BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 14, Number 1, March 1986

AN INEQUALITY INVOLVING THE RECTANGULAR DISTRIBUTION

BY

CHIN-YUAN HU (胡金源)

Abstract. We show an inequality related to the rectangular distribution on (0, 1) and use this inequality to obtain an upper bound for variance of a continuous random variable. The upper bound is attainable only for a rectangular distribution.

1. Introduction and main result. Chernoff (1981) obtained the following inequality (1). Let X be a standard normal random variable. If $g(X)$ is absolutely continuous with finite variance then

(1) $$\operatorname{Var}[g(X)] \leq E[g'(X)]^2$$

with equality iff $g(X)$ is linear in X. The similar inequality for various distributions have been discussed by Cacoullos and Papathanasiou and other people. Here, we show an inequality related to the rectangular distribution on $(0, 1)$ and use this inequality to obtain the following main result.

THEOREM 1. *Let X have a positive and continuous density function f_X. Assume that the variance of X is positive and finite. Then*

(2) $$\operatorname{Var}[X] \leq E\{F_X(X)[1-F_X(X)]/[2f_X^2(X)]\}$$

where F_X is the distribution function of X, and equality holds iff X has a rectangular distribution.

Theorem 1 gives the upper bound for variance of a continuous random variable, and this bound is attainable only for a rectangular

Received by the editors January 15, 1985 and in revised form July 10, 1985.
Key words and phrases: Inequality, Rectangular distribution.

distribution. The lower bound is omitted here, because a direct application of the Cauchy-Schwarz inequality gives the desired result. Note that the upper bound (2) is infinite for most cases. For example, the normal and Cauchy distributions, etc.. It is effective for all positive and continuous density function f_X on a finite closed interval.

2. **Proof.** In the following, the rectangular distribution on $(0, 1)$ is denoted by $R(0, 1)$, and the derivative is denoted by prime. Lemma 1 is not difficult to derive (see, for example, Cacoullos and Papathanasiou).

LEMMA 1. *Let X be distributed as $R(0, 1)$ and $g(x)$ an absolutely continuous function defined on* $[0, 1]$. *Assume that $g(X)$ has a finite variance. Then*

$$\mathrm{Var}\,[g(X)] \leq E\{X(1-X)[g'(X)]^2/2\} \tag{3}$$

and equality holds iff $g(X)$ is linear in X.

Note that $E[g'(X)]^2 < +\infty$ is a sufficient condition for the existence of the upper bound (3).

Let X be a random variable with continuous distribution function F_X. For $0 < y < 1$, define $F_X^{-1}(y) = \mathrm{Inf}\,\{x : F_X(x) = y\}$. The following lemmas are well-known.

LEMMA 2. *Let X be a random variable with continuous distribution function F_X. Then, the random variable $Y = F_X(X)$ is distributed as $R(0, 1)$.*

LEMMA 3. *Let random variable Y be distributed as $R(0, 1)$, and let F_X be a continuous distribution function. Then, $F_X^{-1}(Y)$ has distribution function F_X.*

Let X have a positive and continuous density function f_X. Then, $F_X^{-1}(y)$ is absolutely continuous (Hájek and Sidak p. 34). For $0 < y < 1$, define $G(y) = y(1-y)[(F_X^{-1}(y))']^2/2$.

Proof of Theorem 1. Let random variable Y be distributed as $R(0, 1)$. Then, X and $F_X^{-1}(Y)$ have the same distribution, so do Y and $F_X(X)$ by Lemmas 2 and 3. Furthermore, $F_X(X)[1-F_X(X)]/[2f_X^2(X)]$ and $G(Y)$ have the same distribution since the density function f_X is positive and continuous. Now, using Lemma 1, we get

$$\begin{aligned} \operatorname{Var}[X] &= \operatorname{Var}[F_X^{-1}(Y)] \\ &\leq E[G(Y)] \\ &= E\{F_X(X)[1-F_X(X)]/[2f_X^2(X)]\} \end{aligned}$$

and equality holds iff $F_X^{-1}(Y)$ is linear in Y. X has a positive variance, and therefore equality holds iff X has a rectangular distribution. The proof is complete.

Acknowledgements. The author would like to thank the referee for a very careful reading of the original version, which resulted in substantial improvements. Thanks go to the editor Chii-Ruey Hwang.

References

1. T. Cacoullos and V. Papathanasiou, *On upper bounds for the variance of functions of random variables*, To appear in the Statistics and Probability letters, (1985).

2. H. Chernoff, *A note on an inequality involving the normal distribution*, Ann. Proba., **9** (1981), 533-535.

3. J. Hájek and Z. Sidak, *The theory of rank tests*, Academic Press., (1967), 33-34.

Department of Mathematics
Providence College
Taichung, Taiwan 400, R.O.C.