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OSCILLATORY PROPERTIES OF THE SOLUTIONS OF A CLASS OF NEUTRAL TYPE INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The paper studies the oscillatory properties of the solutions of integro-differential equations of the type

$$x^{(n)}(t+\tau)+\lambda x^{(n)}(t)+\int_{\alpha(t)}^{\beta(t)} K(t,s,x(s))\,ds=0,$$

$n\geq 1$; $\lambda,\tau>0$; $\alpha(t),\beta(t)$; $\mathcal{G}_0\to \mathbf{R}^1$; $\mathcal{G}_0\subset \mathbf{R}^1$; $\alpha(t)\leq\beta(t)$ and suggests some ways to generalizing the obtained results.

In this paper we study the oscillatory properties of the solutions of integro-differential equations of the type

$$(1)\qquad x^{(n)}(t+\tau)+\lambda x^{(n)}(t)+\int_{\alpha(t)}^{\beta(t)} K(t,\ s,\ x(s))\,ds=0$$

where $n\geq 1$; $\lambda,\ \tau>0$; $\alpha,\ \beta:\mathcal{G}_0\to\mathbf{R}$; $\alpha(t)\leq\beta(t)$ for each $t\in\mathcal{G}_0$; $K:\mathbf{D}_k\to\mathbf{R}$, $\mathbf{D}_k\subseteq\mathbf{R}^3$; the letter $\mathcal{G}$ (possibly with some index) will denote an interval ending in $+\infty$ on the right and closed on the left. The functions $K(t,s,u)$, $\alpha(t)$ and $\beta(t)$ will be assumed, for the sake of simplicy, to be continuous, and besides $\limsup_{t\to+\infty}\beta(t)=+\infty$.

DEFINITION 1. The function $x:\mathcal{G}\to\mathbf{R}$, $\mathcal{G}\subseteq\mathcal{G}_0$ is said to be a regular solution of the equation (1) if: for each $t\in\mathcal{G}$, $s\in[\alpha(t),\ \beta(t)]$ we have $(t,\ s,\ x(s))\in\mathbf{D}_k$; there exists an interval $\mathcal{G}_1\subseteq\mathcal{G}$ such that $\alpha(\mathcal{G}_1)\subseteq\mathcal{G}$; the functions $x(t)$, $x'(t),\cdots,x^{(n-1)}(t)$ are locally absolutely continuous; the equation (1) is an identity for almost every $t\in\mathcal{G}$, for which $\alpha(t)\in\mathcal{G}$; $\sup\{t\,|\,x(t)\neq 0\}=+\infty$.

DEFINITION 2. The regular solution is said to be oscillating, if for every $t_1\in\mathcal{G}$ we have $\sup_{t\in[t_1,+\infty)}x(t)>0$ and

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$\inf_{t\in[t_1,+\infty)} x(t) < 0$.

We shall say that conditions (P) are fulfilled if the following conditions hold:

P1. $\operatorname{sgn} K(t, s, u) = \operatorname{sgn} u$ for every point $(t, s, u) \in D_k$.

P2. For every $u_0 > 0$ the following inequality holds:

$$\liminf_{\substack{(t,s,u)\in D_k,\, |u|\ge u_0 \\ t,s\to+\infty}} |K(t, s, u)| > 0 .$$

THEOREM 1. *Let* $\lim_{t\to+\infty}\alpha(t) = +\infty$ *and let for every sufficiently large* $t_0 \in \mathcal{G}_0$ *the following relation be valid:*

$$(2) \qquad \sum_{i=0}^{+\infty} \Big(\inf_{t_0+2i\tau\le s\le t_0+2(i+1)\tau} \operatorname{meas}\{t \mid t_0 \le t < +\infty,\ \alpha(t) \le s \le \beta(t)\}\Big) = +\infty .$$

Let conditions (P) *are fulfilled. Then for* n *even each regular solution* $x : \mathcal{G} \to \mathbf{R}$ *of the equation* (1) *oscillates, while for* n *odd each regular solution of the equation* (1) *either oscillates or, if not, tends to zero for* $t \to +\infty$.

Proof. Let us assume that there exists a non-oscillating solution $x(t)$ of the equation (1) and, for definitness, assume that $x(t) \ge 0$ for $t \in \mathcal{G}$.

We shall suppose that n is an even number. Than equation (1), for $t \ge t_0$, yields $x^{(n)}(t+\tau) + \lambda x^{(n)}(t) \le 0$ and hence there exists an odd number l, $1 \le l \le n$ such that for $t \ge t_0$ the following inequalities hold:

$$(3) \qquad \begin{aligned} &[x(t+\tau) + \lambda x(t)]^{(i)} \ge 0, && i = 0, \cdots, l \\ &(-1)^{l+i}[x(t+\tau) + \lambda x(t)]^{(i)} \ge 0, && i = l+1, \cdots, n \end{aligned}$$

(Cf. [1], Lemma 14.3, p. 289).

