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ALMOST ALL POINTS ARE EVENTUALLY PERIODIC WITH MINIMAL PERIOD 3

BY

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Abstract. We give a class of continuous functions f from [0, 1] into itself which are chaotic in the sense of Li and Yorke, but with the property that almost all (in the sense of Lebesgue) points of [0, 1] are mapped onto the period 3 point $x=0$ of f after only a finite number of iterations under f.

1. **Introduction.** The word "chaos" was introduced by Li and Yorke [8] to describe the dynamical complexity of a continuous map on an interval. This word was soon adopted by several other people. ([1], [4], [6], [7], and [9]). In [4], Coppel even lists 8 different statements about chaos, each is equivalent to the other. The simplest is that a continuous function on an interval is chaotic if it has a periodic point whose period is not an integral power of 2. But as indicated in [2, pp. 21–22], the chaotic behavior of such chaotic functions may be essentially unobservable. In fact, in [5], we introduce a class C of piecewise linear continuous functions f from [0, 1] onto itself which are chaotic in the sense of Li and Yorke, but with the property that almost all (in the sense of Lebesgue) points of [0, 1] are mapped onto the same unstable fixed point $x=1$ of f after only a finite number of iterations under f. Therefore, from a physical point of view, such chaotic functions are not chaotic after all.

We remark that the instability of the fixed point $x=1$ of the above f can be removed by slightly modifying the function f and the above class C can be extended, by using the results in [5] and Theorem 1 in [3], to contain continuous functions from [0, 1] into itself which are not necessarily piecewise linear or surjective. For

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example: Let $0 \leq a < b < c \leq s \leq t \leq 1$ with $a^2 + (s-c)^2 \neq 0$ and let $f(x)$ be a continuous function from [0, 1] into itself with the following five properties:

(i) $f(x) \geq c$ on $[0, a]$,

(ii) $f(x) = s = f(a)$ on $[c, t]$,

(iii) $f(b) = 0$,

(iv) $f(x)$ is linear on $[a, b]$ and $[b, c]$,

(v) $x > f(x) \geq c$ on $[t, 1]$ if $t < 1$.

Then the following can be shown to be true.

(1) If $s = t$, then the fixed point $x = s$ of $f(x)$ attracts almost all points of [0, 1].

(2) If $s < t$, then almost all points of [0, 1] are mapped onto the same fixed point $x = s$ of $f(x)$ after only a finite number of iterations under f. If we also have $c < s$, then $x = s$ is (super) stable.

By modifying the above example, it is also easy to construct a continuous function from [0, 1] into itself with finitely or infinitely many stable fixed points such that almost all points of [0, 1] are mapped onto one of these stable fixed points after only a finite number of iterations. For example: In the above example, let $0 < a$ and $c < s = 1$ and redefine $f(x)$ on $[0, a]$ by letting $f(x) = 0$ on $[0, a/2]$ and $f(x)$ linear on $[a/2, a]$. Then we "glue" together the "scaled-down" copies of this new f.

In this note, we give a class of continuous functions f from [0, 1] into itself which are chaotic in the sense of Li and Yorke, but with the property that almost all points of [0, 1] are mapped onto the period 3 point $x = 0$ of f after only a finite number of iterations under f. This result confirms the observation we made in [5, §4], but the open question posed there remains open. (This open question will be restated in §2 following the main theorem). We state our main result in Section 2 and give its proof in Section 3.

2. **Statement of Main Theorem.** Let $f(x)$ be a continuous function from [0, 1] into itself. For every x in [0, 1], let

$$L_f(x) = \bigcap_{n=1}^{\infty} CL\{f^m(x) \mid m \text{ is any integer greater than or equal to } n\},$$

where CL denotes the closure. It is obvious that $L_f(x)$ is compact and nonempty for every x in [0, 1]. In the following, we denote $L_{f_\alpha}(x)$ as $L_\alpha(x)$. Now we state the following theorem.

THEOREM. *Let $0 < a < b < 1$ and define $f(x)$: $[0, 1] \to [0, 1]$ as follows: $f(x) = -(x-a)/a$ on $[0, a]$, $f(x) = 0$ on $[a, b]$, and $f(x) = (x-b)/(1-b)$ on $[b, 1]$. For $0 \leq \alpha \leq 1$, let $f_\alpha(x) = \alpha f(x)$, $\alpha_1 = [b + \sqrt{b^2 + 4a(1-b)}]/2$, and $\alpha_2 = [b + \sqrt{b^2 + 4b(1-b)}]/2$. Then, for $\alpha_1 \leq \alpha \leq \alpha_2$, $x = \alpha$ is a period 3 point of $f_\alpha(x)$ and, for almost all points x_0 of [0, 1], there exists a natural number m (depending on x_0) such that $f_\alpha^m(x_0) = \alpha$. In particular, $L_\alpha(x) = L_\alpha(\alpha)$ for almost all x in [0, 1].*

The above result confirms the observation we made in [5, §4], but the open question posed there remains open. In the following, we include this open question for completeness.

