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OSCILLATION AND NONOSCILLATION CRITERIA FOR A CLASS OF FORCED DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. The authors consider an n-th order forced nonlinear functional differential equation and give sufficient conditions for all solutions to be oscillatory. A nonoscillation theorem is also proved, and examples illustrating the results are given.

1. **Introduction.** In this paper we discuss the oscillation and nonoscillation of solutions of a class of forced higher order nonlinear functional equations with general deviating arguments. The results presented below generalize those of Graef, Grammatikopoulos and Spikes [4], and are new even for the reduced case of ordinary differential equations. Improvements over other known results are indicated and examples of the theorems are included.

The differential equation to be considered in this paper is

$$L_n x(t) + F(t, x(g_1(t)),\cdots, x(g_m(t))) = h(t), \tag{1}$$

where $n \geq 2$ and L_n is the disconjugate differential operator

$$L_n = \frac{1}{p_n(t)}\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1(t)}\frac{d}{dt}\frac{\cdot}{p_0(t)}. \tag{2}$$

We assume that p_i, g_j, $h : [a, \infty) \to R$ and F: $[a, \infty) \times R^m \to R$ are continuous, $p_i(t) > 0$, $0 \leq i \leq n$, and $g_j(t) \to \infty$ as $t \to \infty$, $1 \leq j \leq m$. In addition we assume that

$$\int_a^\infty p_i(t)dt = \infty \quad \text{for} \quad 1 \leq i \leq n-1. \tag{3}$$

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We introduce the notation:

$$(4)\qquad L_0\,x(t)=\frac{x(t)}{p_0(t)},\ L_i\,x(t)=\frac{1}{p_i(t)}\frac{d}{dt}L_{i-1}\,x(t),\ \ 1\le i\le n.$$

The domain $\mathfrak{D}(L_n)$ of L_n is defined to be the set of all functions $x:[T_x,\infty)\to R$ such that $L_i\,x(t)$, $0\le i\le n$, exist and are continuous on $[T_x,\infty)$. In what follows by a proper solution of equation (1) we mean a function $x\in\mathfrak{D}(L_n)$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)|:t\ge T\}>0$ for every $T\ge T_x$. We make the standing hypothesis that equation (1) does possess proper solutions. A proper solution $x(t)$ is called nonoscillatory if there exists $t_1\ge a$ such that $x(t)\ne 0$ for $t\ge t_1$; the solution is called oscillatory if for any given $t_1\ge a$ there exists t_2 and t_3 satisfying $t_1<t_2<t_3$, $x(t_2)>0$ and $x(t_3)<0$; and it is called a Z-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

Let $i_k\in\{1,\cdots,n-1\}$, $1\le k\le n-1$, and $t,s\in[a,\infty)$. We define

$$(5)\qquad \begin{aligned}&I_0=1\\ &I_k(t,s;\,p_{i_k},\cdots,p_{i_1})=\int_s^t p_{i_k}(r)\,I_{k-1}(r,s;\,p_{i_{k-1}},\cdots,p_{i_1})\,dr.\end{aligned}$$

For convenience of notation for $0\le i\le n-1$ we let

$$(6)\qquad J_i(t,s)=p_0(t)\,I_i(t,s;\,p_1,\cdots,p_i),\ \ J_i(t)=J_i(t,a).$$

2. A nonoscillation result.

THEOREM 1. *Suppose that* (3) *holds and let* $f:[a,\infty)\to R$ *be a continuous function. If for any* $T\ge a$ *we have*

$$(7)\qquad \begin{aligned}&F(t,u_1,\cdots,u_m)\le f(t)\quad \textit{and}\\ &\lim_{t\to\infty}\frac{1}{J_{n-1}(t,T)}\int_T^t J_{n-1}(t,r)\,p_n(r)[h(r)-f(r)]dr\\ &=+\infty\end{aligned}$$

or

$$(8)\qquad \begin{aligned}&F(t,u_1,\cdots,u_m)\ge f(t)\quad \textit{and}\\ &\lim_{t\to\infty}\frac{1}{J_{n-1}(t,T)}\int_T^t J_{n-1}(t,r)\,p_n(r)[h(r)-f(r)]\,dr\\ &=-\infty,\end{aligned}$$

then all proper solutions of equation (1) *are nonoscillatory.*

Proof. Let $x(t)$ be a proper solution of (1) and suppose that (7) holds Integrating the equation

$$L_n x(t) = h(t) - F(t, x(g_1(t)), \cdots, x(g_m(t)))$$

n-times from T to t, we obtain

$$x(t)/p_0(t) = \sum_{i=0}^{n-1} c_i I_i(t,T; p_1, \cdots, p_i) + \int_T^t I_{n-1}(t, r; p_1, \cdots, p_{n-1}) p_n(r) [h(r) - F(r, x(g_1(r)), \cdots, x(g_m(r)))] dr, \tag{9}$$

