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AN ASYMPTOTIC CHARACTERIZATION OF BIAS REDUCTION BY THE HIGHER ORDER JACKKNIFE

BY

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Abstract. In this paper we give a general expression for the bias of the higher order jackknife estimator $J^{(k)}(\hat{\theta})$ in terms of the bias of $\hat{\theta}$, and compare the bias of $J^{(k)}(\hat{\theta})$ with those of $\hat{\theta}$ and $J^{(k-1)}(\hat{\theta})$ asymptotically. We have extended some results of Adams, Gray and Watkins (1971) who considered the two special cases: $k=1, 2$.

1. **Introduction.** Quenouille ([8], [9]) introduced a method to reduce the bias of an estimator $\hat{\theta}$, which eliminates bias term of order $1/n$. This method was later utilized by Tukey [12] to develop a general method for obtaining approximate confidence intervals in not quite large sample size. At that time, Tukey adopted the name "jackknife" for his method. Since then a large number of papers have been written on this method, see Miller [6] and Parr and Schucany [7]. Schucany, Gray and Owen [10] first suggested the higher order jackknife $J^{(k)}$ which eliminates bias terms of order $1/n, 1/n^2, \cdots, 1/n^k$. Recently, Burnham and Overton [3] applied the higher order jackknife $J^{(k)}$ $(k \le 5)$ to the estimation of the size of a closed population when capture probabilities vary among animals; they also proposed a procedure to decide which order k should be used for their study and found it being an available procedure.

By the direct computation of determinant, Adams, Gray and Watkins [1] characterized the bias reduction properties of the first two jackknife estimators $J^{(1)}$ and $J^{(2)}$. It does not seem easy to

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generalize their result along the same line of argument. In this paper we use the Gaussian elimination and mean value theorem to extend their result to any higher order jackknife $J^{(k)}$. We first give a general expression for the bias of $J^{(k)}(\hat{\theta})$ in terms of that of $\hat{\theta}$, and then compare the bias of $J^{(k)}(\hat{\theta})$ with those of $\hat{\theta}$ and $J^{(k-1)}(\hat{\theta})$ asymptotically. Our expression is simlar to, but different from the formula obtained by Sharot [11] who used a combinatorial approach.

2. **Definitions and Lemmas.** In this section we give some definitions and lemmas which are necessary in this paper. In the definitions below we shall follow the line development given in Gray and Schucany [4].

Definition 1. Let $\hat{\theta}$ be an estimator of θ based on the random sample $X_1, X_2, \cdots, X_n$. For $1 \leq k \leq n-1$ and $2 \leq j \leq k+1$, let $\hat{\theta}^{i_2\cdots i_j}$ be the estimator obtained by restricting $\hat{\theta}$ to the samples obtained by deleting at random $j-1$ of the observations. Define $\hat{\theta}_1 = \hat{\theta}$ and

$$\hat{\theta}_j = \overline{\hat{\theta}^{i_2 i_3 \cdots i_j}}, \qquad j = 2, 3, \cdots, k+1,$$

the average over the $\binom{n}{j-1}$ resulting statistics, and define the k-th order jackknife estimator $J^{(k)}(\hat{\theta})$ of θ from $\hat{\theta}$ by

$$J^{(k)}(\hat{\theta}) = \frac{\begin{vmatrix} \hat{\theta}_1 & \hat{\theta}_2 & \cdots & \hat{\theta}_{k+1} \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{n-k} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n^k} & \frac{1}{(n-1)^k} & \cdots & \frac{1}{(n-k)^k} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{n-k} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n^k} & \frac{1}{(n-1)^k} & \cdots & \frac{1}{(n-k)^k} \end{vmatrix}}.$$

Note that if the expectation $E[\hat{\theta}] = \theta + B(n, \theta)$ and $\hat{\theta}_j$ is defined as above, then

$$E[\hat{\theta}_j - \theta] = B(n - j + 1, \theta), \qquad j = 1, 2, 3, \cdots, k + 1.$$

DEFINITION 2. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimators of θ defined on a sample of size n and let $B_1(n, \theta) = E[\hat{\theta}_1 - \theta]$, $B_2(n, \theta) = E[\hat{\theta}_2 - \theta]$. If the limit

$$L = \lim_{n\to\infty} \frac{B_1(n, \theta)}{B_2(n, \theta)}$$

exists, then

(i) if $|L| = 1$, we say that $\hat{\theta}_1$ and $\hat{\theta}_2$ are same order bias estimators of θ, and denote this by $\hat{\theta}_1$ S. O. B. E. $\hat{\theta}_2$;

