(k)}, \tag{6}$$

where

$$s_k^{(k)} = t_1 t_2 \cdots t_k. \tag{7}$$

Since it is also known that

$$a_\alpha^{(k-1)} = a_\alpha^{(k)} - a_{\alpha+1}^{(k)}\, t_k \tag{8}$$

the equation (5) can also be written in the following form:

$$\begin{aligned} &e^{2i\theta}\, t_0 z + \bar{z} \\ &\qquad = e^{2i\theta}\, t_0[a_1^{(k)} - a_2^{(k)} t_k] + (-1)^{k-2}\, s_{k-1}^{(k-1)}[a_{k-1}^{(k)} - a_k^{(k)} t_k]. \end{aligned}$$

Thus the points of intersection of this line and the circle (3) are given by the values of t satisfying the equation:

$$\begin{aligned} &e^{2i\theta}\, t_0[a_1^{(k)} - a_2^{(k)}\, t] + \left[\bar{a}_1^{(k)} - \bar{a}_2^{(k)}\, \frac{1}{t}\right] \\ &\qquad = e^{2i\theta}\, t_0[a_1^{(k)} - a_2^{(k)}\, t_k] + (-1)^{k-2} s_{k-1}^{(k-1)}[a_{k-1}^{(k)} - a_k^{(k)}\, t_k], \end{aligned}$$

that is,

$$\begin{aligned} -\, e^{2i\theta}\, t_0 a_2^{(k)}\, t^2 + \left[e^{2i\theta}\, t_0\, a_2^{(k)}\, t_k + (-1)^{k-1}\, s_k^{(k)}\, a_{-1}^{(k)} \frac{1}{t_k}\right] t& \\ + (-1)^k\, s_k^{(k)}\, a_{k-1}^{(k)} = 0&. \end{aligned} \tag{9}$$

The two roots of this equation are $t = t_k$ and

$$t = (-1)^{k-1} \frac{s_k^{(k)}\, a_{k-1}^{(k)}}{e^{2i\theta}\, a_2^{(k)}\, t_0\, t_k}. \tag{10}$$

Thus the second point of intersection of the line (5) and the circle (3) is given by

$$z = a_1^{(k)} + (-1)^k \frac{s_k^{(k)}\, a_{-1}^{(k)}}{e^{2i\theta}\, t_0\, t_k}. \tag{11}$$

This is the coordinate of the point $(12\cdots(k-1))_{t_0\theta}$.

The line passing through this point and making the angle ϕ with the line k:

$$z t_k + \bar{z} = z_k t_k$$

is given by the equation:

$$\begin{aligned} &e^{2i\phi} t_k z + \bar{z} \\ &\quad = e^{2i\phi} t_k \left[a_1^{(k)} + (-1)^k \frac{s_k^{(k)} a_{k-1}^{(k)}}{e^{2i\theta} t_0 t_k} \right] \\ &\qquad + \bar{a}_1^{(k)} + (-1)^k \frac{e^{2i\theta} t_0 t_k}{s_k^{(k)}} \bar{a}_{k-1}^{(k)}, \end{aligned}$$

that is

(12)
$$\begin{aligned} &e^{2i\phi} t_k z + \bar{z} \\ &\quad = e^{2i\phi} a_1^{(k)} t_k + (-1)^k e^{2i(\phi-\theta)} \frac{s_k^{(k)} a_{k-1}^{(k)}}{t_0} \\ &\qquad + (-1)^{k-1} s_k^{(k)} a_k^{(k)} - e^{2i\theta} a_2^{(k)} t_0 t_k. \end{aligned}$$

Similarly, the line passing through the point $(23\cdots k)_{l_0\theta}$ and making the angle ϕ with the line 1 has the equation:

$$\begin{aligned} &e^{2i\phi} t_1 z + \bar{z} \\ &\quad = e^{2i\phi} a_1^{(k)} t_1 + (-1)^k e^{2i(\phi-\theta)} \frac{s_k^{(k)} a_{k-1}^{(k)}}{t_0} \\ &\qquad + (-1)^{k-1} s_k^{(k)} a_k^{(k)} - e^{2i\theta} a_2^{(k)} t_0 t_1. \end{aligned}$$

Subtracting these two equations side by side, we have

$$e^{2i\phi} z(t_k - t_1) = e^{2i\phi} a_1^{(k)}(t_k - t_1) - e^{2i\theta} a_2^{(k)} t_0 (t_k - t_1).$$

Thus the point of intersection of these two lines is given by

(13)
$$z = a_1^{(k)} - a_2^{(k)} e^{2i(\theta-\phi)} t_0.$$

Since this expression is symmetric with respect to $t_1, \cdots, t_k$, it follows that the k lines stated above meet at this point. This is a point on the circle (3) corresponding to the parameter value $t = e^{2i(\theta-\phi)} t_0$. This point is denoted as $R_{l_0\theta\phi}(12\cdots k)$.

