BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 13, Number 2, June 1985

HYPOANALYTICITY WITH VANISHING LEVI FORM

BY

CHIN-HUEI CHANG (張清輝)

Introduction. In [4] Baouendi, Chang and Treves introduced the hypoanalytic structure on a C^∞ manifold and they geve an application to show the hypoanalyticity of an *RC* structure based on the Levi form (see (1.9)). In this paper we show that a characteristic vector (see (1.3)) on which the Levi form vanishes, yet its product with the second commutation bracket of some vector field in the structure is non zero, is not in the hypoanalytic wave-front set of any *RC* distribution of the manifold (see 1.8)). Similar results to [4] and ours were obtained in [5], [6]. However, our technique gives the result like Corollary 1 here which partially answers a question raised in [10].

1. Preliminaries and statement of the result. For the general theory of hypoanalytic structure the reader is referred to [4]. Here we only recall briefly what is needed.

Let Ω be an open connected subset in $\boldsymbol{R}^{m+n}$. In Ω are defined m complex valued C^∞ functions $Z_1, \cdots, Z_m$ with their differentials linearly independent. We call them *basic hypoanalytic functions.* The differentials $dZ_1, \cdots, dZ_m$ generate a closed m-dimensional subbundle, called T', of the complex cotangent bundle $\boldsymbol{C}T^*\Omega$. The orthogonal of T', denoted by $T'^{\perp}$, for the duality between tangent and cotangent vectors is an n-dimensional locally integrable Frobenius Lie algebra over Ω. The integer n is called the *codimension* of this hypoanalytic structure.

We define the mapping

$$Z = (Z_1, \cdots, Z_m) : \Omega \longrightarrow \boldsymbol{C}^m .$$

Suppose H is a holomorphism defined on $Z(\Omega)$. Then $H_1(Z), \cdots, H_m(Z)$ define an equivalent hypoanalytic structure on Ω. We are also allowed to make C^∞ change of coordinates in Ω.

Received by the editors August 31, 1984 and in revised form October 8, 1984.

A distribution h defined in Ω is called a solution if for any C^∞ section L of $T'^\perp$ we have

$$Lh = 0 \tag{1.1}$$

in the distribution sense. By the result of [9] (Chap. II) that given any C^0 solution h we can find a C^0 function $\tilde{h}$ (with Ω contracted if necessary) defined in $Z(\Omega)$ such that

$$h = \tilde{h} \circ Z. \tag{1.2}$$

A C^∞ submanifold X of Ω is called ***maximally real*** if $dZ_1|_X, \cdots, dZ_m|_X$ are linearly independent and span CT^*X. It is easy to see that $\dim X = m$. Any solution h has a well-defined trace on X which is denoted by h_X. The characteristic set of T' is defined by

$$T^0 = T' \cap (T^*\Omega\backslash 0)\,. \tag{1.3}$$

Let π_X be the natural quotient map from $T^*\Omega|_X$ into T^*X. We see that π_X is injective on $T^0|_X$. We say that a point of $T^*X\backslash 0$ is ***characteristic*** if it belongs to $\pi_X(T^0|_X)$ and noncharacteristic otherwise. We say that a solution h is hypoanalytic at $\omega \in \Omega$ if $\tilde{h}$ (see (1.2)) is the restriction of a holomorphic function defined in a neighborhood of $Z(\omega)$. We write $WF_{ha}(h)$ for the hypoanalytic wave-front set of a solution h. For the definition of WF_{ha} one can find it in [4] (Chap. II Sec. 1.4).

Let $(\omega, \theta) \in T_\omega$ and h be a solution. By the result of [4], in order to show that $(\omega, \theta) \notin WF_{ha}(h)$ we can restrict ourselves to $(\omega, \theta) \in T^0_\omega$. In fact, let X be an arbitrary maximally real submanifold containing ω, it suffices to show

$$\pi_X(\theta) \notin WF_{ha}(h_X). \tag{1.4}$$

We restate a sufficient condition from [4] (Theorem. 2.2Chap. II) for a point (ω, θ) not contained in the hypoanalytic wave-front set of a solution h. For simplicity we shall take ω to be the origin of R^{m+n}. Let X be a maximally real submanifold containing 0. Let A be an open cone in C^m containing $\pi_X(\theta)$. For $\zeta \in C^m$ we set

