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ON THE FIRST EIGENFUNCTION OF CONVEX SYMMETRIC DOMAINS

BY

CHAO-LIANG SHEN (沈昭亮)

Abstract. Let Ω be a $4n$ symmetric convex domain in $\boldsymbol{R}^2$. Let u be the first eigenfunction of the fixed membrane problem on Ω. In this article we study the behavior of u by observing the asymptotic behavior of the twisted derivative u_{xy} along the line $x = y$. We prove that if $n \geq 2$, or if $n = 1$ and

$$\frac{\max\{\operatorname{dist}((0, 0), B) : B \in \partial\Omega\}}{R} - 1 < \frac{\pi J_4(\pi/\sqrt{2})}{2j_0},$$

where R is the inradius of Ω, j_0 is the first positive zero of the Bessel function $J_0(x)$, then for $a \neq 0$, $|a|$ small, $u_{xy}(a, a) > 0$. As an application of this result we show that under previous assumptions the function $\kappa(y)$ of the curvature of the curves $x \mapsto u(x, y)$ at $x = 0$ attains a local maximum at $y = 0$. We also use the similar method to derive an isoperimetric estimate of the coefficients of the Fourier-Bessel series expansion of u.

1. **Introduction.** Recently the study of the first eigenfunctions of convex domains has attracted the attention of mathematicians, and many interesting qualitative and quantitative results have been obtained. Let us mention some which are related to this paper. In [PS], PAYNE and STAKGOLD proved some interesting a priori estimates of the first eigenfunction. It follows from an idea of SERRIN [S] and from the result of GIDAS-NI-NIRENBERG'S paper [GNN] that sufficient symmetry of the convex domain implies that the center of symmetry in the maximum point of the first eigenfunction among several applications of their results. In [BL-1, 2] BRASCAMP and LIEB proved the logarithmic concavity of the GREEN function of the heat operator on a convex domain, and it follows from their results (also see [APP]) that we know that the first eigenfunction is logarithmically concave with convex level curves. The curvature of these level curves is estimated in

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the paper [APP] of ACKER, PAYNE and PHILIPPIN. Since the restriction of an eigenfunction to a *nodal* domain becomes a first eigenfunction, if we know more about the graph of the first eigenfunction of an arbitrary domain we might know more about the other eigenfunctions and their nodal sets.

In this paper we study the behavior of the graph of the first eigenfunction u of a convex $(4n)$-symmetric domain Ω (see §2 for definition and basic properties) near the center of symmetry. We derive an estimate of the second-order twisted derivative u_{xy} near the origin (see §3), and we find that under a suitable condition on the boundary of Ω, we can determine the sign of u_{xy} (which is not obvious even though we know the hessian of u is negative definite near the origin). Using this estimate we prove the curvature function $\kappa(y)$ of the sections $x \to u(x, y)$ at $x = 0$ attains a local maximum at $y = 0$ if Ω is $4(n+1)$-symmetric, $(n \geq 1)$, or Ω is 4-symmetric, and satisfies condition (3.5). We also obtain an isoperimetric estimate of the coefficients of the FOURIER-BESSEL series expansion of the first eigenfunction (see (3.10)). Since $y \to u(0, y)$ and $x \to u(x, 0)$ serve as the flame of the graph of u, if we know more about $\kappa(y)$ (i.e., not just in a neighborhood of $y = 0$) we will know more about the graph of u. We will study related problems in a subsequent paper.

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2. **Preliminaries.** Throughout this paper we shall only consider convex domains in $\boldsymbol{R}^2$. Thus when we say Ω *is a domain*, we always mean Ω *is a convex domain in* $\boldsymbol{R}^2$ *with smooth boundary.* Unless specified, m, n and k shall denote positive integers.

Let Ω be a bounded domain in $\boldsymbol{R}^2$, symmetric with respect to the x-axis. We say Ω is *m-symmetric* if it is invariant under rotation about the origin by angle $2\pi/m$. For a real-valued function $f(x, y)$ on an m-symmetric domain Ω we define the function $f_{(m)}(x, y)$ to be $f(x\cos(2\pi/m) - y\sin(2\pi/m), x\sin(2\pi/m) + y\cos(2\pi/m))$. Then f is *m-symmetric* if $f_{(m)}(x, y) = f(x, y)$ in Ω. Given an *m-symmetric* domain Ω there exists a natural and non-trivial *m-symmetric* function as shown in the following lemma.