For the $(k+1)$-line $\{12\cdots k(k+1)\}$ we have $k+1$ such points: $R_{l_0\theta\phi}(12\cdots k)$, $R_{l_0\theta\phi}(12\cdots(k-1)(k+1)), \cdots$, $R_{l_0\theta\phi}(23\cdots k(k+1))$. Since, by (8) the expression (13) can also be written in the form:

$$
(14)\qquad
\begin{aligned}
z &= [a_1^{(k+1)} - a_2^{(k+1)} t_{k+1}] \\
&\qquad - [a_2^{(k+1)} - a_3^{(k+1)} t_{k+1}]\, e^{2i(\theta-\phi)}\, t_0, \\
&= [a_1^{(k+1)} - a_2^{(k+1)} e^{2i(\theta-\phi)}\, t_0] \\
&\qquad - [a_2^{(k+1)} - a_3^{(k+1)} e^{2i(\theta-\phi)}\, t_0]\, t_{k+1},
\end{aligned}
$$

it follows that the point $R_{l_0\theta\phi}(12\cdots k)$ is on the following circle:

$$
(15)\qquad z = [a_1^{(k+1)} - a_2^{(k+1)} e^{2i(\theta-\phi)}\, t_0] - [a_2^{(k+1)} - a_3^{(k+1)} e^{2i(\theta-\phi)}\, t_0]\, t.
$$

Since this expression is symmetric with respect to $t_1, \cdots, t_{k+1}$, we can conclude that the $(k+1)$ points: $R_{l_0\theta\phi}(12\cdots k)$, $R_{l_0\theta\phi}(12\cdots(k-1)(k+1)), \cdots$, $R_{l_0\theta\phi}(23\cdots k(k+1))$ all lie on this circle which is denoted as $C_{l_0\theta\phi}(12\cdots k(k+1))$.

By the above discussion, we know that the circle $C_{l_0\theta\phi}(12\cdots k)$ corresponding to the k-line $\{12\cdots k\}$ has the equation:

$$
z = [a_1^{(k)} - a_2^{(k)} e^{2i(\theta-\phi)}\, t_0] - [a_2^{(k)} - a_3^{(k)} e^{2i(\theta-\phi)}\, t_0]\, t\,.
$$

This equation and (13) show that the point $R_{l_0\theta\phi}(12\cdots k)$ is the center of the circle $C_{l_0\theta\phi}(12\cdots k)$. Thus the centers of circles $C_{l_0\theta\phi}(12\cdots k)$, $C_{l_0\theta\phi}(12\cdots(k-1)(k+1)), \cdots$, $C_{l_0\theta\phi}(2\cdots k(k+1))$ all lie on the circle $C_{l_0\theta\phi}(12\cdots k(k+1))$, and the Proposition 1 is shown.

Now, let θ', ϕ' be another pair of fixed angles, and let $C_{l_0\theta'\phi'}(1234)$, $C_{l_0\theta'\phi'}(12345), \cdots$, $C_{l_0\theta'\phi'}(12\cdots n)$ be the system of center circles with reference to the line l_0 corresponding to this pair of angles. Then, from the expression of (15), it follows that the two systems of center circles: $C_{l_0\theta\phi}(1234), \cdots$, $C_{l_0\theta\phi}(12\cdots n)$ and $C_{l_0\theta'\phi'}(1234), \cdots$, $C_{l_0\theta'\phi'}(12\cdots n)$ coincide if and only if $e^{2i(\theta-\phi)} = e^{2i(\theta'-\phi')}$, that is, $e^{2i[(\theta-\phi)-(\theta'-\phi')]} = 1$ holds. Thus, we have the following:

COROLLARY 1. *The two systems of center circles* $C_{l_0\theta\phi}(1234), \cdots$, $C_{l_0\theta\phi}(12\cdots n)$ *and* $C_{l_0\theta'\phi'}(1234), \cdots$, $C_{l_0\theta'\phi'}(12\cdots n)$ *coincide if and only if* $(\theta-\phi) \equiv (\theta'-\phi') \bmod \pi$ *holds.*

From the equation (15), we also have the following:

COROLLARY 2. *The system of center circles* $C_{l_0\theta\phi}(1234), \cdots$, $C_{l_0\theta\phi}(12\cdots n)$ *coincides with the system of center circles* $C_{l_0'}(1234), \cdots$,

$C_{l_0'}(12\cdots n)$ *if the line* l_0' *has the turn* $t_0' = -e^{2i(\theta-\phi)}t_0$, *where* t_0 *is the turn of the line* l_0.

