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TEST OF INDEPENDENCE IN INCOMPLETE SAMPLES

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Abstract. Suppose that we have a sample from a p-dimensional normal population $N(\mu, \Sigma)$. Let S denote the sample covariance matrix. We partition μ, Σ and S as

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \begin{matrix} p_1\times 1 \\ p_2\times 1 \end{matrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

Suppose that we have an extra sample from $N(\mu_2, \Sigma_{22})$ and denote the sample covariance matrix by V. For testing the hypothesis $H_0 : \Sigma_{12} = 0$ versus $H_1 : \Sigma_{12} \neq 0$, Eaton and Kariya (1975, 1983) shows that, among the class of tests invariant under the group of affine transformations, the one based on

$$\frac{n}{p_1} \operatorname{tr} S_{11}^{-1} S_{12}(S_{22} + V)^{-1} S_{21} - \operatorname{tr} S_{22}(S_{22} + V)^{-1}$$

where n is the degree of freedom associated with S, is uniquely locally best. In this paper, we show that it is also locally minimax within the class of tests of the same level.

1. **Introduction.** In multivariate analysis, we often come across samples which provide, in addition to n observations on all the characters, m independent observations on some of the characters. This type of incomplete sample usually arises due to accidents or limited sampling budgets. For making inference about the distribution of these characters, one way is to ignore those extra observations. But a more reasonable procedure would be to find a method which exploits the extra observations.

Suppose that we have a sample of size n from a p-dimensional normal distribution $N(\mu, \Sigma)$ and partition μ and Σ as

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$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \begin{matrix} p_1 \times 1 \\ p_2 \times 1 \end{matrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

with $p = p_1 + p_2$. Let the sample covariance matrix be analogously partitioned as

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

Suppose that we also have a sample of size m from $N(\mu_2, \Sigma_{22})$ and denote the sample covariance matrix of this extra sample by V. For testing the hypothesis H_0: $\Sigma_{12}=0$ versus the alternatives H_1: $\Sigma_{12} \neq 0$, Lee and Geisser (1972) shows that the likelihood ratio statistic is $\prod_{i=1}^{p}(1-r_i^2)$, where the r's are the sample canonical correlation coefficients. This is identical to the likelihood ratio statistic when there is no extra observations. Eaton and Kariya (1975, 1983) argue that this problem is invariant under the group of affine transformations and there does not exist a uniformly most powerful invariant test. They thus obtain a unique locally best invariant test which is based on

$$(1.1) \qquad U = \frac{n'}{p_1} \operatorname{tr} S_{11}^{-1} S_{12}(S_{22}+V)^{-1} S_{21} - \operatorname{tr} S_{22}(S_{22}+V)^{-1}.$$

where n' is the degree of freedom associated with S. It is then natural to ask whether this test is locally minimax. However, except for the trivial case when $p_1 = p_2 = 1$, they were unable to find one with this property.

In this paper, we follow the method used by Schwartz (1967) to show that the test U is locally minimax. In Section 2, some definitions and notations are introduced. In Section 3, the important theorem by Hunt and Stein is stated. In Section 4, the local minimaxity of U is established.

2. **Definition and notation.** Let $(\mathfrak{X}, \mathscr{A})$ be a measurable space. For each point (δ, η) in the parametric space Ω, where $\delta \geq 0$ and η is of orbitrary dimension and its range may depend on δ, suppose that $p(\cdot; \delta, \eta)$ is a probability density function on $(\mathfrak{X}, \mathscr{A})$ with respect to some σ-finite measure. We are interested in testing the hypothesis H_0: $\delta=0$ versus H_1: $\delta=\lambda$ where λ is

a positive specified constant. Denote by Q_α the class of all level α tests, $0<\alpha<1$. For fixed α, consider critical regions of the form

$$R=\{X:\ U(X)>C_\alpha\}$$

where U is bounded and positive and has a continuous distribution function for each (δ, η), equicontinuous in (δ, η) for $\delta\leq$ some δ_0 and where

$$P_{0,\eta}(R)=\alpha \quad\text{and}\quad P_{\lambda,\eta}(R)=\alpha+h(\lambda)+g(\lambda, \eta)$$

where $g(\lambda, \eta)=o(h(\lambda))$ uniformly in η with $h(\lambda)>0$ for $\lambda>0$ and $h(\lambda)=o(1)$.