CONJECTURE. *Let $f(x)$, $f_\alpha(x)$ and $L_\alpha(x)$ be defined as in the above theorem. Then for almost all α in the parameter space [0, 1], $L_\alpha(x) = L_\alpha(\alpha)$ for almost all x in [0, 1].*

3. Proof of Main Theorem.

For $\alpha_1 \leq \alpha \leq \alpha_2$, it is easy to see that $x = \alpha$ is a period 3 point of $f_\alpha(x)$, its preimage is 0 and the preimage of 0 is the interval $[a, b]$. So, from now on, we assume that $\alpha_1 \leq \alpha \leq \alpha_2$.

Let $S_1 = \{x \in [0, 1]\setminus[a, b] \mid a \leq f_\alpha(x) \leq b\}$ and, for $n = 2, 3, 4, \cdots$, let $S_n = \{x \in [0, 1]\setminus[a, b] \mid f_\alpha^n(x) \in [a, b]$ and $f_\alpha^k(x) \notin [a, b]$ for all $1 \leq k \leq n-1\}$. Then $S_1, S_2, S_3, \cdots$, are pairwise disjoint and for the rest of the proof, if suffices to show that

$$(b-a) + \sum_{n=1}^{\infty} \mu(S_n) = 1,$$

where μ is the Lebesgue measure.

For each natural number n, it is easy to see that there are finitely many disjoint closed intervals $I_{n1}, I_{n2}, \cdots; I_{nr_n}$ (I_{n1} and I_{nr_n} may consist of exactly one point) on which $f_\alpha^n(x)$ are constant and $f_\alpha^n(x)$ is linear on each component of the complement of all these closed intervals. Furthermore, these closed intervals I_{nj} satisfy the following two properties:

(a) $x<y$ for all x in I_{nj}, y in I_{nk}, and for all $1 \le j < k \le r_n$.

(b) $f_\alpha^n(x)$ is zero on I_{nk} exactly for all k odd, or for all k even.

Since we are only interested in the sizes of the sets S_n, we will use these constant values of $f_\alpha^n(x)$ on these intervals I_{nj} to represent the graph of $y=f_\alpha^n(x)$. For example: the graph of $y=f_\alpha^2(x)$ looks like Fig. 1. So, in our notations, $f_\alpha(\alpha)\,0\,\alpha\,0\,f_\alpha(\alpha)$ will represent the graph of $y=f_\alpha^2(x)$. For simplicity, let us use the numbers 1, 2, and 3 to denote 0, $f_\alpha(\alpha)$, and α respectively. Thus, under our new notations, 21312 will represent the graph of $y=f_\alpha^2(x)$.

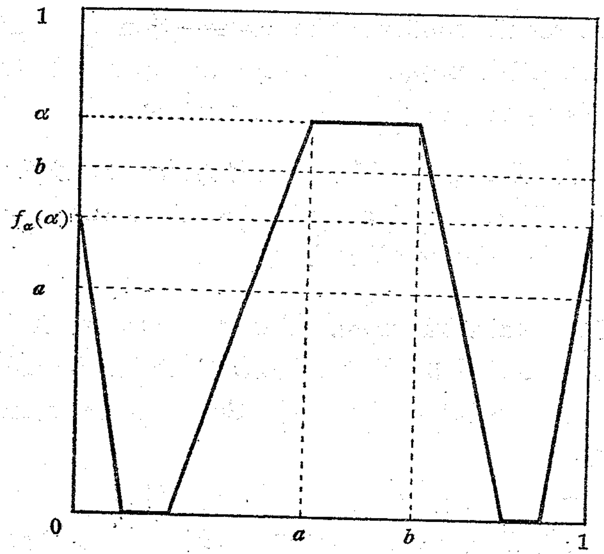

FIG. 1. The graph of $y=f_\alpha^2(x)$.

In Fig. 1, the graph of $y=f_\alpha^2(x)$, from left to right, has 4 line segments with nonzero slopes, say, m_1, m_2, m_3, and m_4 respectively. From now on, we will simply say that m_1 is the slope of 21, m_2 the slope of 13, m_3 the slope of 31, and m_4 the slope of 12.