(ii) if $L = 0$, we say that $\hat{\theta}_1$ is a lower order bias estimator than $\hat{\theta}_2$, and denote this by $\hat{\theta}_1$ L. O. B. E. $\hat{\theta}_2$; and

(iii) if $0 < |L| < 1$, we say that $\hat{\theta}_1$ is a better same order bias estimator than $\hat{\theta}_2$, and denote this by $\hat{\theta}_1$ B. S. O. B. E. $\hat{\theta}_2$.

DEFINITION 3. Let $\{a_n\}$ be a sequence of real numbers, then for any n we denote $\Delta^k a_n$, $k = 0, 1, 2, \cdots, n-1$, by

$$\Delta^k a_n = \begin{cases} a_n, & \text{if } k = 0 \\ \Delta a_n = a_n - a_{n-1}, & \text{if } k = 1 \\ \Delta(\Delta^{k-1} a_n), & \text{if } k = 2, 3, \cdots, n-1. \end{cases}$$

To prove the theorems in the next section, we need the following lemmas. As stated below, Lemmas 1, 2 and 3 are well-known.

LEMMA 1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, where $\{b_n\}$ is monotone. Then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\Delta a_n}{\Delta b_n},$$

provided that the second limit exists, finite or infinite.

Proof. See Bromwich ([2], p. 143) or Knopp ([5], p. 109).

LEMMA 2. *For any positive integers m, n and k, where $k \le n$,*

$$\sum_{i=0}^{k} (-1)^i \binom{m+i}{i} \binom{n}{k-i} = \frac{(n-m-1)(n-m-2)\cdots(n-m-k)}{k!}.$$

Proof. This lemma can be proved by using the factorial binomial theorem, see Yaglom and Yaglom ([13], p. 136).

LEMMA 3. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Then for any positive integers k and n satisfying $1 \le k \le n-1$,*

$$\Delta^k(a_n b_n) = \sum_{i=0}^{k} \binom{k}{i} (\Delta^{k-i} a_{n-i}) \Delta^i b_n .$$

Proof. This lemma can be proved easily by induction.

LEMMA 4. *For any $p \neq 0$ and any positive integer i,*

$$\lim_{n\to\infty} \frac{\Delta^i n^{-p}}{n^{-p-i}} = (-p)(-p-1)\cdots(-p-i+1) = (-1)^i \binom{p+i-1}{i} i! .$$

Proof. Since for any $n > i$

$$\frac{\Delta^i n^{-p}}{n^{-p}} = \frac{1}{n^{-p}} \sum_{j=0}^{i} (-1)^j \binom{i}{j} (n-j)^{-p} = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \left(1 - \frac{j}{n}\right)^{-p},$$

it can be proved by mean value theorem that

$$\frac{\Delta^i n^{-p}}{n^{-p-i}} = f^{(i)}\left(-\theta_n \frac{i}{n}\right), \quad \text{for some } 0 < \theta_n < 1,$$

where $f^{(i)}$ is the i-th derivative function of $f(x) = (1+x)^{-p}$. Therefore

$$\lim_{n\to\infty} \frac{\Delta^i n^{-p}}{n^{-p-i}} = \lim_{n\to\infty} f^{(i)}\left(-\theta_n \frac{i}{n}\right) = f^{(i)}(0) = (-p)(-p-1)\cdots(-p-i+1) .$$

LEMMA 5. *Let $\{a_n\}$ be a sequence of real numbers and $p > 0$ such that*

$$\lim_{n\to\infty} n^p a_n = c \neq 0 \quad \textit{or} \quad \pm\infty ,$$

and

$$\lim_{n\to\infty} n^{p+k} \Delta^k a_n \quad \textit{exists},$$

where k is a fixed positive integer. Then for $i = 1, 2, \cdots, k$,

$$\lim_{n\to\infty} \frac{n^i \Delta^i a_n}{a_n} = (-p)(-p-1)\cdots(-p-i+1)$$
$$= (-1)^i \binom{p+i-1}{i} i!\,.$$