LEMMA 2.1. *Let u be an eigenfunction corresponding to the first eigenvalue λ_1 of the DIRICHLET eigenvalue problem on Ω: $\Delta u + \lambda u = 0$ in Ω, $u = 0$ on $\partial\Omega$. Then u is m-symmetric if Ω is m-symmetric.*

Proof. Let $\nu(x, y) = u_{(m)}(x, y)$. Then it is easy to see that $\nu = 0$ on $\partial\Omega$, $\nu(0, 0) = u(0, 0)$, and $\Delta\nu = (\Delta u)_{(m)}$ in Ω. Since Ω is *m-symmetric* and u is an eigenfunction we must have $(\Delta u)_{(m)} + \lambda_1 u_{(m)} = 0$ in Ω. By the fact that λ_1 is simple (see [CH]) and $u(0, 0) = \nu(0, 0)$, u and ν must agree in Ω. Thus u is *m-symmetric.*

Given an *m-symmetric* domain Ω, an eigenfunction u corresponding to the first eigenvalue λ_1 of the Dirichlet eigenvalue problem is said to be *normalized* if $u(0, 0) = 1$. We shall call the normalized eigenfunction of λ_1, *the first eigenfunction* of Ω, and the notation u is reserved for the first eigenfunction. Concerning the first eigenfunction of an *m-symmetric* domain we have the following lemma:

LEMMA 2.2. *If Ω is a $(2n)$-symmetric domain, then $u(x, y) = u(x, -y) = u(-x, y)$, $u_x(x, y) = u_x(x, -y) = -u_x(-x, y)$, $u_y(-x, y) = u_y(x, y)$, $u_y(x, -y) = -u_y(x, y)$, $u_y(x, 0) = 0$, $u_{xy}(x, 0) = 0$ for (x, y), $(x, 0)$ in Ω. If Ω is $(4n)$-symmetric, then $u_x(0, y) = 0$, $u_y(x, 0) = 0$, $u_{xy}(x, 0) = 0$, $u_{xy}(0, y) = 0$, $u_x(a, b) = u_y(b, a)$, $u_{xy}(a, b) = u_{xy}(b, a)$, $u_{xx}(a, b) = u_{yy}(b, a)$ for all $(x, 0)$, $(0, y)$, (a, b) in Ω.*

The properties listed in Lemma 2.2 are an immediate consequence of the symmetry of u, so we omit the proof.

In [Theorem 2.1, GNN], GIDAS, NI, and NIRENBERG studied the first order derivative with respect to x of the first eigenfunction of a domain Ω which is symmetric with respect to the y-axis. They proved that if (x, y) is in Ω which lies on the left-hand side of the y-axis, then $\partial u/\partial x(x, y) > 0$ and $\partial u/\partial x(x, y) = 0$ only when $x = 0$. Employing this result we see immediately that the center of symmetry of a $(4n)$-symmetric domain is the maximum point of the first eigenfunction. It also follows from their result that we can compare the two first-order derivatives of u at the same point.

LEMMA 2.3. *Suppose the domain Ω is $(4n)$-symmetric. Then $u_x(a, b) \lesseqgtr u_y(a, b)$ if and only if $a \gtreqless b$, where $(a, b) \in \Omega$.*

Proof. Since Ω is $(4n)$-symmetric, it is symmetric with respect to the line $x = y$ in $\boldsymbol{R}^2$. By [Theorem 2.1, GNN] we see that the directional derivative $(D_{-\pi/4})\, u(a, b) > 0$ if and only if $b - a > 0$. Since $(D_{-\pi/4})\, u = \sqrt{2}/2\, [u_x(a, b) - u_y(a, b)]$, we are done.

The following result is also a consequence of [Theorem 2.1, GNN].