DEFINITION. ψ^* is locally minimax of level α for testing the hypothesis H_0: $\delta=0$ versus H_1: $\delta=\lambda$ as $\lambda\rightarrow 0$ if

$$\lim_{\lambda\rightarrow 0}\frac{\inf_\eta P_{\lambda,\eta}(R)-\alpha}{\sup_{\psi_\lambda\in Q_\alpha}\inf_\eta P_{\lambda,\eta}(\psi_\lambda \text{ reject } H_0)-\alpha}=1.$$

Since a minimax procedure is usually a Bayes procedure with constant risk with respect to some prior, Giri and Kiefer (1964) establishes the following lemma.

LEMMA 1. *If* (1) U *satisfies*

$$P_{0,\eta}(U>C_\alpha)=\alpha \quad \textit{and} \quad P_{\lambda,\eta}(U>C_\alpha)=\alpha+h(\lambda)+g(\lambda, \eta)$$

where $g(\lambda, \eta)=o(h(\lambda))$ *uniformly in* η, $h(\lambda)>0$ *for* $\lambda>0$ *and* $h(\lambda)=o(1)$, *and*

(2) *there exist probability density functions* $\xi_{0,\lambda}$ *and* $\xi_{1,\lambda}$ *on sets* $\delta=0$ *and* $\delta=\lambda$ *respectively for which*

$$\frac{\int p(x;\ \lambda, \eta)\ \xi_{1,\lambda}(d\eta)}{\int p(x;\ 0, \eta)\ \xi_{0,\lambda}(d\eta)}=1+h(\lambda)[q(\lambda)+r(\lambda)\,U(x)]+B(x, \lambda) \tag{2.1}$$

where $0<K_1<r(\lambda)<K_2<\infty$ *for* λ *sufficiently small and* $q(\lambda)=o(1)$ *and* $B(x, \lambda)=o(h(\lambda))$ *uniformly in* x, *then* U *is locally minimax of level* α *for testing the hypothesis* H_0: $\delta = 0$ *versus the alternatives* H_1: $\delta = \lambda$ *as* $\lambda \to 0$.

NOTATION. Throughout this paper, we denote

$GL(j)$: the group of all $j \times j$ nonsinglar matrices,

$G_T(j)$: the subgroup of nonsingular lower triangular matrices, and

$G_T^+(j)$: the subgroup of $G_T(j)$ consisting of matrices with all diagonal elements being positive.

3. **Hunt-Stein Theorem.** In statistical analysis, one way to simplify the inference problems is through the invariance principle. It allows us to consider a subclass of statistics which are invariance under some group of transformations. A natural question arises as under what conditions an optimum invariant statistic is also optimum among the class of all statistics. Hunt and Stein showed that under some conditions on the transformation group G, there is an invariant test of level α which minimizes the maximum error of second kind among all test of level α.

THEOREM (Hunt-Stein)

Let $\mathscr{B}$ *be a* σ*-field of subsets of* G *such that for any* $A \in \mathscr{A}$, *the set of pairs* (x, g) *with* $gx \in A$ *is in* $\mathscr{A} \times \mathscr{B}$ *and for any* $B \in \mathscr{B}$, $g \in G$, *the set* $Bg \in \mathscr{B}$. *Suppose that there exists a sequence of distribution functions* ν_n *on* $(G, \mathscr{B})$ *which is asymptotically right invariant is the sense that for any* $g \in G$, $B \in \mathscr{B}$

$$\lim_{n\to\infty} |\nu_n(Bg) - \nu_n(B)| = 0 .$$

Then, for testing H_0: $\theta \in \Omega_0$ *versus* H_1: $\theta \in \Omega_1$, *with* $\Omega_0 \cap \Omega_1 = \phi$, *if there is a test* ϕ_0 *maximizing* $\inf_{\Omega_1} E(\phi(x)|\theta)$ *over all level* α *tests, there is an invariant test with this property.*

Hence we need only to look for a transformation group satisfying Hunt-Stein condition and then a minimax test invariant under the group.

It has been shown (Lehman (1959)) that the groups $G_T(p)$ and $G_T^+(p)$ satisfy the condition whereas $GL(p)$ does not.