$$\langle\zeta\rangle = (\zeta_1^2 + \cdots + \zeta_m^2)^{1/2}.$$

We assume that for any $\zeta \in A$

$$|\mathrm{Re}\,\zeta| > |\mathrm{Im}\,\zeta|.$$

Let V^C be a neighborhood of the origin in C^m. We take any function $g \in C_0^\infty(\Omega)$ such that g equals to 1 near 0. Let w be a local coordinate of X containing 0. We write $z(w) = Z|_X$. For any solution h and any positive number κ we define

$$F^\kappa(gh;\ \tau,\ \zeta) = \int_X e^{-i\zeta\cdot z(w) - \kappa\langle\zeta\rangle[\tau - z(w)]^2} g(w)\, h(w)\, dz(w) \tag{1.5}$$

with $\zeta \in A$ and $\tau \in V^C$. For $z = (z_1, \cdots, z_m) \in C^m$ we have used the notation

$$[z]^2 = z_1^2 + \cdots + z_m^2.$$

Let Z_w denote the Jacobian matrix of Z_k with respect to w. Let κ_* be some positive constant multiple of the maximum of the second derivatives of $z(w)$ under any suitable norm in supp g. Suppose Im $Z_w(0) = 0$ and for some $\kappa > \kappa_*$ there exist A, V^C, g and positive constants c, R such that

$$|F^\kappa(gh;\ \tau,\ \xi)| \le c\, e^{-|\zeta|/R} \tag{1.6}$$

then $(0, \theta) \notin WF_{ha}(h)$.

THEOREM. *Let $(\omega, \theta) \in T^0_\omega$. Suppose the following conditions are satisfied:*

$$\mathrm{Im}\langle \theta,\ [L,\ \bar{L}]_\omega\rangle = 0 \tag{1.7}$$

for all C^∞ section L of $T'^\perp$ and there exists one section L such that

$$\langle\theta,\ [L,\ [L,\ \bar{L}]]_\omega\rangle \ne 0, \tag{1.8}$$

where $[A, B]_\omega$ means taking the Lie bracket of the vector fields A, B then evaluating at ω. Then $(\omega, \theta) \notin WF_{ha}(h)$ for any solution h.

REMARK. Conditions (1.7) and (1.8) are also satisfied for $(\omega, -\theta)$. Therefore in this case $(\omega, \theta) \notin WF_{ha}(h)$ implies $(\omega, -\theta) \notin WF_{ha}(h)$ and vice versa.

The Theorem 6.1 of [4] says that if for a covector $(\omega, \theta) \in T^0_\omega$ there exists a C^∞ section L of $T'^\perp$ such that

$$\operatorname{Im}\langle \theta, [L, \bar{L}]_\omega \rangle < 0 \tag{1.9}$$

then $(\omega, \theta) \notin WF_{ha}(h)$ for any solution h. Combine this result and our theorem we can state the following corollary.

COROLLARY 1. *Fix* $\omega \in \Omega$. *Suppose for all* θ *such that* $(\omega, \theta) \in T^0_\omega$ *we have either* (1.7), (1.8) *or* (1.9) *holds for* θ. *Then any solution* h *is hypoanalytic at* ω.

We say a hypoanalytic structure is *real analytic* T' and $T'^\perp$ are in the real analytic category. By the Theorem 5.3 of [1] and Corollary 1 we have.

COROLLARY 2. *Suppose the hypoanalytic structure is real analytic and the assumptions of Corollary* 1 *are satisfied. Then* $T'^\perp$ *is analytic hypo-elliptic at* ω.

2. **Reductions.** From now on the point ω under consideration is chosen to be the origin of $\boldsymbol{R}^{m+n}$ and Ω is an open connected set containing 0. Let $Z_1, \cdots, Z_m$ be basic hypoanalytic functions. The rank of the mapping Z at 0 is $m+r$ where $0 \le r \le (m, n)$. By the Proposition 2.4 of Chap. I of [9] there is a coordinate (x, y) with $x \in \boldsymbol{R}^m$, $y \in \boldsymbol{R}^n$ such that in Ω we have

$$Z_j(x, y) = x_j + i\varphi_j(x, y), \qquad j = 1, \cdots, m, \tag{2.1}$$

where $i = \sqrt{-1}$ and φ_j are real function satisfying

$$\begin{aligned} &\varphi_j(x, y) = y_j, \qquad 1 \le j \le r; \\ &\varphi_j(0, 0) = \operatorname{grad}\varphi_j(0, 0) = 0, \qquad r+1 \le j \le m. \end{aligned} \tag{2.2}$$

In this coordinate $T'^\perp$ can be generated by the following set of vector fields.