LEMMA 2.4. *If Ω is $4n$-symmetric, then $u_{xx}(0, y) \leq 0$, $u_{yy}(x, 0) \leq 0$ for all $(0, y)$, $(x, 0)$ in Ω.*

Proof. Since $u_x(x, y) \gtreqless 0$ if $x \lesseqgtr 0$, $x = 0$ is the maximum point of the function $x \rightarrow u(x, y)$. Thus $u_{xx}(0, y) \leq 0$. $u_{yy}(x, 0) \leq 0$ is proved similarly.

Circular domains are $(4n)$-symmetric and are often used as a comparison object in isoperimetric problems as well as eigenvalue estimates. Suppose Ω is a $(4n)$-symmetric domain with *inradius* R (i.e., the radius of the maxiumum inscribed disk, which is tangent to $\partial\Omega$ at points with minimum curvature). Then it follows from the eigenvalue comparison theorem that we have,

LEMMA 2.5. *Let j_0 denote the first positive zero of the Bessel function $J_0(r)$. Let λ_1 be the first eigenvalue of the $4n$-symmetric domain Ω. Let R be the inradius of Ω. Then we have the following inequality*

(2.1) $$\frac{\pi}{\sqrt{2}} \leq \sqrt{\lambda_1}\, R \leq j_0 .$$

The equality on the left-hand side of (2.1) *holds if* Ω *is a square. The equality on the right-hand side of* (2.1) *holds if* Ω *is a disk.*

Proof. Since Ω is convex and $(4n)$-symmetric with inradius R, it is contained in a square with side length $2R$. By the First Monotonicity Principle ([§3.7, We], or [Ch. 6, CH]), we have

$$\frac{\pi^2}{2R^2} \leq \lambda_1 \leq \left(\frac{j_0}{R}\right)^2 .$$

(2.1) follows from the latter inequality.

Recall for nonnegative integer m, the BESSEL function $J_m(z)$ has the series expansion $\sum_{k=0}^{\infty}((-1)^k/k!(k+m)!)[z/2]^{2k+m}$. The positive zeros of the BESSEL functions and their derivatives have many interesting properties which can be found in [CHAPTER XV, W]. The following is one of them:

LEMMA 2.6. $m>0$. *Let* j'_m *denote the first positive zero of the function* $J'_m(z)$. *Then* $j'_m > m$.

We use the previous lemma to prove the following theorem which shall be used in the next section to obtain certain estimates for the coefficients of the FOURIER-BESSEL series of the first eigenfunction.

PROPOSITION 2.7. *Suppose* Ω *is a* $(4n)$*-symmetric domain with inradius* R *and first eigenvalue* λ_1. *Then, for* $k \geq 1$,

(2.2) $$J_{4k}(\sqrt{\lambda_1}\, R) \geq J_{4k}(\pi/\sqrt{2}\,) .$$

Proof. Since $k \geq 1$, $j_0 = 2.4048$, $4k > j_0$. It follows from Lemma 2.6 that $J_{4k}(z)$ is an increasing function in the interval $(0, j_0)$. The inequality (2.2) follows from the monotonicity of J_{4k} and inequality (2.1).

3. **The curvature function** $\kappa(y)$. Throughout this section we shall assume Ω to be a $(4n)$-symmetric convex domain with inradius R. λ_1 will denote the first eigenvalue of Ω, $\mu_1 = \sqrt{\lambda_1}$.

Let u be the first eigenfunction of the domain Ω. To study the graph of u, as $y \to u(0, y)$ and $x \to u(x, 0)$ serve as the flames, the curves C_y defined by $x \to u(x, y)$ (y is fixed) serve as the skeleton of the graph of u, it is important to know the change of the curvature $\kappa(y)$ of the curvature of C_y at $x=0$. Since $u_x(0, y)=0$ (Lemma 2.2), the curvature $\kappa(y)$ of C_y at $x=0$ is given by $\kappa(y)=|u_{xx}(0, y)|$. Because of $u_{xx}(0, y)<0$ (Lemma 2.4), to compare $\kappa(y)$ it suffices to compare $u_{xx}(0, y)$. Since, by Lemma 2.2, $u_{xxy}(0, 0)=0$, we see that $u_{xx}(0, y)$ is locally increasing or decreasing depending upon the sign of $u_{xxyy}(0, 0)$ if the latter is not zero. Thus the behavior of the function $\kappa(y)$ in a neighborhood of $y=0$ is determined by the sign of $u_{xxyy}(0, 0)$.