Proof. From the assumptions and Lemma 1, it can be seen that

$$\lim_{n\to\infty} \frac{a_n}{n^{-p}} = \lim_{n\to\infty} \frac{\Delta a_n}{\Delta n^{-p}}$$
$$= \cdots$$
$$= \lim_{n\to\infty} \frac{\Delta^i a_n}{\Delta^i n^{-p}}$$
$$= \cdots$$
$$= \lim_{n\to\infty} \frac{\Delta^k a_n}{\Delta^k n^{-p}}$$
$$= c\,.$$

Then for any i, $i = 1, 2, \cdots, k$, we have

$$\lim_{n\to\infty} \frac{n^i \Delta^i a_n}{a_n} = \lim_{n\to\infty} \frac{(\Delta^i a_n)/(\Delta^i n^{-p})}{a_n/(n^{-p})} \cdot \frac{\Delta^i n^{-p}}{n^{-p-i}}$$
$$= \lim_{n\to\infty} \frac{\Delta^i n^{-p}}{n^{-p-i}}$$
$$= (-p)(-p-1)\cdots(-p-i+1)\,.$$

The last equality follows from Lemma 4.

3. **Main theorems.** In this section we shall give four main results. In order to characterize the bias reduction properties of $J^{(k)}$, we need a general expression for the bias of $J^{(k)}(\hat\theta)$ in terms of the bias of $\hat\theta$. This is accomplished by the following theorem which reduces to the results of Adams, Gray and Watkins ([1], Theorem 1) when $k = 1, 2$.

THEOREM 1. *Let $\hat\theta$ be an estimator of θ based on the random sample $X_1, X_2, \cdots, X_n$ and let $E[\hat\theta - \theta] = B(n, \theta)$. Then*

$$E[J^{(k)}(\hat\theta) - \theta] = \frac{1}{k!} \Delta^k (n^k B(n, \theta))\,,$$

where $1 \le k \le n-1$.

Proof. From the definition of $J^{(k)}(\hat{\theta})$, we have

$$E[J^{(k)}(\hat{\theta})]=\frac{\begin{vmatrix} E\hat{\theta}_1 & E\hat{\theta}_2 & \cdots & E\hat{\theta}_{k+1} \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{n-k} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n^k} & \frac{1}{(n-1)^k} & \cdots & \frac{1}{(n-k)^k} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{n-k} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n^k} & \frac{1}{(n-1)^k} & \cdots & \frac{1}{(n-k)^k} \end{vmatrix}}.$$

Reviewing $E\hat{\theta}_j=\theta+B(n-j+1,\theta)$, we know $E[J^{(k)}(\hat{\theta})]=\theta+R$, where

$$R=\frac{\begin{vmatrix} B(n,\theta) & B(n-1,\theta) & \cdots & B(n-k,\theta) \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{n-k} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n^k} & \frac{1}{(n-1)^k} & \cdots & \frac{1}{(n-k)^k} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{n-k} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n^k} & \frac{1}{(n-1)^k} & \cdots & \frac{1}{(n-k)^k} \end{vmatrix}}.$$

Multiplying both denominator and numerator of R by $n^k(n-1)^k\cdots(n-k)^k$, we obtain

$$R=\frac{\begin{vmatrix} n^kB(n,\theta) & (n-1)^kB(n-1,\theta) & \cdots & (n-k)^kB(n-k,\theta) \\ n^{k-1} & (n-1)^{k-1} & \cdots & (n-k)^{k-1} \\ \vdots & \vdots & & \vdots \\ n & n-1 & \cdots & n-k \\ 1 & 1 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} n^k & (n-1)^k & \cdots & (n-k)^k \\ n^{k-1} & (n-1)^{k-1} & \cdots & (n-k)^{k-1} \\ \vdots & \vdots & & \vdots \\ n & n-1 & \cdots & n-k \\ 1 & 1 & \cdots & 1 \end{vmatrix}}.$$