$$L_j = \frac{\partial}{\partial y_j} + \sum_{k=1}^{m} \lambda_j^k(x, y) \frac{\partial}{\partial x_k}, \qquad j = 1, \cdots, n. \tag{2.3}$$

In this coordinate the submanifold $X = \{(x, y);\ y = 0\}$ is obviously maximally real. Let $Z_x(x, y)$ denote the $m \times m$ matrix of the first order partial derivatives of $Z_j(x, y)$ with respect to x_k. We have

(2.4) $$Z_x(0, 0) = I.$$

Let ξ, η denote the covariables of x and y respectively. We set $x = (x', x'')$ and $y = (y', y'')$ with $x' = (x_1, \cdots, x_r)$, $x'' = (x_{r+1}, \cdots, x_m)$, $y' = (y_1, \cdots, y_r)$ and $y'' = (y_{r+1}, \cdots, y_n)$. Similar notations apply for ξ and η. It is easy to see that

(2.5) $$T_0^0 = \{(0, 0, \xi, \eta); \xi' = \eta = 0, \xi'' \in \boldsymbol{R}^{m-r} \backslash 0\}.$$

To avoid triviality. i. e., $T_0^0 = \phi$, we assume that $m > r$. By the representation theory of a solution in Chap. II Sec. 4 of [9] we may assume that the solution is continuously differentiable near the origin.

We first prove the theorem when $r = n$ (so $m > n$). In this case (2.2) reads $\varphi_j = y_j$, $j = 1, \cdots, n$ and $\varphi_j(0) = \text{grad } \varphi_j(0) = 0$, $j = n + 1, \cdots, m$. We shall always assume this condition in the rest of our proof.

Let $L \in T'^{\perp}$. We use the notations $L^+ = L$, $L^- = \bar{L}$ and $L^{\pm}$ means either L^+ or L^-. Let $L \in T'^{\perp}$ and $(0, \theta) \in T_0^* \backslash 0$. For nonnegative integer l we set

(2.6) $$Q_l(\theta, L) = \langle \theta, [L^{\pm}, [L^{\pm}, [\cdots, [L^{\pm}, [L, \bar{L}] \cdots]_0 \rangle,$$

where the number of brackets equals l and it is understood that $Q_0(\theta, L) = \langle \theta, \bar{L}|_0 \rangle$. The ***weight*** of a covector $(0, \theta) \in T_0^* \backslash 0$ is defined to be the positive integer l such that for all $L \in T'^{\perp}$ we have

$$Q_j(\theta, L) = 0, \qquad \text{for all } j < l - 1,$$

and

$$Q_{l-1}(\theta, L) \neq 0$$

for some $L \in T'^{\perp}$. It follows that noncharacteristic covector has weight 1. It is easy to see that

$$T_0^* \backslash 0 = \sum_{l=1}^{\infty} T_l^0$$

where T_0^l consists of covectors of weight l and all but finitely many summands are empty. As we identify T_0^* with $\boldsymbol{R}^{m+n}$ by sending $(0, \theta)$ to θ, it is easy to see that for any integer $r \geq 1$ the set $T_r = \{0\} \cup (\cup_{l \geq r}^{\infty} T_0^l)$ is a subspace of T_0^*. In a coordinate satisfying (2.1), (2.2) we have

$$T_0^1 = \Big\{ \sum_{j=1}^{m} a_j \, dx_j + \sum_{k=1}^{n} b_k \, dy_k ; \; a_j, \, b_k \in \boldsymbol{R}, \quad \sum_{j=1}^{n} (|a_j| + |b_j|) \neq 0 \Big\}. \tag{2.7}$$

In order to transform the functions φ_j, $j = n+1, \cdots, m$ in a more manageable form, we introduce the following notation. We say that two real valued C^∞ functions f_1, f_2 defined on Ω are *equivalent* if there exists a holomorphic function F defined in a neighborhood of the origin of C^m such that

$$f_1(x, y) - f_2(x, y) = \operatorname{Re} F(Z_1(x, y), \cdots, Z_m(x, y))$$

and we write

$$f_1 \sim f_2.$$

Now assume that T_0^2 is non-empty. Let

$$T'_{2,1} = T_2/T_3.$$

We can find a non-zero element $\theta \in T'_{2,1}$. In virtue of (2.7) we can make a linear change of coordinates in x'' (with corresponding transformation in $Z_{n+1}, \cdots, Z_m$) such that