Let $g(a)=u_{xy}(a, a)$. Then $g'(a)=u_{xxy}(a, a)+u_{xyy}(a, a)$, $g''(a)=u_{xxxy}(a, a)+2u_{xxyy}(a, a)+u_{xyyy}(a, a)$. It follows from Lemma 2.2 that $g'(0)=0$ and $g''(0)=2u_{xxyy}(0, 0)$. Since $g(0)=0$, $g'(0)=0$, the following Proposition 3.1 will tell us that if we have certain information about the sign of $u_{xy}(a, a)$ when $|a|$ is small we may see something about the curvature information of the graph of u over the domain Ω. Usually the behavior, in particular the sign of u_{xy} near $(0, 0)$, is not easy to understand, even though we know that $(0, 0)$ is the maximum point of u and the hessian is negatively definite in a neighborhood of the origin. Fortunately we only need to know the behavior of u_{xy} along the line $x=y$, the following approach will be sufficient.

Introducing polar coordinate (r, θ) in $\boldsymbol{R}^2$ the LAPLACIAN Δ takes the form $\Delta=\partial^2/\partial r^2+(1/r)(\partial/\partial r)+(1/r^2)(\partial^2/\partial\theta^2)$. Since Ω is $(4n)$-symmetric, for fixed r, $0<r\le R$, The function $u(r, \theta)=:u(r\cos\theta, r\sin\theta)$, which is a periodic function of θ, admits the following FOURIER series expansion:

$$(3.1') \qquad u(r, \theta)=\sum_{m=0}^{\infty} A_{4m}(r)\cos 4m\theta\,.$$

Applying the polar LAPLACIAN to (3.1′) and using the fact that $\Delta u+\lambda_1 u=0$ in Ω, we see that $A_{4m}(r)=a_{4m}J_{4m}(\mu_1 r)$, i.e.,

$$(3.1) \qquad u(r, \theta)=\sum_{m=0}^{\infty} a_{4m}J_{4m}(\mu_1 r)\cos 4m\theta\,,$$

where a_{4m} is a constant, J_{4m} is the $(4m)^{th}$ BESSEL function of the first kind. Furthermore, by the regularity of u (recall that we assume the boundary of Ω to be smooth,) we can interchange the order of differentiation (resp., integration) and summation in the *FOURIER-BESSEL series* expansion (3.1) of u. The coefficient a_{4k} is determined by the following formula (see [T]) ($k \geq 1$):

$$a_{4k} J_{4k}(\mu_1 r) = \frac{1}{\pi} \int_0^{2\pi} u(r, \theta) \cos 4k\theta \, d\theta . \tag{3.2}$$

Note that $a_0 = 1$ by the normality of u and $J_0(0) = 1$, $J_k(0) = 0$ ($k \geq 1$). Using the FOURIER-BESSEL series of u we can prove the following asymptotic formula for $u_{xy}(a, a)$:

PROPOSITION 3.1. *Let u be the first eigenfunction of the $(4n)$-symmetric convex domain Ω. Then we have*

$$u_{xy}(a, a) = \left\{ \frac{1}{8} - \frac{a_4}{16} \right\} \lambda_1^2 a^2 + o(a^3) \quad as \quad a \to 0 . \tag{3.3}$$

Furthermore, $u_{xxyy}(0, 0) > 0$ if and only if $u_{xy}(a, a) > 0$ for $|a|$ near 0.

Proof. Expressing $u_{xy}(x, y)$ in polar coordinate we have

$$\begin{aligned} u_{xy}(x, y) &= [u_{rr} r_x + u_{r\theta} \theta_x] r_y \\ &\quad + [u_{r\theta} r_x + u_{\theta\theta} \theta_x] \theta_y + u_r r_{xy} + u_\theta \theta_{xy} \\ &= \left[u_{rr} \cos\theta - \frac{1}{r} u_{r\theta} \sin\theta \right] \sin\theta \\ &\quad + \frac{1}{r} \left[u_{r\theta} \cos\theta - \frac{1}{r} u_{\theta\theta} \sin\theta \right] \cos\theta \\ &\quad - \frac{1}{r} u_r \sin\theta \cos\theta - \frac{1}{r^2} u_\theta (\cos^2\theta - \sin^2\theta) . \end{aligned}$$