where c_i, $0 \leq i \leq n-1$, are constants. Since (3) implies that

$$\lim_{t\to\infty} \frac{I_i(t, T; p_1, \cdots, p_i)}{I_{n-1}(t, T; p_1, \cdots, p_{n-1})} = 0, \ 0 \leq i \leq n-2,$$

there exist constants $K > 0$ and $T' > T$ such that

$$\sum_{i=0}^{n-1} c_i I_i(t, T; p_1, \cdots, p_i) \geq -KI_{n-1}(t, T; p_1, \cdots, p_{n-1}) \tag{10}$$

for $t \geq T'$. From (9) and (10) we have

$$x(t) \geq \int_T^t J_{n-1}(t, r) p_n(r)[h(r) - f(r)] dr - KJ_{n-1}(t, T)$$

for $t \geq T'$. An application of condition (7) shows that $x(t)$ is eventually positive. The proof in case (8) holds is similar.

Theorem 1 generalizes Theorem 3 in [4]. We should also note that Theorem 1 can obviously be extended to equations where the functions F and h depend on $x, x', \cdots, x^{(n-1)}$ with each $x^{(i)}$ depending on several different deviating arguments $g_1, g_2, \cdots, g_m$. With this in mind, Theorem 1 would also generalize the nonoscillation results in [1], [2], and [6]. Nonoscillation results under much different types of conditions can be found for example in [3] and [5].

EXAMPLE 1. Consider the differential equation

$$\left(\frac{1}{t}\left(\frac{1}{t} x'(t)\right)'\right)' + \frac{t^{12\alpha}+1}{t^{12\alpha-1}} \frac{x^2(t^\alpha)}{1+x^2(t^\alpha)} = 49t, \quad t \geq 1, \tag{11}$$

where $\alpha > 0$ is a constant. It is a matter of straightforward computation to verify that condition (7) is satisfied for this equation. Therefore, by Theorem 1, all proper solutions of equation (11) are nonoscillatory. In fact, $x(t) = t^6$ is one such solution.

3. Oscillation Theorems.

THEOREM 2. *Suppose that* (3) *holds and*

$$u_1 F(t, u_1, \cdots, u_m) \geq 0 \text{ if all } u_j \geq 0 \text{ or } u_j \leq 0 \text{ for } j = 1, 2, \cdots, m. \tag{12}$$

If for all large T we have

$$\liminf_{t\to\infty} \frac{1}{J_{n-1}(t, T)} \int_T^t J_{n-1}(t, r)\, p_n(r)\, h(r)\, dr = -\infty \tag{13}$$

and

$$\limsup_{t\to\infty} \frac{1}{J_{n-1}(t, T)} \int_T^t J_{n-1}(t, r)\, p_n(r)\, h(r)\, dr = +\infty, \tag{14}$$

then all proper solutions of equation (1) *are oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory or Z-type solution of (1), say $x(t) \geq 0$ for $t \geq t_0 \geq a$. There exists $T \geq t_0$ such that $g_j(t) \geq t_0$ for $t \geq T$, $j = 1, \cdots, m$. From equation (1) and (12) we have

$$L_n x(t) = h(t) - F(t, x(g_1(t)), \cdots, x(g_m(t))) \leq h(t), \ t \geq T.$$

Proceeding as in the proof of Theorem 1, if we integrate the above inequality n-times, we see that there exist $K > 0$ and $T' > T$ such that

$$x(t) \leq \int_T^t J_{n-1}(t, r)\, p_n(r)\, h(r)\, dr + K J_{n-1}(t, T)$$

for $t \geq T'$. Condition (13) then yields a contraction to the assumption that $x(t) \geq 0$ for $t \geq T_0$. The proof in case $x(t) \leq 0$ for $t \geq t_0$ is similar.

EXAMPLE 2. Consider equation

$$\left(\frac{1}{t}\left(\frac{1}{t}\, x'(t)\right)'\right)' + \frac{e^{3\pi}}{t^5}\, x(te^{-\pi/2}) = 6t[6 \sin(\ln t) + 7 \cos(\ln t)] \tag{15}$$

for $t \geq 1$. All conditions of Theorem 2 are satisfied, and so every proper solution of equation (15) is oscillatory. It is easy to see that $x(t) = t^6 \sin(\ln t)$ is an oscillatory solution of (15).

Theorem 2 generalizes Theorem 4 in [4] and Theorem 2.1 in [8]. In the sense of the remarks following Theorem 1, it also extends Theorem 9 in [7].

THEOREM 3. *Suppose that* (3) *holds and*

$$(16) \quad \begin{aligned} u_1 F(t, u_1, \cdots, u_m) \leq 0 \quad &\text{if all} \quad u_j \geq 0 \\ &\text{or} \quad u_j \leq 0 \quad \text{for} \quad j = 1, 2, \cdots, m. \end{aligned}$$

If conditions (13) *and* (14) *hold for all large* T, *then every proper solution* $x(t)$ *of equation* (1) *such that*

$$(17) \qquad x(t) = O(J_{n-1}(t)) \quad \text{as} \quad t \to \infty$$

is oscillatory.