Then taking the same column operations in each of the above determinants, we have

$$R=\frac{\begin{vmatrix}\Delta[n^kB(n,\theta)] & \Delta[(n-1)^kB(n-1,\theta)] & \cdots & \Delta[(n-k+1)^kB(n-k+1,\theta)] & (n-k)^kB(n-k,\theta)\\ \Delta n^{k-1} & \Delta(n-1)^{k-1} & \cdots & \Delta(n-k+1)^{k-1} & (n-k)^{k-1}\\ \vdots & \vdots & & \vdots & \vdots\\ \Delta n & \Delta(n-1) & \cdots & \Delta(n-k+1) & n-k\\ 0 & 0 & \cdots & 0 & 1\end{vmatrix}}{\begin{vmatrix}\Delta n^k & \Delta(n-1)^k & \cdots & \Delta(n-k+1)^k & (n-k)^k\\ \Delta n^{k-1} & \Delta(n-1)^{k-1} & \cdots & \Delta(n-k+1)^{k-1} & (n-k)^{k-1}\\ \vdots & \vdots & & \vdots & \vdots\\ \Delta n & \Delta(n-1) & \cdots & \Delta(n-k+1) & (n-k)\\ 0 & 0 & \cdots & 0 & 1\end{vmatrix}}$$

$$=\frac{\begin{vmatrix}\Delta[n^kB(n,\theta)] & \Delta[(n-1)^kB(n-1,\theta)] & \cdots & \Delta[(n-k+1)^kB(n-k+1,\theta)]\\ \Delta n^{k-1} & \Delta(n-1)^{k-1} & \cdots & \Delta(n-k+1)^{k-1}\\ \vdots & \vdots & & \vdots\\ \Delta n & \Delta(n-1) & \cdots & \Delta(n-k+1)\end{vmatrix}}{\begin{vmatrix}\Delta n^k & \Delta(n-1)^k & \cdots & \Delta(n-k+1)^k\\ \Delta n^{k-1} & \Delta(n-1)^{k-1} & \cdots & \Delta(n-k+1)^{k-1}\\ \vdots & \vdots & & \vdots\\ \Delta n & \Delta(n-1) & \cdots & \Delta(n-k+1)\end{vmatrix}}.$$

Continuing the above procedure $k-1$ times, we finally obtain that

$$R=\frac{\Delta^k[n^kB(n,\theta)]}{\Delta^k n^k}=\frac{1}{k!}\Delta^k[n^kB(n,\theta)].$$

Thus the proof of this theorem is completed.

In the next theorem we compare the bias of the higher order jackknife estimator $J^{(k)}(\hat{\theta})$ with that of $\hat{\theta}$ from an asymptotic point of view. Two special cases $(k=1, 2)$ of this theorem were also obtained by Adams, Gray and Watkins ([1], Theorem 3).

THEOREM 2. *Let $\hat{\theta}$ be an estimator of θ based on the random sample $X_1, X_2, \cdots, X_n$ and let $E[\hat{\theta}-\theta]=B(n,\theta)$. Assume that there exists a $p>0$ such that*

$$\lim_{n\to\infty} n^p B(n,\theta)=c(\theta)\neq 0 \quad or \quad \pm\infty,$$

and

$$\lim_{n\to\infty} n^{p+k}\Delta^k B(n,\theta) \quad exists,$$

where k is a positive integer. Then

(i) *if $p\in\{1, 2,\cdots, k\}$, then $J^{(k)}(\hat{\theta})$ L.O.B.E. $\hat{\theta}$;*

(ii) *if $p<k+1$ and $p\notin\{1, 2,\cdots, k\}$, then $J^{(k)}(\hat{\theta})$ B.S.O.B.E. $\hat{\theta}$;*

(iii) *if* $p=k+1$, *then* $J^{(k)}(\hat{\theta})$ *S.O.B.E.* $\hat{\theta}$;

(iv) *if* $p>k+1$, *then* $\hat{\theta}$ *B.S.O.B.E.* $J^{(k)}(\hat{\theta})$.

Proof. From Theorem 1 and Lemmas 3–5, we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{E(J^{(k)}(\hat{\theta})-\theta)}{E(\hat{\theta}-\theta)}\\
&\quad=\lim_{n\to\infty}\frac{(1/k!)\,\Delta^k(n^kB(n,\theta))}{B(n,\theta)}\\
&\quad=\frac{1}{k!}\lim_{n\to\infty}\frac{\sum_{i=0}^{k}\binom{k}{i}[\Delta^{k-i}(n-i)^k]\,\Delta^iB(n,\theta)}{B(n,\theta)}\\
&\quad=\frac{1}{k!}\lim_{n\to\infty}\sum_{i=0}^{k}\binom{k}{i}\frac{\Delta^{k-i}(n-i)^k}{n^i}\cdot\frac{n^i\Delta^iB(n,\theta)}{B(n,\theta)}\\
&\quad=\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\binom{k}{k-i}(k-i)!\cdot(-1)^i\binom{p+i-1}{i}i!\\
&\quad=\sum_{i=0}^{k}(-1)^i\binom{k}{k-i}\binom{p+i-1}{i}.
\end{aligned}$$