Then equation (1) yields

$$(4) \qquad \int_{t_0}^{+\infty} \Big(\int_{\alpha(t)}^{\beta(t)} K(t, s, x(s))\, ds\Big)\, dt < +\infty .$$

On the other hand, inequalities (3) imply $\liminf_{t\to+\infty}(x(t+\tau)+\lambda x(t)) \ge c > 0$ ($x(t)$ being a regular solution) and hence, in virtue of Lemma 1 [2] there exists $\gamma > 0$ such that the intersection of the set $E = \{t \mid x(t) \geq \gamma\}$ with each interval $[s, s + 2\tau]$, for s sufficiently large, has a measure not less than τ. Then condition P2 implies that there exists a point $t_1 \in \mathcal{G}$ such that for $t \geq t_1$, $s \geq t_1$, $u \geq \gamma$, $(t, s, u) \in D_k$, the number $\inf K(t, s, u) \equiv \delta$ is positive, hence, without loss of generality, we may consider that $t_1 = t_0$ and $x(t) \geq 0$ for all $s \geq \alpha(t)$, $t \geq t_0$.

Making use of Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{t_0}^{+\infty}\left(\int_{\alpha(t)}^{\beta(t)} K(t, s, x(s))\,ds\right)dt \\
&\quad \geq \int_{t_0}^{+\infty}\left(\int_{\max(\alpha(t), t_0)}^{\max(\beta(t), t_0)} K(t, s, x(s))\,ds\right)dt \\
&\quad = \int_{t_0}^{+\infty}\left(\int_{\{t \mid t_0 \leq t_0 < +\infty,\ \alpha(t) \leq s \leq \beta(t)\}} K(t, s, x(s))\,dt\right)ds \\
&\quad \geq \int_{[t_0, +\infty)\cap E}\left(\int_{\{t \mid t_0 \leq t < +\infty,\ \alpha(t) \leq s \leq \beta(t)\}} K(t, s, x(s))\,dt\right)ds \\
&\quad \geq \delta \int_{[t_0, +\infty)\cap E} \operatorname{meas}\{t \mid t_0 \leq t < +\infty,\ \alpha(t) \leq s \leq \beta(t)\}\,ds \\
&\quad = \delta \sum_{i=0}^{+\infty} \int_{[t_0+2i\tau,\ t_0+2(i+1)\tau]\cap E} \operatorname{meas}\{t \mid t_0 \leq t \leq +\infty, \\
&\qquad\qquad \alpha(t) \leq s \leq \beta(t)\}\,ds.
\end{aligned}$$

Condition (2) and the properties of the set E imply that the final expression is not less than

$$\delta \sum_{i=0}^{+\infty} \tau \cdot \inf_{t_0+2i\tau \leq s \leq t_0+2(i+1)\tau} \operatorname{meas}\{t \mid t_0 \leq t < +\infty,\ \alpha(t) \leq s \leq \beta(t)\} = +\infty$$

which contradicts inequality (4).

Let n be an odd number. Then there exists an even number l, $0 \leq l \leq n-1$ such that inequalities (3) ([1], Lemma 14.3, p. 289) hold. Therefore, either $\lim_{t\to+\infty}(x(t+\tau) + \lambda x(t)) = 0$ and hence $\lim_{t\to+\infty} x(t) = 0$, or $\lim_{t\to+\infty}(x(t+\tau) + \lambda x(t)) = c > 0$, which case is dealt with as the one when is even. Thus Theorem 1 is proved.

REMARK 1. It is easily verified that for condition (2) to hold, it is sufficient that for each sufficiently large number $t_0 \in \mathcal{G}_0$ we have

$$(5) \qquad \int_{t_0}^{+\infty}\left(\inf_{t_0 \leq \sigma \leq s} \operatorname{meas}\{t \mid t_0 \leq t < +\infty,\ \alpha(t) \leq \sigma \leq \beta(t)\}\,ds = +\infty\right).$$

REMARK 2. Condition (5), for example, holds if the function $\beta(t)$ satisfies the Lipschitz condition and $\beta(t) - \alpha(t) \geq c > 0$.

The condition $\alpha(+\infty) = +\infty$, which was essential for the proof of Theorem 1, excludes the important case $\alpha(t) = \text{const}$.

In order to include the latter case as well, condition (2) is to be strengthen. We give one of the possible variants for strengthening condition (2).

THEOREM 2. *Let the following conditions be fulfilled:*

1. *Conditions* (P) *hold.*
2. *For each* $M > 0$ *the following inequality holds*

$$\sup_{\substack{(t,s,u)\in D_k \\ |s|\leq M,\, |u|\leq M}} |K(t, s, u)| < +\infty. \tag{6}$$

3. *For each sufficiently large* $t_0 \in \mathcal{G}_0$ *the following relation holds*

$$\lim_{T\to+\infty}\left(T^{-1}\sum_{i=0}^{+\infty} \inf_{t_0+2i\tau\leq s\leq t_0+2(i+1)\tau} \operatorname{meas}\{t\,|\,t_0 \leq t \leq T,\ \alpha(t) \leq s \leq \beta(t)\}\right) = +\infty. \tag{7}$$