$$\theta = dx_{n+1}.$$

By the reductions in Chap. II Sec. 5 of [4] we may also assume that

$$\varphi_{n+1}(x, y) = P_{n+1}(x', y) + R_{n+1}(x, y),$$

where $P_{n+1}(x', y)$ is a homogeneous polynomial of degree two with $P_{n+1}(x', y) \not\sim O(|(x, y)|^3)$ (this follows from the fact that $\theta \in T_0^2$), and $R_{n+1}(x, y) = O(|(x, y)|^3)$. Let $T_{2,1} = [dx_{n+1}, T_3]$ be the subspace spanned by dx_{n+1} and T_3. We can carry on this process recursively as follows. Suppose that we have at the origin the covectors $dx_{n+1}, \cdots, dx_{n+j}$ all in T_0^2 and we have for $1 \leq r \leq j$ the subspaces $T'_{2,r}$, $T_{2,r}$ where

$$T_{2,r} = [dx_{n+r}, T_{2,r-1}], \qquad T'_{2,r} = T_2/T_{2,r-1}.$$

We have also for $1 \leq r \leq j$

$$\varphi_{n+r}(x, y) = P_{n+r}(x', y) + R_{n+r}(x, y),$$

where $P_{n+r}(x', y)$ is a homogeneous polynomial of degree two, $P_{n+r}(x', y) \not\sim O(|(x, y)|^3)$ and $R_{n+r}(x, y) = O(|(x, y)|^3)$. Moreover, for any real numbers $a_1, \cdots, a_j$ such that

$$\sum_{r=1}^{j} a_r \varphi_{n+r}(x, y) \sim O(|(x, y)|^3)$$

we must have $a_1 = \cdots = a_j = 0$. Now if $T'_{2, j+1} = T_2/T_{2,j} \neq \{0\}$, we can find a non-zero element $\theta \in T'_{2, j+1}$ and make a linear change of coordinates in $x_{n+j+1}, \cdots, x_m$ (with corresponding transformation in $Z_{n+1}, \cdots, Z_m$) such that $\theta = dx_{n+j+1}$. By the same reason as for φ_{n+1} we may assume that

$$\varphi_{n+j+1}(x, y) = P_{n+j+1}(x', y) + R_{n+j+1}(x, y),$$

where $P_{n+j+1}(x', y)$ is a homogeneous polynomial of degree two, $P_{n+j+1}(x', y) \not\sim O(|(x, y)|^3)$ and $R_{n+j+1}(x, y) = O(|(x, y)|^3)$. We should remark that in doing so $P_{n+r}(x', y)$, $1 \leq r \leq j$, are unchanged and still $R_{n+r}(x, y) = O(|(x, y)|^3)$, $1 \leq r \leq j$. If there are real numbers $a_1, \cdots, a_{j+1}$ such that

$$\sum_{r=1}^{j+1} a_r \varphi_{n+r}(x, y) \sim O(|(x, y)|^3),$$

then $a_{j+1} \neq 0$ by our assumption on $\varphi_{n+r}(x, y)$, $1 \leq r \leq j$. But this implies that dx_{n+j+1} lies in T_3 modulo $T_{2,j}$ which is a contradiction to our choice of dx_{n+j+1}. Since T_0^* is finite dimensional, this process must stop for some integer $k \leq m - n$. Therefore we have a coordinate system in Ω satisfying (2.1), (2.2), (2.7) and the above conclusions which we summarize as follows: The space T_2/T_3 is spanned by the k covectors $dx_{n+1}, \cdots, dx_{n+k}$ and we have

$$T_0^2 = \left\{\sum_{j=1}^{m-n} a_j \, dx_{n+j};\ a_j \in \boldsymbol{R},\ \sum_{j=1}^{k} |a_j| \neq 0\right\}, \tag{2.7$'$}$$

for $j = n + 1, \cdots, n + k$

$$\varphi_j(x, y) = P_j(x', y) + R_j(x, y), \tag{2.8}$$

where $P_j(x', y)$ are homogeneous polynomials of degree two and $R_j(x, y) \in C^\infty(\Omega)$, $R_j(x, y) = O(|(x, y)|^3)$. Also for any real numbers $a_1, \cdots, a_k$ such that

$$(2.8)' \qquad \sum_{j=1}^{k} a_j \varphi_{n+j}(x, y) \sim O(|(x, y)|^3)$$

we shall have $a_1 = \cdots = a_k = 0$. Note that in this coordinates

$$[dx_{n+k+1}, \cdots, dx_m] = T_3.$$

In other words, $[dx_{n+k+1}, \cdots, dx_m]$ consists of all the covectors at the origin of weight bigger than two.