Now let us study Fig. 1 closely and see how to obtain the representation for the graph of $y=f_\alpha^3(x)$ from that of $y=f_\alpha^2(x)$. In Fig. 1, the line segment between 1 and 3 (i. e., the second, from left to right, line segment with nonzero slope) crosses the lines $y = a$ and $y = b$. So $f_\alpha(x)$ "divides" this line segment into two line segments with nonzero slopes and a horizontal line segment which lies on the x-axis. Thus, by our notations, we obtain 312 from 13 (under $f_\alpha(x)$). We use $13 \to 312$ to indicate this. Similarly, we have $31 \to 213$, $12 \to 31$, and $21 \to 13$. Recall that, under $f_\alpha(x)$, we have $1 \to 3$, $2 \to 1$, and $3 \to 2$. But from above, we see that if 1 and 3 come together, we should insert 1 between the respective images of 1 and 3 under $f_\alpha(x)$. Therefore, we obtain that the representation for the graph of $y = f_\alpha^3(x)$, from that of $y = f_\alpha^2(x)$, is 1312131. Similarly, the representation for the graph of $y = f_\alpha^4(x)$, from that of $y = f_\alpha^3(x)$, is 31213131213, and so on. Note that in the representation for the graph of $y = f_\alpha^3(x)$, there are two 13s (and two 31s). They generally have different slopes. But, in the following, we will see that there is no need to distinguish them.

Now let $m_1 = \alpha/a$ and $m_2 = \alpha/(1-b)$. So $-m_1$ is the slope of $f_\alpha(x)$ on $[0, a]$ and m_2 is the slope of $f_\alpha(x)$ on $[b, 1]$. As discussed above, we have, under $f_\alpha(x)$, $13 \to 312$. It is easy to see that the slope of 31 (or 12 respectively) in 312 is equal to that of 13 times m_1 (or m_2 respectively). Similarly, from $31 \to 213$, we obtain that the slope of 21 (or 13 respectively) in 213 is equal to that of 31 times m_2 (or m_1 respectively). From $12 \to 31$, we obtain that the slope of 31 is equal to that of 12 times m_1. From $21 \to 13$, we obtain that the slope of 13 is equal to that of 21 times m_1.

For every natural number n, let $a_{n1} a_{n2} \cdots a_{nr_n}$, where $a_{nj} = 1, 2$, or 3, be the representation for the graph of $y = f_\alpha^n(x)$, and let w_n (v_n respectively) be the sum of the absolute values of the inverses of all slopes of 13 and 31 (12 and 21 respectively) in $a_{n1} a_{n2} \cdots a_{nr_n}$. Then it follows from above that, for $n \geq 1$, $w_{n+1} = (w_n + v_n)/m_1$, $v_{n+1} = (w_n)/m_2$, and $\mu(S_n) = (b-a)\, w_n + (f_\alpha(\alpha) - a)\, v_n$. Note that $w_1 = 1/m_1 + 1/m_2$ and $v_1 = 0$. So,

$$w_{n+1} = (w_n)/m_1 + (v_n)/m_1 = (w_n)/m_1 + (w_{n-1})/(m_1 m_2).$$

Summing both sides and let $w = \sum_{n=1}^{\infty} w_n$, we obtain that $w - w_1 - w_2 = (w - w_1)/m_1 + w/(m_1 m_2)$. Thus,

$$w - w/m_1 - w/(m_1 m_2) = w_1 + w_2 - (w_1)/m_1 = w_1 = 1/m_1 + 1/m_2$$

since $w_2 = (w_1)/m_1$. It follows that

$$
\begin{aligned}
w &= (m_1 + m_2)/(m_1 m_2 - m_2 - 1) \\
&= \alpha(a + 1 - b)/[\alpha^2 - a\alpha - a(1 - b)]
\end{aligned}
$$

since $m_1 = \alpha/a$ and $m_2 = \alpha/(1 - b)$. Therefore,

$$
\begin{aligned}
(b - a) &+ \sum_{n=1}^{\infty} \mu(S_n) \\
&= (b - a) + (b - a) \sum_{n=1}^{\infty} w_n + (f_a(\alpha) - a) \sum_{n=1}^{\infty} v_n \\
&= (b - a) + (b - a)\, w \\
&\qquad + [\alpha(\alpha - b)/(1 - b) - a] \sum_{n=1}^{\infty} (w_n)/m_2 \\
&= (b - a) + (b - a)\, w \\
&\qquad + [\alpha^2 - b\alpha - a(1 - b)]/(1 - b) \cdot [(1 - b)/\alpha]\, w \\
&= (b - a) + w[(b - a)\,\alpha + \alpha^2 - b\alpha - a(1 - b)]/\alpha \\
&= (b - a) + w[\alpha^2 - a\alpha - a(1 - b)]/\alpha \\
&= (b - a) + (a + 1 - b) \\
&= 1 .
\end{aligned}
$$

This completes the proof.

Added in Proof. The method used in this paper to prove the main theorem has been generalized to treat other related problems. See [10, 11, 12] for details.

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