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ON BOUNDARY VALUE PROBLEMS FOR $y''' = f(x, y, y' y'')$

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The purpose of this paper is to clarify certain statements made in [6]. We shall show that under the same restrictions the results which can be deduced from our earlier work [1 – 4] are better (non-comparable) than those they claim to have "more liberal" or better than ours.

For the third order nonlinear differential equation

$$y''' = f(x, y, y', y'') \tag{1}$$

we assume as in [6] some of these conditions

(a) f is continuous on $[x_1, x_3] \times R^3$

(b) the solutions of initial value problems for (1) exist uniquely on $[x_1, x_3]$

(c) f satisfies condition A at x_2 in (x_1, x_3), that is

(i) $y_1 \geq y_2$, $z_1 < z_2$ implies $f(x, y_1, z_1, w) < f(x, y_2, z_2, w)$ on $(x_1, x_2]$

(ii) $y_1 \leq y_2$, $z_1 < z_2$ implies $f(x, y_1, z_1, w) < f(x, y_2, z_2, w)$ on $[x_2, x_3)$

(d) f satisfies Lipschitz condition

$$|f(x, y_1, z_1, w_1) - f(x, y_2, z_2, w_2)| \leq L_0|y_1 - y_2| + L_1|z_1 - z_2| + L_2|w_1 - w_2| \text{ on } [x_1, x_3] \times R^3$$

THEOREM 1 [6]. *Assume the following*:

(i) *Conditions* (a) – (c)

(ii) *for every* $m \in R$ *there exist solutions of* (1) *satisfying*

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(2) $$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_2) = m$$

as well as solutions satisfying

(3) $$y(x_2) = y_2, \quad y'(x_2) = m, \quad y(x_3) = y_3.$$

Then, the boundary value problem

(4) *Equation* (1), $y(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3$

has a unique solution.

THEOREM 2 [2]. *Assume* (a) *and* (d) *are satisfied. Then, each of the boundary value problems* (1), (2); (1), (3) *has a unique solution provided*

(5) $$\frac{3}{160} L_0 h_i^3 + \frac{33}{320} L_1 h_i^2 + \frac{3}{8} L_2 h_i < 1$$

where $h_i = x_{i+1} - x_i$, $i = 1, 2$.

REMARK 1. The restriction (5) on the length of the interval is not comparable with

(6) $$\frac{1}{60} L_0 h_i^3 + \frac{1}{6} L_i h_i^2 + \frac{2}{3} L_2 h_i < 1$$

obtained in [Theorem 3.1, 6].

THEOREM 3. *Assume the following:*

(i) *conditions* (a), (c) *and* (d)

(ii) *inequality* (5) *is satisfied.*

Then, the boundary value problem (4) *has a unique solution.*

Proof. The proof is immediate from theorems 1 and 2.

REMARK 2. In Theorem 3 the condition (ii) can be replaced by (ii)′ inequality (6) is satisfied [Theorem 4.1, 6]. Thus as in remark 1, we find "under the same conditions on f i.e. (a), (c) and (d)" the length restriction obtained in Theorem 3 is non-comparable with the estimate they have in their Theorem 4.1. Hence, their claim that they obtain "more liberal" interval of existence and uniqueness is not correct, in fact it should be "non-comparable".

In [9] Krishnamoorthy announced the following:

THEOREM 4. *Assume the following*:

(i) *conditions* (a) *and* (d)

$$(7)\quad \text{(ii)}\qquad \frac{9}{160} L_0 h_i^3 + \frac{1}{6} L_1 h_i^2 + \frac{1}{2} L_2 h_i < 1,\qquad h_i = x_{i+1} - x_i \quad (i = 1, 2).$$

Then, the boundary value problem (4) *has a unique solution.*

The proof of this result is given in [3].

REMARK 3. Theorem 4 does not require additional condition A on the function f as in Theorem 3 or in their Theorem 4.1. Thus, their claim on the restriction on the length of the interval when f is independent of y' and y'' is better, that is

$$(8)\qquad \frac{1}{60} L_0 h_i^3 < 1,\quad h_i = x_{i+1} - x_i \quad (i = 1, 2)$$

compare to

$$(9)\qquad \frac{9}{160} L_0 h_i^3 < 1,\quad h_i = x_{i+1} - x_i \qquad (i = 1, 2)$$

is subject to additional condition A. However, for this case in [10, p. 86] Krishnamoorthy has pointed out same restriction as (8).

THEOREM 5 [1]. *Let f be independent of y' and y'' and*, (a) *and* (d) *are satisfied. Then, each of the boundary value problems* (1), (2); (1), (3) *has a unique solution provided $h_i < l_1$, $h_i = x_{i+1} - x_i$ $(i = 1, 2)$ where l_1 is the first positive root of the equation*

$$2 \sin\left(\frac{\sqrt{3}}{2} L_0^{1/3} l - \frac{\pi}{6}\right) + \exp\left(-\frac{3}{2} L_0^{1/3} l\right) = 0\,.$$

This result is best possible.

THEOREM 6. *Assume the following*:

(i) *f is independent of y' and y''*

(ii) *conditions* (a), (c) *and* (d)

(iii) *$h_i < l_1$, $h_i = x_{i+1} - x_i$ $(i = 1, 2)$ and l_1 is same as in Theorem* 5.

Then, the boundary value problem (4) *has a unique solution.*

Proof. The proof is obvious from theorems 1 and 5.

REMARK 4. Thus Theorem 6 provides best possible result in this particular case. An easy computation provides $L_0 h_i \gtrsim 4.234\cdots$ compare to $L_0 h_i \gtrsim 3.9148\cdots$, $h_i = x_{i+1} - x_i$ $(i = 1, 2)$ from (8), In fact, it is possible to find best possible result even if f satisfies generalized Lipschitz condition [4].

REMARK 5. For f continuous and satisfies the Lipschitz condition on $(a, b) \times R^3$ where $[x_1, x_3] \subset (a, b)$, Jackson [7, 8] has obtained best possible length estimates for the existence and uniqueness of each boundary value problem (1), (2); (1), (3); (4). The question whether (a, b) can be replaced by $[x_1, x_3]$ remains undecided. This is possible for the particular cases as in Theorem 5, also for several other cases see [4].

THEOREM 7. *Assume the following*:

(i) *condition* (a)

(ii) *for all* $(x, y, z, w) \in [x_1, x_3] \times R^3$

$$|f(x, y, z, w)| \leq L + L_0|y| + L_1|z| + L_2|w|.$$

Then, each of the boundary value problems (1), (2); (1), (3) *has a solution provided*

$$\frac{2}{81} L_0 h_i^3 + \frac{1}{6} L_1 h_i^2 + \frac{2}{3} L_2 h_i < 1, \qquad (10)$$

$$h_i = x_{i+1} - x_i \quad (i = 1, 2).$$

This result is a particular case of Theorem 6 proved in [5].

THEOREM 8. *Assume the following*:

(i) *conditions* (a), (b) *and* (c)

(ii) *condition* (ii) *of Theorem* 7

(iii) *inequality* (10) *is satisfied.*

Then, the boundary value problem (4) *has a unique solution.*

Proof. The proof follows from theorems 1 and 7.

REMARK 6. In Theorem 8 we are able to relax Lipschitz condition however (10) is weaker than (5) or (6) and it will be desirable to know if (10) can be improved further.

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