Assuming above constructions, suppose we have for some covector $(0, \theta) \in T_0^0$ such that conditions (1.7), (1.8) are satisfied. This implies that $T_0^3 \neq \phi$ and the integer k in (2.7)$'$ is less than $m - n$. By a linear change of coordinates in $x_{n+k+1}, \cdots, x_m$ we may assume (and so in the rest of our paper) that

$$(2.9) \qquad \theta = dx_{n+k+1}.$$

This change of coordinates of course changes $Z_{n+1}, \cdots, Z_m$, but properties (2.1), (2.2), (2.7) $-$ (2.8)$'$ all retain, In virtue of the Lemma 5.1 and equations (5.12), (5.13) of [4] we may assume in addition to (2.1), (2.2), (2.7), (2.9) that

$$\varphi_j(x, y) = l_{j,1}(x'')l_{j,2}(x', y) + O(|(x, y)|^3),$$
$$j = n + k + 1, \cdots, m,$$

where $l_{j,1}$, $l_{j,2}$ are linear functions. By performing a holomorphism on $Z(\Omega)$

$$H_j(Z) = Z_j, \qquad j = 1, \cdots, n + k$$

$$H_j(z) = Z_j - i\, l_{j,1}(Z'')\, l_{j,2}(Z', -iZ'), \qquad j = n + k + 1, \cdots, m$$

and a change of coordinates in $x_{n+k+1}, \cdots, x_m$ we have

$$(2.10) \qquad \varphi_j(x, y) = O(|(x, y)|^3), \qquad j = n + k + 1, \cdots, m.$$

By the reduction in Chap. II Sec. 5 of [4] we may assume that

$$\varphi_j(x, 0) = O(|x|^3), \qquad j = n + 1, \cdots, m.$$

By the following holomorphism on $Z(\Omega)$

$$F_j(Z) = Z_j, \qquad j = 1, \cdots, n + k,$$

$$F_j(Z) = Z_j - i \sum_{l,s,t=1}^{m} \varphi_{j, x_l, x_s, x_t}(0) Z_l Z_s Z_t, \qquad j = n + 1, \cdots, m$$

and a change of coordinates in $x_{n+1}, \cdots, x_m$ we have

(2.11) $$\varphi_j(x, 0) = O(|x|^4), \qquad j = n+1, \cdots, m.$$

In summary we have

LEMMA 2.1. *With the assumptions of the Theorem and $r = n$ there exist a coordinate (x, y) in Ω and basic hypoanalytic functions $Z_1, \cdots, Z_m$ such that* (2.1), (2.2), (2.7) – (2.11) *hold.*

In the sequel we shall always assume the conclusions of Lemma 2.1 hold.

Let L_j, L_s, L_l be in the form (2.3), $j, s, l \in \{1, \cdots, n\}$ we have for $\theta = dx_{n+k+1}$

(2.12) $$\langle \theta, [L_j, [L_s, \bar{L}_l]]_0 \rangle = 16\, \varphi_{n+k+1, \bar{z}_j, \bar{z}_s, z_l}(0).$$

Simple computation shows that (2.12) has the form

$$\sum_{l=1}^{m} a_{\alpha(l)} (\varphi_{n+k+1}\, \varphi_l)^{(\alpha(l))}(0), \qquad a_{\alpha(l)} \in C$$

where $\alpha(l)$ is a multi-index in N^{m+n} or length four. Conditions (2.8), (2.10) imply that this sum depends linearly on the third order terms in (x', y) of the Taylor expansions of φ_{n+k+1}. To get $a_{\alpha(l)}$ it suffices to take φ_{n+k+1} monomial of degree three in (x', y) and make the computations.

We have in particular

$$\varphi_{n+k+1} = P_3(x', y) + r_1(x, y)$$

where $P_3(x', y)$ is a homogeneous polynomial of degree three and $r_1(x, y) = O(|(x, y)|^3)$. Also the third order part in the Taylor expansions of $r_1(x, y)$ contains no terms in (x', y) only. It follows from (1.8) and (2.12) that $P_3(x', y)$ cannot be equivalent to higher degree terms. By making the transformation

$$H_j(Z) = Z_j, \qquad j = 1, n+1, \cdots, m,$$

$$H_j(Z) = Z_j + \alpha_j Z_1, \qquad \alpha_j \in C,\ j = 2, \cdots, n$$

and corresponding change of coordinates in (x', y), with suitable choices of α_j, $j = 2, \cdots, n$ we can conclude that

$$\varphi_{n+k+1}(x, y) = \mathrm{Re}(a z_1 |z_1|^2) + r_2(x, y)$$

with $a \in C \backslash 0$ and $r_2(x, y)$ does not contain terms homogeneous of degree three in (x_1, y_1) in its Taylor expansions. By rotating z_1