Thus

$$u_{xy}\left[\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right] = \frac{1}{2} \left\{ \left[u_{rr} - \frac{1}{r} u_r \right] \Big|_{\theta=\pi/4} - \frac{1}{r^2} u_{\theta\theta} \Big|_{\theta=\pi/4} \right\} .$$

Using the fact that $J_{4m}(z) \sim (z/2)^{4m}/(4m)!$ as $z \to 0$ ($m \geq 1$) (see [Ch. 5, L]) and the regularity of (3.1), we have

$$2u_{xy}\left[\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right] = \left[u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta}\right]\Big|_{\theta=\pi/4}$$
$$\sim \lambda_1[J_0''(\mu_1 r) - a_4 J_4''(\mu_1 r)]$$
$$- \frac{\mu_1}{r}[J_0'(\mu_1 r) - a_4 J_4'(\mu_1 r)]$$
$$- \frac{1}{r^2}[16 a_4 J_4(\mu_1 r)] \quad \text{as} \quad r \to 0.$$

Using the following formulas (notation "$TD \geq n$" means "terms with degree $\geq n$")

(3.4)
$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - (TD \geq 6),$$
$$J_4(x) = \frac{x^4}{384} - (TD \geq 6),$$
$$J_0'(x) = -\frac{x}{2} + \frac{x^3}{16} - (TD \geq 5),$$
$$J_0''(x) = -\frac{1}{2} + \frac{3}{16}x^2 - (TD \geq 4),$$
$$J_0^{(3)}(x) = \frac{3}{8}x - (TD \geq 3),$$
$$J_0^{(4)}(x) = \frac{3}{8} - (TD \geq 2),$$
$$J_4'(x) = \frac{x^3}{96} - (TD \geq 5),$$
$$J_4''(x) = \frac{x^2}{32} - (TD \geq 4),$$
$$J_4^{(3)}(x) = \frac{x}{16} - (TD \geq 3),$$
$$J_4^{(4)}(x) = \frac{1}{16} - (TD \geq 2),$$

we see that

$$u_{xy}\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right) = \left\{\frac{1}{8} - \frac{a_4}{16}\right\}\frac{\lambda_1^2 r^2}{2} + o(r^3), \quad r \to 0.$$

(3.3) follows immediately. Furthermore, as $u_{xxyy}(0, 0) = \{1/8 - a_4/16\}\lambda_1^2$, we see that $u_{xxyy}(0, 0) > 0$ if and only if $u_{xy}(a, a) > 0$ for $a \neq 0$, a near 0.

COROLLARY 3.2. *If the convex domain* Ω *is* $(4n)$*-symmetric with* $n>1$, *then* $\kappa(y)$ *attains local maximum at* $y=0$.

Proof. By the assumption on Ω, we have $a_4=0$. It follows from proposition 3.1 that $u_{xxyy}(0,0)>0$. Since $\kappa(y)=-u_{xx}(0,y)$, $\kappa'(0)=0$, we see that $y=0$ is a local maximum of $\kappa(y)$.

COROLLARY 3.3. *If* Ω *is* 8*-symmetric, then* $u_{xx}(x,0)>u_{yy}(x,0)$ *for* $|x|$ *small enough.*

Proof. If Ω is 8-symmetric, by calculation, we see that $u_{xy}(r,r)=$ (a positive constant) $[u_{xx}(\sqrt{2r},0)-u_{yy}(\sqrt{2r},0)]$. The assertion follows from $a_4=0$ and Proposition 3.1.

Now we use Proposition 3.1 to study $\kappa(y)$ when Ω is 4-symmetric.

THEOREM 3.4. *Let* Ω *be a* 4*-symmetric convex domain in* $\boldsymbol{R}^2$ *with inradius* R. *If*

$$\frac{\max\{\mathrm{dist}(0,0),B):B\in\partial\Omega\}}{R}-1<\frac{\pi J_4(\pi/\sqrt{2})}{2j_0}, \tag{3.5}$$

then $u_{xy}(a,a)>0$ *for* $|a|$ *near zero. Furthermore* $y=0$ *is a local maximum of* $\kappa(y)$.