The right hand side of the last equality is 0 provided p is replaced by any one of $1,2,\cdots,k$. Hence $(p-1)(p-2)\cdots(p-k)$ is a factor of $\sum_{i=0}^{k}(-1)^i\binom{k}{k-i}\binom{p+i-1}{i}$. Finally, after checking the order of p and the constant term we know that

$$\begin{aligned}
\lim_{n\to\infty}\frac{E[J^{(k)}(\hat{\theta})-\theta]}{E[\hat{\theta}-\theta]}&=\sum_{i=0}^{k}(-1)^i\binom{k}{k-i}\binom{p+i-1}{i}\\
&=\frac{(-1)^k}{k!}\cdot(p-1)(p-2)\cdots(p-k).
\end{aligned}$$

Thus the conclusions of this theorem follow immediately from Definition 2.

In Theorem 2 we have proved that in many cases $J^{(k)}$ is an effective bias removal tool. Moreover, it should now be clear that $J^{(k)}$ can be more effective than $J^{(k-1)}$. To establish an asymptotic characterization of this property we show the next two respective extensions of two results of Adams, Gray and Watkins ([1], Theorems 4 and 6) who had considered the special case $k=2$.

THEOREM 3. *Let* $B(n,\theta)$ *and* p *be defined as in Theorem* 2, *and assume* $p\notin\{1,2,\cdots,k-1\}$. *Then*

(i) *if* $p=k$, *then* $J^{(k)}(\hat{\theta})$ *L.O.B.E.* $J^{(k-1)}(\hat{\theta})$;

(ii) *if* $p<2k$ *and* $p \notin \{1, 2, \cdots, k\}$, *then* $J^{(k)}(\hat{\theta})$ *B.S.O.B.E.* $J^{(k-1)}(\hat{\theta})$;

(iii) *if* $p=2k$, *then* $J^{(k)}(\hat{\theta})$ *S.O.B.E.* $J^{(k-1)}(\hat{\theta})$;

(iv) *if* $p>2k$, *then* $J^{(k-1)}(\hat{\theta})$ *B.S.O.B.E.* $J^{(k)}(\hat{\theta})$.

Proof. The conclusions of this theorem follow from the fact that

$$\begin{aligned}
\lim_{n\to\infty} \frac{E[J^{(k)}(\hat{\theta})-\theta]}{E[J^{(k-1)}(\hat{\theta})-\theta]}
&= \lim_{n\to\infty} \frac{(1/k!)\,\Delta^k[n^k B(n,\theta)]}{(1/(k-1)!)\,\Delta^{k-1}[n^{k-1} B(n,\theta)]} \\
&= \frac{1}{k} \lim_{n\to\infty} \frac{\{\Delta^k[n^k B(n,\theta)]\}/B(n,\theta)}{\{\Delta^{k-1}[n^{k-1} B(n,\theta)]\}/B(n,\theta)} \\
&= \frac{-1}{k} \cdot \frac{(p-1)(p-2)\cdots(p-k+1)(p-k)}{(p-1)(p-2)\cdots(p-k+1)} \\
&= \frac{-1}{k}(p-k).
\end{aligned}$$

In the above theorem it was necessary to assume that $p \notin \{1, 2, \cdots, k-1\}$. For $p=k-1$, we have the following theorem. And it is possible to consider the cases $p=1, 2, \cdots, k-2$ similarly.