we may assume that a is real. It is easy to see that

$$\mathrm{Re}((z_1|z_1|^2) \sim 4x_1 y_1^2 \sim \frac{4}{3} x_1^3,$$

$$\mathrm{Im}(z_1|z_1|^2) \sim 4x_1^2 y_1 \sim \frac{4}{3} y_1^3,$$

$$\mathrm{Re}(z_1|z_1|^2) = \mathrm{Im}(iz_1|iz_1|^2).$$

By a dilation in z_1 we may therefore assume that

$$\varphi_{n+k+1}(x, y) = y_1^3 + r_3(x, y)$$

where $r_3(x, y)$ has the same property as $r_2(x, y)$. Applying again the transformations to get (2.10), (2.11), we have a coordinate (x, y) and basic hypoanalytic functions satisfying conclusions of Lemma 2.1 and moreover

$$\varphi_{n+k+1}(x, y) = y_1^3 + R_{n+k+1}(x, y)$$

where $R_{n+k+1}(x, y)$ keeps the property $r_3(x, y)$ has.

For the final reduction we make the following change of coordinates and transformations, for $\lambda > 0$ to be chosen later:

$$\begin{aligned} x_1 &= \lambda \tilde{x}_1, & y_1 &= \lambda \tilde{y}_1, \\ y_j &= \lambda^2 \tilde{y}_j, & j &= 2, \cdots, n, \\ x_j &= \lambda^2 \tilde{x}_j, & j &= 2, \cdots, n+k, n+k+2, \cdots, m, \\ x_j &= \lambda^3 \tilde{x}_j, & j &= n+k+1. \end{aligned}$$

and

$$H_1(Z) = \frac{1}{\lambda} Z_1,$$

$$H_j(Z) = \frac{1}{\lambda^2} Z_j, \qquad j = 2, \cdots, n+k, n+k+2, \cdots, m,$$

$$H_j(Z) = \frac{1}{\lambda^3} Z_j, \qquad j = n+k+1.$$

It follows from the construction of R_{n+k+1} and (2.10) that we have the final form of φ_j's.

LEMMA 2.2. *With the assumptions of Lemma* 2.1 *we have in addition to the conclusions of Lemma* 2.1 *that*

$$\begin{aligned} \varphi_{n+k+1}(x, y) &= y_1^3 + R_{n+k+1}(\lambda, x, y), \\ \varphi_j(x, y) &= R_j(\lambda, x, y), \; j = n+k+2, \cdots, m \end{aligned} \tag{2.13}$$

where R_j *is a* C^∞ *function in* $(0, \delta) \times \Omega$ *and* $\lim_{\lambda\to 0} R_j(\lambda, x, y) = 0$, $j = n+k+1, \cdots, m$.

3. **Proof of the Theorem.** We first prove the Theorem when $r = n$. We assume the conclusions of Lemma 2.2. The maximally real submanifold X is given by

$$X = \{y = 0\}.$$

All we need to show is the estimate (1.6).

By contraction we may assume that $\Omega = V \times W$ with V(W, resp.) an open ball in $\mathbf{R}^m$ ($\mathbf{R}^n$, resp.) centered at the origin. We may and shall assume that for each $y \in W$ the mapping $x \longmapsto Z(x, y)$ from V into $\mathbf{C}^m$ is injective. So the mapping

$$(x, y) \longmapsto (Z(x, y), y) \tag{3.1}$$

is a diffeomorphism of Ω onto a C^∞ submanifold M of $\mathbf{C}^m \times \mathbf{R}^n$.

Let h be a C^1 solution and $h = \tilde{h} \circ Z$. Since h is a solution, the m-form $h\, dZ (dZ = dZ_1 \cdots dZ_m)$ is closed in Ω. Consequently, its push-forward via (3.1), $\tilde{h}\, dz$, is closed on M. We associated with h the following closed m-form on M:

$$H = H^\kappa(z, \tau, \zeta) = e^{-i\zeta\cdot z - \kappa\langle\zeta\rangle[z-\tau]^2}\, \tilde{h}(z)\, dz.$$