4. A locally mimimax test. Let $X_1, \cdots, X_n$ be a random sample from the p-dimensional $N(\mu, \Sigma)$ distribution where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \begin{matrix} p_1 \times 1 \\ p_2 \times 1 \end{matrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

with $p = p_1 + p_2$. Let $Y_1, \cdots, Y_m$ be a random sample from $N(\mu_2, \Sigma_{22})$, independent of the X's. In this section, we are to find a locally minimax test for the hypothesis H_0: $\Sigma_{12} = 0$ versus the alternatives H_1: $\Sigma_{12} \neq 0$, that is, a test that satisfies the two conditions of Lemma 1. We consider the group

$$G = \left\{ g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} : g_1 \in G_T(p_1),\ g_2 \in G_T(p_2) \right\}$$

of transformations where

$$g \circ X = \begin{pmatrix} g_1 X_{(1)} \\ g_2 X_{(2)} \end{pmatrix} \quad \text{with} \quad X = \begin{pmatrix} X_{(1)} \\ X_{(2)} \end{pmatrix} \begin{matrix} p_1 \times 1 \\ p_2 \times 1 \end{matrix}$$

The testing problem is invariant under G. The induced group on the parameter space acts as

$$\bar{g} \circ \mu = \begin{pmatrix} g_1 \mu_1 \\ g_2 \mu_2 \end{pmatrix}, \qquad \bar{g} \circ \Sigma = g \Sigma g'$$

and

$$\bar{g} \circ \mu_2 = g_2 \mu_2, \qquad \bar{g} \circ \Sigma_{22} = g_2 \Sigma_{22} g_2'.$$

This group G satisfies the conditions of the Hunt-Stein Theorem.

However the maximal invariant under G is a very complicated set of random variables, one can hardly write down its joint probability density function and then find a locally minimax test. So we use the Stein's representation of the probability ratio of the maximal invariant:

$$\frac{\int_G p(gx;\ \lambda,\ \eta)\ d\nu(g)}{\int_G p(gx;\ 0,\ \eta)\ d\nu(g)}$$

where ν denotes the left Haar measure on G.

Let $\Gamma = \Sigma^{-1}$ and (Γ_{ij}) $i, j = 1, 2$ be the partition of Γ corresponding to the partition (Σ_{ij}) of Σ. Then there exists $\bar{g} \in G$, where

$$\bar{g} = \begin{pmatrix} \bar{g}_1 & 0 \\ 0 & \bar{g}_2 \end{pmatrix} \quad \text{with} \quad \bar{g}_1 \in G_T^+(p_1) \quad \text{and} \quad \bar{g}_2 \in G_T^+(p_2)$$

such that $\bar{g}_i \Gamma_{ii} \bar{g}_i' = I_{p_i}$, $i = 1, 2$. Let $\Sigma^* = (\bar{g}')^{-1} \circ \Sigma$ and $\Gamma^* = (\Sigma^*)^{-1}$. The distribution of the maximal invariant under Σ is the same as under Σ^* for which $\Gamma_{ii}^* = I_{p_i}$. Let $\delta = \operatorname{tr} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1} = \operatorname{tr} \Gamma_{12}^* \Gamma_{21}^*$, then our problem is equivalent to testing

$$H_0:\ \delta = 0, \quad \text{i.e.,} \quad \Omega_0 = \{\theta \mid \theta = (0, \eta)\}$$

versus

$$H_1:\ \delta \neq 0, \quad \text{i.e.,} \quad \Omega_\lambda = \{\theta \mid \theta = (\lambda, \eta),\ \lambda > 0\}\,.$$

We also note that G acts transitively on the sample space, so that the maximal invariant has a single distribution under H_0.

Now we are to use Lemma 1 to find a locally minimax invariant test. Since under H_0, $\delta = \operatorname{tr} \Gamma_{12}^* \Gamma_{21}^* = 0$, Ω_0 contains only one point, that is $\theta_0 = (0, 0)$, hence the null prior distribution $\xi_{0,\lambda}$ can only be taken as having mass one at the point θ_0. The left hand side of (2.1) is thus

$$\frac{\int p(x;\ \lambda, \eta)\, \xi_{1,\lambda}(d\eta)}{p(x;\ 0, 0)} = \int \frac{p(x;\ \lambda, \eta)}{p(x;\ 0, 0)}\, \xi_{1,\lambda}(d\eta)\,,$$

where the integrand is the probability ratio of the maximal invariant.