THEOREM 4. *Let* $B(n,\theta)$ *and* $p=k-1$ *be defined as in Theorem 2. Assume that there exists a* $k_1>0$ *such that*

$$\lim_{n\to\infty} n^{k_1}[n^{k-1} B(n,\theta) - c(\theta)] = c_1(\theta) \neq 0 \quad or \quad \pm\infty$$

and

$$\lim_{n\to\infty} n^{k_1+k}\,\Delta^k[n^{k-1} B(n,\theta)] \quad exists.$$

Then

(i) *if* $k_1=1$, *then* $J^{(k)}(\hat{\theta})$ *L.O.B.E.* $J^{(k-1)}(\hat{\theta})$;

(ii) *if* $0<k_1<k+1$ *but* $k_1\neq 1$, *then* $J^{(k)}(\hat{\theta})$ *B.S.O.B.E.* $J^{(k-1)}(\hat{\theta})$;

(iii) *if* $k_1=k+1$, *then* $J^{(k)}(\hat{\theta})$ *S.O.B.E.* $J^{(k-1)}(\hat{\theta})$;

(iv) *if* $k_1>k+1$, *then* $J^{(k-1)}(\hat{\theta})$ *B.S.O.B.E.* $J^{(k)}(\hat{\theta})$.

Proof. Consider

$$
\begin{aligned}
\Delta^k[n^k B(n,\theta)]
&= \Delta^k[n(n^{k-1}B(n,\theta))] \\
&= \sum_{i=0}^{k}\binom{k}{i}[\Delta^{k-i}(n-i)]\,\Delta^i[n^{k-1}B(n,\theta)] \\
&= (n-k)\,\Delta^k[n^{k-1}B(n,\theta)] + k\,\Delta^{k-1}[n^{k-1}B(n,\theta)] \\
&\qquad + \sum_{i=0}^{k-2}\binom{k}{i}[\Delta^{k-i}(n-i)]\,\Delta^i[n^{k-1}B(n,\theta)] \\
&= (n-k)\,\Delta^k[n^{k-1}B(n,\theta)] + k\,\Delta^{k-1}[n^{k-1}B(n,\theta)]\,.
\end{aligned}
$$

By the assumptions we have

$$
\lim_{n\to\infty}\frac{\Delta^k[n^{k-1}B(n,\theta)]}{\Delta^k n^{-k_1}} = \lim_{n\to\infty}\frac{\Delta^{k-1}[n^{k-1}B(n,\theta)]}{\Delta^{k-1}n^{-k_1}} \neq 0\,,
$$

and hence

$$
\begin{aligned}
\lim_{n\to\infty}&\frac{(n-k)\,\Delta^k[n^{k-1}B(n,\theta)]}{\Delta^{k-1}[n^{k-1}B(n,\theta)]} \\
&= \lim_{n\to\infty}\frac{(n-k)\,\Delta^k n^{-k_1}}{\Delta^{k-1}n^{-k_1}}\cdot\frac{\Delta^k[n^{k-1}B(n,\theta)]/(\Delta^k n^{-k_1})}{\Delta^{k-1}[n^{k-1}B(n,\theta)]/(\Delta^{k-1}n^{-k_1})} \\
&= \lim_{n\to\infty}\frac{(n-k)\,\Delta^k n^{-k_1}}{\Delta^{k-1}n^{-k_1}} \\
&= \lim_{n\to\infty}\frac{[\Delta^k(n^{-k_1})]/n^{-k-k_1}}{[\Delta^{k-1}(n^{-k_1})]/n^{-k-k_1+1}}\cdot\frac{n^{-k-k_1}(n-k)}{n^{-k-k_1+1}} \\
&= \frac{(-k_1)(-k_1-1)\cdots(-k_1-k+1)}{(-k_1)(-k_1-1)\cdots(-k_1-k+2)} \\
&= -k_1-k+1\,.
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
\lim_{n\to\infty}&\frac{E[J^{(k)}(\hat\theta)-\theta]}{E[J^{(k-1)}(\hat\theta)-\theta]} \\
&= \lim_{n\to\infty}\frac{(1/k!)\,\Delta^k[n^{k-1}B(n,\theta)]}{(1/(k-1)!)\,\Delta^{k-1}[n^{k-1}B(n,\theta)]} \\
&= \lim_{n\to\infty}\frac{1}{k}\left\{k+\frac{(n-k)\,\Delta^k[n^{k-1}B(n,\theta)]}{\Delta^{k-1}[n^{k-1}B(n,\theta)]}\right\} \\
&= \frac{1}{k}(k-k_1-k+1) \\
&= \frac{1}{k}(1-k_1)\,,
\end{aligned}
$$

and the conclusions of this theorem follow immediately from Definition 2.

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