Proof. As in the proof of Theorem 2 we let $x(t)$ be a nonoscillatory or Z-type solution of (1), say $x(t) \geq 0$ for $t \geq t_0 \geq a$, and let us assume moreover that $x(t)$ satisfies (17). Letting $T \geq t_0$ be such that $x(g_j(t)) \geq 0$ for $t \geq T$, $j = 1, \cdots, m$, and integrating equation (1) we obtain

$$x(t) \geq \int_T^t J_{n-1}(t, r)\, p_n(r)\, h(r)\, dr - kJ_{n-1}(t, T)$$

for all sufficiently large t. Condition (14) then gives a contradiction to the boundedness of $x(t)/J_{n-1}(t)$. A similar proof holds if $x(t) \leq 0$ for $t \geq t_0$.

EXAMPLE 3. The conditions of Theorem 3 are satisfied for the differential equation

$$(18) \qquad \begin{aligned} &\left(\frac{1}{t}\left(\frac{1}{t}\, x'(t)\right)'\right)' - \frac{1}{t^5 e^{3\pi}}\, x(te^{\pi/2}) \\ &\qquad = 6t[6 \sin(\ln t) + 7 \cos(\ln t)], \ t \geq 1\,. \end{aligned}$$

It follows that every proper solution $x(t)$ of (18) satisfying $x(t) = O(t^4)$ as $t \to \infty$ is oscillatory. We note that equation (18) has an unbounded oscillatory solution $x(t) = t^6 \sin(\ln t)$ such that $x(t)/t^4$ is unbounded.

The final theorem in this paper is an oscillation result which places conditions of F which are similar to those used to obtain nonoscillation in Theorem 1.

THEOREM 4. *Suppose that* (3) *holds and there are continuous functions* $f_1, f_2\colon [a, \infty) \to R$ *such that*

$$f_1(t) \le F(t, u_1, \cdots, u_m) \le f_2(t) \quad \text{for all} \quad (t, u_1, \cdots, u_m) \in [a, \infty) \times R^m. \tag{19}$$

If for all large T *we have*

$$\liminf_{t\to\infty} \frac{1}{J_{n-1}(t, T)} \int_T^t J_{n-1}(t, r)\, p_n(r)[h(r) - f_1(r)]\, dr = -\infty \tag{20}$$

and

$$\limsup_{t\to\infty} \frac{1}{J_{n-1}(t, T)} \int_T^t J_{n-1}(t, r)\, p_n(r)[h(r) - f_2(r)]\, dr = +\infty, \tag{21}$$

then any proper solution $x(t)$ *of equation* (1) *is oscillatory and* $x(t)/p_0(t)$ *is unbounded.*

Proof. Let $x(t)$ be a proper solution of (1) defined for $t \ge t_0 \ge a$. From (1) and (19) we have

$$h(t) - f_2(t) \le L_n x(t) \le h(t) - f_1(t)$$

for $t \ge T$, where $T \ge t_0$ is such that $g_j(t) \ge t_0$ for $t \ge T$, $j = 1, \cdots, m$. Integrating the above inequality n-times we obtain that there exist $K > 0$ and $T' \ge T$ such that

$$\int_T^t J_{n-1}(t, r)\, p_n(r)[h(r) - f_2(r)]\, dr - KJ_{n-1}(t, T) \le x(t) \le \int_T^t J_{n-1}(t, r)\, p_n(r)[h(r) - f_1(r)]\, dr + KJ_{n-1}(t, T)$$

for $t \ge T'$. Conditions (20) and (21) imply that $x(t)$ satisfies

$$\limsup_{t\to\infty} \frac{x(t)}{J_{n-1}(t)} = +\infty \quad \text{and} \quad \liminf_{t\to\infty} \frac{x(t)}{J_{n-1}(t)} = -\infty\,.$$

If follows that $x(t)$ is oscillatory and $x(t)/p_0(t)$ is unbounded.

EXAMPLE 4. Consider the equation

$$\left(\frac{1}{t}\left(\frac{1}{t}x'(t)\right)'\right)' - \frac{(1+t^{12})^{1/2}}{t^5 e^{3\pi}} \frac{x(te^{\pi/2})}{[1+x^2(t)+e^{-6\pi}x^2(te^{\pi/2})]^{1/2}} = 6t[6\cos(\ln t) - 7\sin(\ln t)],\ t \geq 1. \tag{22}$$

As easily verified, Theorem 4 is applicable to equation (22), so that all of its proper solutions are oscillatory and unbounded. One such solution is $x(t) = t^6 \cos(\ln t)$.

Theorems 3 and 4 generalize Theorems 5 and 6 in [4] respectively. In the spirit of the remarks following Theorem 1 we also have that Theorem 3 generalizes Theorem 2 in [2] and Theorem 2.2 in [8].

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