Since we may without loss of generality assume that $\mu = 0$, the joint density, with respect to Lebesgue measure Q, of X and Y under Σ^* is

$$(2\pi)^{-pn/2}\, |\Sigma^*|^{-n/2} \operatorname{etr}\Big[-\frac{1}{2}(\Sigma^*)^{-1} XX'\Big]$$
$$\cdot\, (2\pi)^{-p_2 m/2}\, |\Sigma_{22}^*|^{-m/2} \operatorname{etr}\Big[-\frac{1}{2}(\Sigma_{22}^*)^{-1} YY'\Big].$$

The measure with volumn element $|XX'|^{-n/2}|YY'|^{-m/2}dQ$ is invariant under all linear transformations. Hence with respect to this measure, the joint density is

$$f_{\Sigma^*}(X, Y) = (2\pi)^{-pn/2-p_2m/2}|\Sigma^*|^{-n/2} \cdot |\Sigma^*_{22}|^{-m/2}|XX'|^{n/2}|YY'|^{m/2} \cdot \mathrm{etr}\Big[-\frac{1}{2}(\Sigma^*)^{-1}XX' - \frac{1}{2}(\Sigma^*_{22})^{-1}YY'\Big],$$

(4.1) or

$$f_{(\delta,\Gamma^*_{12})}(X, Y) = (2\pi)^{-(pn+p_2m)/2}|\Gamma^*|^{n/2} \cdot |I_{p_2} - \Gamma^*_{21}\Gamma^*_{12}|^{m/2}|S|^{n/2}|V|^{m/2} \cdot \mathrm{etr}\Big\{-\frac{1}{2}[S_{11} + (S_{22} + V) + 2\Gamma^*_{12}S_{21} - \Gamma^*_{21}\Gamma^*_{12}V]\Big\}$$

where $S = XX'$, $V = YY'$.

Since each invariant statistic can be expressed as a function of the maximal invariant, we consider only the probability ratio of the maximal invariant, which is, by Stein's representation

$$\frac{\int_G f_{(\delta,\Gamma^*_{12})}(gX, g_2Y)\,d\nu(g)}{\int_G f_{(0,0)}(gX, g_2Y)\,d\nu(g)}. \tag{4.2}$$

Since there exists a $\hat{g} \in G$,

$$\hat{g} = \begin{pmatrix} \hat{g}_1 & 0 \\ 0 & \hat{g}_2 \end{pmatrix}$$

such that $\hat{g}_1 S_{11}\hat{g}_1' = I_{p_1}$ and $\hat{g}_2(S_{22} + V)\hat{g}_2' = I_{p_2}$, multiplying g on the right by $\hat{g}$ in the Stein's formula leaves (4.2) unchanged, so that (4.2) can be rewritten as

$$C^{-1}|\Gamma^*|^{n/2}|I_{p_2} - \Gamma^*_{21}\Gamma^*_{12}|^{m/2} \int_G |gg'|^{n/2}|g_2g_2'|^{m/2}\,\mathrm{etr}\Big\{-\frac{1}{2}[g_1g_1' + g_2g_2' + 2\Gamma^*_{12}g_2\hat{g}_2S_{21}\hat{g}_1'g_1' - \Gamma^*_{21}\Gamma^*_{12}g_2\hat{g}_2V\hat{g}_2'g_2']\Big\}\,d\nu(g) \tag{4.3}$$

where

$$C=\int_G |gg'|^{n/2}\,|g_2 g_2'|^{m/2}\,\mathrm{etr}\left\{-\frac{1}{2}\,[g_1 g_1'+g_2 g_2']\right\}d\nu(g)$$

and

$$d\nu(g)=\prod_{i=1}^{2}\prod_{j\ge l} dg_{i(jl)}\cdot\prod_{i=1}^{2}\prod_{j=1}^{p_i}|g_{i(jj)}|^{-j}.$$

hence

$$C=(2\pi)^{[p_1(p_1+1)+p_2(p_2+1)]/4}\cdot\sum_{j=1}^{p_1}E|g_{1(jj)}|^{n-j}\cdot\sum_{j=1}^{p_2}E|g_{2(jj)}|^{m+n-j}$$

where E denotes the expectation taken with respect to the standard normal distribution.

To calculate the integration in (4.3), we expand the last two terms in the exponent into power series and delete all odd power terms in g_1 or g_2, which contribute nothing to the whole integration, we thus obtain