3. Constructions of Hirano's another system of circles. Hirano's another system of circles: $B(1234),\cdots, B(12\cdots n)$ is constructed as follows: For the 4-line $\{1234\}$, let $R_4(123)$ be the point $R_{l_0}(123)$ defined above taking the line 4 as the line l_0. Then, we have four such points: $R_4(123)$, $R_3(124)$, $R_2(134)$, and $R_1(234)$. These points are on a circle denoted as $B(1234)$. For the 5-line $\{12345\}$, we can define similary a circle $B(12345)$ as follows: Let $R_5(1234)$ be the point $R_{l_0}(1234)$ taking the line 5 as the line l_0. Then, the five such points: $R_5(1234)$, $R_4(1235)$, $R_3(1245)$, $R_2(1345)$, and $R_1(2345)$ lie on one circle denoted as $B(12345)$. And so on. Thus for the n-line $\{12\cdots n\}$, we have a system of circles: $B(1234)$, $B(12345),\cdots,B(12\cdots n)$.

Corresponding to Hirano's this system of circles, we can show the following:

PROPOSITION 2. *Suppose that the* n*-line* $\{12\cdots n\}$ *is given. Let* θ, ϕ *be two fixed angles such that* $\theta-\phi\equiv\pi/2 \bmod \pi$. *Let* $R_{4\theta\phi}(123)$ *be the point* $R_{l_0\theta\phi}(123)$ *defined above taking the line* 4 *as the line* l_0. *Then the four points similarly defined:* $R_{4\theta\phi}(123)$, $R_{3\theta\phi}(124)$, $R_{2\theta\phi}(134)$, $R_{1\theta\phi}(234)$, *all lie on a circle denoted as* $B_{\theta\phi}(1234)$. *Next, let* $R_{5\theta\phi}(1234)$ *be the point* $R_{l_0\theta\phi}(1234)$ *defined above taking the line* 5 *as the line* l_0. *Then the five points similarly defined:* $R_{5\theta\phi}(1234)$, $R_{4\theta\phi}(1235)$, $R_{3\theta\phi}(1245)$, $R_{2\theta\phi}(1345)$, *and* $R_{1\theta\phi}(2345)$, *all lie on one circle denoted as* $B_{\theta\phi}(12345)$. *And so on. In general, for the* n*-line* $\{12\cdots n\}$, *let* $R_{n\theta\phi}(12\cdots(n-1))$ *be the point* $R_{l_0\theta\phi}(12\cdots(n-1))$ *taking the line* n *as the line* l_0. *Then the* n *points similarly defined:* $R_{n\theta\phi}(12\cdots(n-1))$, $R_{(n-1)\theta\phi}(12\cdots(n-2)n),\cdots$, $R_{1\theta\phi}(23\cdots n)$, *all lie on one circle denoted as* $B_{\theta\phi}(12\cdots n)$. *Thus, for the* n*-line* $\{12\cdots n\}$ *we have a system of circles* $B_{\theta\phi}(1234)$, $B_{\theta\phi}(12345),\cdots$, $B_{\theta\phi}(12\cdots n)$.

Proof: We give the proof for the general situation here. As proved in Proposition 1, the point $R_{l_0\theta\phi}(12\cdots(n-1))$ has the coordinate:

$$z = a_1^{(n-1)} - e^{2i(\theta-\phi)} a_2^{(n-1)} t_0. \tag{16}$$

Thus the point $R_{n\theta\phi}(12\cdots(n-1))$ has the coordinate:

$$z = a_1^{(n-1)} - e^{2i(\theta-\phi)} a_2^{(n-1)} t_n. \tag{17}$$

By (8) this expression can also be written as follows:

$$\begin{aligned} z &= [a_1^{(n)} - a_2^{(n)} t_n] - e^{2i(\theta-\phi)} t_n [a_2^{(n)} - a_3^{(n)} t_n] \\ &= a_1^{(n)} - a_2^{(n)} t_n [1 + e^{2i(\theta-\phi)}] + a_3^{(n)} e^{2i(\theta-\phi)} t_n^2. \end{aligned} \tag{18}$$

If $\theta - \phi \equiv \pi/2 \mod \pi$, then $1 + e^{2i(\theta-\phi)} = 0$, so this point $B_{n\theta\phi}(12\cdots(n-1))$ has the coordinate:

$$z = a_1^{(n)} - a_3^{(n)} t_n^2. \tag{19}$$

This shows that the point $B_{n\theta\phi}(12\cdots(n-1))$ is on the following circle:

$$z = a_1^{(n)} - a_3^{(n)} t\,. \tag{20}$$

This is the equation of the circle $B_{\theta\phi}(12\cdots n)$, with $\theta - \phi \equiv \pi/2 \mod \pi$. It is obvious that the other points: $B_{(n-1)\theta\phi}(12\cdots(n-2)n), \cdots, B_{1\theta\phi}(23\cdots n)$ are all on this circle.

Since the circle $B(12\cdots n)$ has the equation $z = a_1^{(n)} - a_3^{(n)} t$, from (20) we can conclude the following:

COROLLARY 3. *Let the two angles θ, ϕ be such that $\theta - \phi \equiv \pi/2 \mod \pi$. Then the system of circles $B_{\theta\phi}(1234), \cdots, B_{\theta\phi}(12\cdots n)$ coincides with the system of circles $B(1234), \cdots, B(12\cdots n)$.*

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