Then each regular solution $x(t)$ *of the equation* (1) *either oscillates or*

$$\liminf_{t\to+\infty} |x(t+\tau) + \lambda x(t)| = 0.$$

Proof. Let $x(t)$ be a non-oscillating solution of the equation (1) and, for definitness, suppose that $x(t) \geq 0$ for $t \geq t_0$. Let us suppose that $\liminf_{t\to+\infty}(x(t+\tau) + \lambda x(t)) > 0$. Then for $T > t_0$ we have the estimate

$$\begin{aligned} &\int_{t_0}^{T}\left(\int_{\alpha(t)}^{\beta(t)} K(t, s, x(s))\, ds\right) dt \\ &\quad\geq \int_{t_0}^{T}\left(\int_{[\alpha(t),\beta(t)]\cap[t_0,+\infty)} K(t, s, x(s))\, ds\right) dt \\ &\qquad + \int_{t_0}^{T}\left(\int_{[\alpha(t),\beta(t)]\setminus[t_0,+\infty)} K(t, s, x(s))\, ds\right) dt. \end{aligned} \tag{8}$$

The first integral in the right-hand side of inequality (8) is estimated as in the proof Theorem 1. It is not less than the expression

$$\delta\sum_{i=0}^{+\infty} \tau\cdot \inf_{t_0+2i\tau\leq s\leq t_0+2(i+1)\tau} \operatorname{meas}\{t\,|\,t_0 \leq t \leq T,\ \alpha(t) \leq s \leq \beta(t)\}.$$

As to the estimate of the second integral in the right-hand side of (8) it is sufficient to note that for $t \in [t_0, +\infty)$ the set $[\alpha(t), \beta(t)]\backslash[t_0, +\infty)$ is uniformly bounded and hence the set of values of $x(s)$ for s varying on $[\alpha(t), \beta(t)]\backslash[t_0, +\infty)$ is also uniformly bounded, whence, by aid of condition (6), we obtain for $T \to +\infty$ its estimate $O(T)$. Therefore, from inequality (8), taking into account (7), we obtain

$$\int_{t_0}^{+\infty}\left(\int_{\alpha(t)}^{\beta(t)} K(t, s, x(s))\, ds\right) dt = +\infty .$$

Hence, from equation (1), $[x(t+\tau) + \lambda x(t)] \xrightarrow{t\to+\infty} -\infty$ which contradicts the final nonnegativity of $x(t)$. Thus Theorem 2 is proved.

REMARK 3. For condition (7) to hold, it is sufficient that for all sufficiently large numbers $t_0 \in \mathcal{J}_0$ the folloing relation hold

$$(9) \qquad \lim_{T\to+\infty}\left(T^{-1}\int_{t_0}^{+\infty}(\inf_{t_0\le\sigma\le s} \text{meas}\,\{t \mid t_0 \le t \le T,\ \alpha(t) \le \sigma \le \beta(t)\}\, ds\right) = +\infty .$$

REMARK 4. If $\sup \alpha(t) < +\infty$ then condition (9) takes the following simple form:

$$\lim_{T\to+\infty}\left(T^{-1}\int_{[t_0, T]\cap[t \mid \beta(t)\ge t_0]} \beta(s)\, ds = +\infty\right),$$

for all sufficiently large t_0. It will be fulfilled in particular if $\lim_{t\to+\infty}\beta(t) = +\infty$.

REMARK 5. It is obvious that if one of the conditions (2), (5), (7) and (9) is fulfilled for some α and β then it is fulfilled for all functions α_1 and β_1, for which $\alpha_1(t) \le \alpha(t)$ $\beta(t) \le \beta_1(t)$ for each $t \in \mathcal{J}_0$.

To end up with we suggest some ways to generalize the obtained results.

For example, instead of equation (1), the following equation may be considered:

$$(10) \qquad x^{(n)}(t+\tau) + \lambda x^{(n)}(t) + \int_{\{s \mid (t,s)\in Q\}} K(t, s, x(s))\, ds = 0,$$

where $Q \subset \boldsymbol{R}^2$ is a given closed set whose intersection with each straight line $t=\text{const}$ is bounded. All of the above proved statements will naturally hold for this case as well, relations of the type $\alpha(t) \leq s \leq \beta(t)$ being substituted by the relation $(t, s) \in Q$.

Another way to generalize is the weakening of condition P2. Considering an equation of the type (10), the existence of some "singular" set $\tilde{Q} \subset Q$ may be assumed such that the condition P2 be fulfilled only for $(t, s) \in Q \backslash \tilde{Q}$. Besides, the set $\tilde{Q}$ should be required to fulfill the condition of not being "oversized". For example, $\tilde{Q} = Q \cap (\boldsymbol{R} \times B)$ may be set, where the set B satisfies the condition

$$\text{meas}(B \cap [s, s+2\tau]) \leq \text{const} < \tau, \qquad B \subseteq \boldsymbol{R},$$

for each $s \in \mathcal{G}_0$. Similarly, condition (2) may be generalized, etc., the proofs remaining unaltered.

References

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