$$\begin{aligned}&|\Gamma^*|^{n/2}\,|I_{p_2}-\Gamma_{21}^*\Gamma_{12}^*|^{m/2}\\&\quad\cdot\int_G |gg'|^{n/2}\,|g_2 g_2'|^{m/2}\,\mathrm{etr}\left(-\frac{1}{2}\,g_1 g_1'-\frac{1}{2}\,g_2 g_2'\right)\\&\qquad\cdot\left\{1+\frac{1}{2}(\mathrm{tr}\,\Gamma_{21}^*\Gamma_{12}^* g_2\hat{g}_2 S_{21}\hat{g}_1' g_1')^2\right.\\&\qquad\qquad\left.+\sum_{u=2}^{\infty}(\mathrm{tr}\,\Gamma_{12}^* g_2\hat{g}_2 S_{21}\hat{g}_1' g_1')^{2u}/(2u)!\right\}\\&\qquad\cdot\left\{1+\frac{1}{2}\,\mathrm{tr}\,\Gamma_{21}^*\Gamma_{12}^* g_2\hat{g}_2 V\hat{g}_2' g_2'\right.\\&\qquad\qquad\left.+\sum_{v=2}^{\infty}\left(\frac{1}{2}\,\mathrm{tr}\,\Gamma_{21}^*\Gamma_{12}^* g_2\hat{g}_2 V\hat{g}_2' g_2'\right)^v\Big/v!\right\}d\nu(g).\end{aligned}$$

Using the inequality that $\mathrm{tr}\,AB\le(\mathrm{tr}\,AA')^{1/2}(\mathrm{tr}\,BB')^{1/2}$ we have

$$\sum_{v=2}^{\infty}\left(\frac{1}{2}\,\mathrm{tr}\,\Gamma_{21}^*\Gamma_{12}^* g_2\hat{g}_2 V\hat{g}_2' g_2'\right)^v\Big/v!=O(\delta^2).$$

and

$$\sum_{u=2}^{\infty} (\mathrm{tr}\, \Gamma_{12}^* g_2 \hat{g}_2 S_{21} \hat{g}_1' g_1')^{2u}/(2u)! = O(\delta^2)$$

Hence the integration becomes

$$\int_G |gg'|^{n/2} |g_2 g_2'|^{m/2} \,\mathrm{etr}\Big(-\frac{1}{2} g_1 g_1' - \frac{1}{2} g_2 g_2'\Big) \cdot \Big\{1 + \frac{1}{2} (\mathrm{tr}\, \Gamma_{12}^* g_2 \hat{g}_2 S_{21} \hat{g}_1' g_1')^2 + \frac{1}{2} \mathrm{tr}\, \Gamma_{21}^* \Gamma_{12}^* g_2 \hat{g}_2 V \hat{g}_2' g_2' + \text{rest terms}\Big\} d\nu(g)$$

$$= C\Big\{1 + \frac{1}{2} \sum_{r=1}^{p_1} \sum_{s=1}^{p_2} W_{sr}^2 \Big[\sum_{l>r} \sum_{k>s} \Gamma_{12(lk)}^{*2} + \sum_{k>s} (n-r+1) \Gamma_{12(rk)}^{*2} + \sum_{l>r} (n+m-s+1) \Gamma_{12(ls)}^{*2} + (n-r+1)(n+m-s+1) \Gamma_{12(rs)}^{*2}\Big] + \frac{1}{2} \sum_{s=1}^{p_2} \sum_{k=1}^{p_2} W_{2(sk)}^2 \Big[\sum_{l>s}^{p_2} \sum_{r=1}^{p_1} \Gamma_{12(rl)}^{*2} + (n+m-s+1) \sum_{r=1}^{p_1} \Gamma_{12(rs)}^{*2}\Big]\Big\}$$

where $W = \hat{g}_2 S_{21} \hat{g}_1' = (W_{sr})$ and $W_2 = \hat{g}_2 V^{1/2} = (W_{2(sk)})$.

Noting that $|\Gamma^*| = |I_{p_2} - \Gamma_{21}^* \Gamma_{12}^*| = 1 - \delta + o(\delta)$, we have the value of (4.3), the probability ratio of the maximal invariant, as

$$R = 1 - \frac{1}{2} (n+m) \delta + \frac{1}{2} \sum_{s=1}^{p_2} \sum_{k=1}^{p_2} W_{2(sk)}^2 \cdot \Big[\sum_{l>s}^{p_2} \sum_{r=1}^{p_1} \Gamma_{12(rl)}^{*2} + (m+n-s+1) \sum_{r=1}^{p_1} \Gamma_{12(rs)}^{*2}\Big] + \frac{1}{2} \sum_{r=1}^{p_1} \sum_{s=1}^{p_2} W_{sr}^2 \Big[\sum_{l>r} \sum_{k>s} \Gamma_{12(lk)}^{*} + (n-r+1) \cdot \sum_{k>s} \Gamma_{12(rk)}^{*2} + (n+m-s+1) \sum_{l>r} \Gamma_{12(ls)}^{*2} + (n-r+1)(n+m-s+1) \Gamma_{12(rs)}^{*2}\Big] + o(\delta).$$