Proof. By proposition 3.1 we see that $y=0$ is a local maximum of $\kappa(y)$ if we have $2>a_4$. Let us try to estimate a_4.

For the first eigenfunction u, using the beautiful *PAYNE-STAKGOLD inequality*: $u(P)\le\mu_1\|u\|_\infty\,\mathrm{dist}(P,\partial\Omega)$ *(see* [*PS*]), *we have*

$$u(R,\theta)\le\mu_1[\max\{\mathrm{dist}((0,0),B):B\in\partial\Omega\}-R] \tag{3.6}$$

for all θ *(remember we have normalized* u *so that* $\|u\|_\infty=1$). *It follows from* (3.6) *and the fact* $\int_0^{2\pi}|\cos 4\theta|\,d\theta=4$, *we have*

$$\begin{aligned}&\pi|a_4|J_4(\mu_1R)\\ &\qquad\le 4\mu_1[\max\{\mathrm{dist}((0,0),B):B\in\partial\Omega\}-R].\end{aligned} \tag{3.7}$$

Applying Proposition 2.7 *and* (2.1) *to* (3.7) *we have*

$$
\begin{aligned}
&\pi|a_4|J_4(\pi/\sqrt{2}) \\
(3.8)\qquad &\leq 4\mu_1 R\left[\frac{\max\{\text{dist}((0,\,0),\,B) : B \in \partial\Omega\}}{R} - 1\right] \\
&\leq 4j_0\left[\frac{\max\{\text{dist}((0,\,0),\,B) : B \in \partial\Omega\}}{R} - 1\right].
\end{aligned}
$$

Thus we obtain the following estimate for a_4:

$$
\begin{aligned}
(3.9)\qquad |a_4| \leq &\frac{4j_0}{\pi J_4(\pi\sqrt{2})} \\
&\cdot\left[\frac{\max\{\text{dist}((0,\,0),\,B) : B \in \partial\Omega\}}{R} - 1\right].
\end{aligned}
$$

It follows from (3.9) *that if* (3.5) *holds, then* $|a_4| < 2$. *Thus it follows from Proposition* 3.1 *and* (3.9) *that if* (3.5) *holds, then we have* $u_{xy}(a, a) > 0$ *if* $|a|$ *is small enough, and hence* $\kappa(y)$ *attains a local maximum at* $y = 0$.

REMARK. We have evidence that in many cases we probably will have $a_4 \leq 0$, but so far we still do not have a complete proof.

The technique we employed in the proof of Theorem 3.4 can also be used to estimate the coefficients a_{4n} of the FOURIER-BESSEL series (3.1) of the first eigenfunction. Although the estimate is not very strong, it is isoperimetric in the sense when Ω is a disk it reduces to $a_{4n} = 0$ for all $n \geq 1$. We state the estimate in the following proposition.

PROPOSITION 3.5. *Let* Ω *be a convex* 4-*symmetric domain. Let* a_{4n} *denote the* $4n$*th coefficient of the FOURIER-BESSEL series* (3.1). *Then*

$$
\begin{aligned}
(3.10)\qquad |a_{4n}| \leq &\frac{4j_0}{\pi J_{4n}(\pi/\sqrt{2})} \\
&\cdot\left[\frac{\max\{\text{dist}((0,\,0),\,B) : B \in \partial\Omega\}}{R} - 1\right].
\end{aligned}
$$

Proof. Applying (3.6) and $\int_0^{2\pi} |\cos 4n\theta|\, d\theta = 4$ we have

$$\begin{aligned}\pi|a_{4n}|\,J_{4n}(\mu_1 R) &\le \int_0^{2\pi} u(R,\,\theta)\,|\cos 4n\theta|\,d\theta \\ &\le 4\mu_1 R\left[\frac{\max\{\mathrm{dist}((0,\,0),\,B):B\in\partial\Omega\}}{R}-1\right] \\ &\le 4j_0\left[\frac{\max\{\mathrm{dist}((0,\,0),\,B):B\in\partial\Omega\}}{R}-1\right].\end{aligned}$$

(3.10) follows immediately from (2.2) and the last inequality.

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Institute of Mathematics
National Tsing Hua University
Hsinchu, Taiwan 300, R. O. C.