Here $\tau, \zeta \in \mathbf{C}^m$ and $|\mathrm{Re}\, \zeta| > |\mathrm{Im}\, \zeta|$. Let t be an arbitrary point in W. We call X_t the maximally real submanifold of Ω defined by $y = t$. Denote by M_t the image of X_t under the mapping (3.1). Let $l(t)$ be the straight line segment in W joining 0 and t and call $\Gamma(t)$ the $(m+1)$-chain which is the image of $V \times l(t)$ under the mapping (3.1). For $g \in C_0^\infty(V)$ we define $\tilde{g}(z, y) = g(x)$ when $z = Z(x, y)$. Then supp $\tilde{g}$ intersects the boundary of $\Gamma(t)$ only on M_0 and M_t. Stokes' theorem we have

$$\int_{M_t} \tilde{g}H - \int_{M_0} \tilde{g}H = \int_{\Gamma(t)} d\tilde{g} \wedge H. \tag{3.2}$$

Note that

$$F^\kappa(gh;\, \tau, \zeta) = \int_{M_0} \tilde{g}H.$$

To get estimate (1.6) we show the same estimate holds for both

$$\int_{M_t} \tilde{g}H \quad \text{and} \quad \int_{\Gamma(t)} d\tilde{g} \wedge H.$$

By Lemma 2.2 we can find positive numbers δ_0 and λ_0 such that for $\lambda \in (0, \lambda_0)$, $\delta \in (0, \delta_0]$, $y_1 \in [-\delta, 0]$ and $|x| \leq 8\delta$

$$\varphi_{n+k+1}(x, -\delta, 0) \leq -\frac{\delta^3}{2}, \tag{3.3}$$

$$|\varphi_{n+k+1}(x, y_1, 0)| \leq 2\delta^3, \tag{3.4}$$

$$|(\varphi_1(x, y_1, 0), \cdots, \varphi_m(x, y_1, 0)| \leq 2\delta\,. \tag{3.5}$$

We can take δ_0 small such that

$$\{(x, y); |x| \leq 8\delta_0, \ |y| \leq \delta_0\} \subset \Omega\,.$$

From now on the number $\lambda \in (0, \lambda_0)$ will be fixed. On X we consider the functions $Z_j(x, 0)$, $j = 1, \cdots, m$ with $x \in V' = \{x \in \boldsymbol{R}^m;\ |x| \leq 8\delta,\ \delta \in (0, \delta_0]\}$. Condition (2.11) and the choice of the maximally real submanifold X imply that for $x \in V'$ the positive number κ_* in §1 is a constant multiple of δ^2. From now on we shall choose δ fixed and small enough such that $\kappa_* < \delta/16$.

We choose $g \in C_0^\infty(V')$ satisfying

$$0 \leq g \leq 1 \quad \text{and} \quad g = 1 \quad \text{for} \quad |x| \leq 6\delta\,. \tag{3.6}$$

Let $t = (-\delta, 0, \cdots, 0) \in \boldsymbol{R}^n$. To estimate $\int_{M_t} \tilde{g}H$ we consider the quantity

$$Q = \mathrm{Re}\{i\zeta \cdot z + \kappa\langle\zeta\rangle [z - \tau]^2\}/|\zeta|\,.$$

It suffices to deal with $\tau = 0$, $\zeta \in A \cap \boldsymbol{R}^m$ and we call this quantity Q_0. We use ξ for ζ real and write $\dot{\xi} = \xi/|\xi|$. Thus for $\theta = dx_{n+k+1}$, $z(x, t)$ on M_t

$$Q_0 = -\varphi_{n+k+1}(x, t) + \kappa(|x|^2 - |(\varphi_1(x, t), \cdots, \varphi_m(x, t))|^2) + (\varphi_1(x, t), \cdots, \varphi_m(x, t)) \cdot (\theta - \dot{\xi}).$$

The positive number κ will be specified later. From (3.3), (3.5) we have

$$Q_0 \geq \frac{\delta^3}{2} - 4\kappa\delta^2 - 2\delta|\theta - \dot{\xi}|.$$

If we make

$$\frac{\delta}{16} < \kappa < \frac{\delta}{8}, \tag{3.7}$$

V^C small enough and A "thin" enough, we can conclude that

$$Q \geq c > 0$$

for some constant c. Therefore for such choices of κ, V^C and A there exist positive constant C_1, R_1 such that

$$\left| \int_{M_t} \tilde{g} H \right| \leq C_1 e^{-|\zeta|/R_1}, \qquad \text{for all} \quad \tau \in V^C,\ \zeta \in A. \tag{3.8}$$

To estimate $\int_{\Gamma(t)} d\tilde{g} \wedge H$ we observe that by (3.4), (3.5) and (3.6) we have on the support of $d\tilde{g}$

$$Q_0 \geq -2\delta^3 + \kappa(36\delta^2 - 4\delta^2) - 2\delta|\theta - \dot{\xi}|.$$

From (3.7) we conclude that there exist positive constants C_2, R_2 such that

$$\left| \int_{\Gamma(t)} d\tilde{g} \wedge H \right| \leq C_2 e^{-|\zeta|/R_2}, \qquad \text{for all} \quad \tau \in V_1^C,\ \zeta \in A_1, \tag{3.9}$$

where V_1^C is chosen small and A_1 "thin" enough.