Now we have to choose a suitable prior $\xi_{1,\lambda}$ on Ω_λ. Using the fact that

$$\sum_{r=1}^{p_1}\sum_{s=1}^{p_2} W_{sr}^2 = \operatorname{tr} S_{11}^{-1} S_{12}(S_{22}+V)^{-1} S_{21}$$

and

$$\sum_{s=1}^{p_2}\sum_{l=1}^{p_2} W_{2(sl)}^2 = \operatorname{tr} V(S_{22}+V)^{-1} = p_2 - \operatorname{tr} S_{22}(S_{22}+V)^{-1},$$

we can take $\xi_{1,\lambda}$ as giving mass

$$k^*\left(\frac{1}{n-r}\right)\left(\frac{1}{n-r+1}\right)$$

to the point

$$\begin{cases} \Gamma^*_{12(ls)} = 0 & l \neq r \\ \Gamma^*_{12(rk)} = \left[\dfrac{\lambda(n+m)(n+m-p_2)}{(n+m-k)(n+m-k+1)\,p_2}\right]^{1/2}, & k = 1, \cdots, p_2 \end{cases}$$

where k^* is the normalizing constant that makes the total mass equal to unity, that is

$$k^* = \left(\sum_{r=1}^{p_1}\left(\frac{1}{n-r}\left(\frac{1}{n-r+1}\right)\right)\right)^{-1} = \frac{n}{p_1}(n-p_1)\,.$$

The mass points may be easily checked to belong to the alternative space $\Omega_\lambda = \{(\delta, \Gamma^*_{12}) \mid \delta = \operatorname{tr} \Gamma^*_{12}\Gamma^*_{21} = \lambda\}$ and the matrices $\begin{pmatrix} I & \Gamma^*_{12} \\ \Gamma^*_{21} & I \end{pmatrix}$ are all positive definte for small λ. With this prior, the ratio in (2.1) becomes

$$\begin{aligned} \int R d\,\xi_{1,\lambda}(\Gamma^*_{12}) &= 1 - \frac{1}{2}(n+m)\lambda + \frac{1}{2}\frac{n}{p_1}\frac{\lambda}{p_2}(n+m)\sum_{r=1}^{p_1}\sum_{s=1}^{p_2} W_{sr}^2 \\ &\quad + \frac{\lambda}{p_2}(n+m)\sum_{s=1}^{p_2}\sum_{l=1}^{p_2} W_{2(sl)}^2 + o(\lambda) \\ &= 1 + \frac{1}{2}\frac{\lambda(n+m)}{p_2}\left(\frac{n}{p_1}\operatorname{tr} S_{11}^{-1} S_{12}(S_{22}+V)^{-1} S_{21} \right. \\ &\qquad \left. - \operatorname{tr} S_{22}(S_{22}+V)^{-1}\right) + o(\lambda)\,. \end{aligned}$$

With

$$U(X, Y) = \frac{n}{p_1}\operatorname{tr} S_{11}^{-1} S_{22}(S_{22}+V)^{-1} S_{21} - \operatorname{tr} S_{22}(S_{22}+V)^{-1},$$

the condition (2.1) in Lemma 1 is satisfied, and U is nothing but the unique locally best invariant test obtained by Eaton and Kariya (1975, 1983).

As to the power conditions in Lemma 1, the result in Eaton and Kariya (1975, 1983) is adopted as

$$P(U > C_\alpha \mid \lambda, \Gamma_{12}^*) = \alpha + \frac{\lambda}{2} E(UI_{(U>C_\alpha)} \mid H_0) + o(h(\lambda))$$

where C_α is the level α critical value of U. Hence, by Lemma 1, we have

THEOREM. The level α test ϕ^* defined by

$$\phi^*(X, Y) = \begin{cases} 1 & \text{if } \dfrac{n}{p_1} \operatorname{tr} S_{11}^{-1} S_{12}(S_{22}+V)^{-1} S_{21} \\ & \qquad - \operatorname{tr} S_{22}(S_{22}+V)^{-1} > C_\alpha \\ 0 & \text{otherwise} \end{cases}$$

is locally minimax, as $\delta \to 0$, for testing H_0: $\Sigma_{12} = 0$ versus H_1: $\Sigma_{12} \neq 0$.

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