We have thus prove the Theorem when $r = n$. To deal with the case $r < n$ we assume that (2.1), (2.2) and (2.3) hold. We shall add $(n-r)$ variables $x_{m+1}, \cdots, x_{m+n-r}$. The set Ω is thus changed to $\Omega' = V \times U \times W \subset \boldsymbol{R}^{m+2n-r}$ with $(x_1, \cdots, x_m) \in V = B(0, \delta_1) \subset \boldsymbol{R}^m$, $(x_{m+1}, \cdots, x_{m+n-r}) \in U = B(0, \delta_2) \subset \boldsymbol{R}^{n-r}$ and $(y_1, \cdots, y_n) \in W = B(0, \delta_3) \subset \boldsymbol{R}^n$ where $B(0, \delta)$ means open ball centered at 0 with radius δ. We then consider a new n-dimensional integrable Lie algebra, $\tilde{T}'^{\perp}$, in Ω' generated by

$$\begin{cases} \tilde{L}_j = L_j, & j = 1, \cdots, r, \\ \tilde{L}_j = L_j - \dfrac{\partial}{\partial x_{m+j-r}}, & r < j \leq n \end{cases} \tag{3.10}$$

where L_j, $j = 1, \cdots, n$ is given in (2.3). We also have an enlarged sytem of basic hypoanalytic functions for the enlarged structure bundle $\tilde{T}'$ given by

$$\begin{cases} Z_j = Z_j, & j = 1, \cdots, m, \\ Z_j = x_j + iy_{r+j-m}, & j = m+1, \cdots, m+n-r. \end{cases}$$

This new system is of maximal rank, i.e., $\tilde{r} = n$. Note that if h is a solution for T' then h considered as a distribution defined in Ω' (independent of x_j, $j = m+1, \cdots, m+n-r$) is a solution for $\tilde{T}'$. We can then apply the previously proved result for $r = n$ to conclude that $(0, \theta) \notin WF_{ha}(h)$. This completes the proof of the Theorem.

Acknowledgement. This work was done mainly when the author was at the Institute for Advanced Study, Princeton. The author would like to thank the hospitality of that institution. He also would like to thank Professors M. S. Baouendi and F. Treves for their encouragement.

BIBLIOGRAPHY

1. M. S. Baouendi and F. Treves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. of Math., **113** (1981), 387-421.

2. M. S. Baouendi and F. Treves, *A local constancy principle for the solutions of certain overdetermined systems of first-order linear partial differential equation*, Math. Ana. and Appl., Part A, Advances in Math. Supplementary Studies Vol. **7A** (1981), 245-262.

3. M. S. Baouendi and F. Treves, *A microlocal version of Bochner's tube theorem*, Ind. Univ. Math. J., **31** (1982), 883-895.

4. M. S. Baouendi, C. H. Chang and F. Treves, *Microlocal hypoanalyticity and extension of CR functions*, J. Diff. Geo., 18 (1983), 331-391.

5. A. Bogges, *CR extendibility near a point where the first Levi form vanishes*, Duke Math. J., 48 (1981), 665-684.

6. A. Boggess and J. C. Polking, *Holomorphic extension of CR functions*, Duke Math. J., **49**(1982), 757-784.

7. H. Lewy, *On the local character of the solution of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. Math., **64** (1956), 514-522.

8. H. Lewy, *On hulls of holomorphy*, Comm. in Pure and Applied Math., **13**(1960) 587-591.

9. F. Treves, *Approximation and Representations of Functions and Distributions Annihilated by a System of Complex Vector Fields*, Ecole Polytechnique, Centre de Mathématiques, (1981).

10. A. Boggess, *The first and second Levi forms and CR extension*, Proc. of Symp. in Pure Math., **41** (1984), 25-38.

11. A. Boggess and J. Pitts. *CR Extension near a point of higher type*. Preprint.

12. M. S. Baouendi and F. Treves, *About the holomorphic extension of CR funcitons on real hypersurfaces in complex space*, Duke Math. J., **51** (1984), 77-107.

Institute of Mathematics
Academia Sinica
Taiwan, R. O. C.