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TWO THEOREMS ON CR-SUBMANIFOLDS OF KAEHLER MANIFOLDS

BY

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A. Bejancu defined the CR-submanifolds of Kaehler manifolds as follows [1]: A submanifold N of a Kaehler Manifold M is called a CR-submanifold if there exists on N a nontrivial holomorphic distribution $\mathcal{D}$ such that its orthogonal complement $\mathcal{D}^\perp$ is totally real (anti-invariant) in M. Thus $J\mathcal{D}_p = \mathcal{D}_p$ and $J\mathcal{D}_p^\perp \subset T_p^\perp N$, where $T_p^\perp N$ is the normal space of N at p.

D. E. Blair and B. Y. Chen proved many interesting fundamental theorems on CR-submanifolds of Kaehler manifolds in their paper [2]. Among them are the following ones:

THEOREM A. *Let N be a submanifold of a Kaehler manifold M and $\mathcal{D}_p$ the maximal holomorphic subspace of $T_p N$. Suppose* $\dim \mathcal{D}_p =$ *constant. Then the holomorphic distribution $\mathcal{D}$ is integrable if and only if the second fundamental form σ satisfies $\sigma(X, JY) = \sigma(JX, Y)$ for all vector fields X and Y belonging to $\mathcal{D}$.*

THEOREM B. *Let N be a submanifold of a complex space form $M(c)$ with $c \neq 0$. Then N is a CR-submanifold if and only if the maximal holomorphic subspaces $\mathcal{D}_p = T_p N \cap J(T_p N)$, $p \in N$ define a nontrivial differentiable distribution $\mathcal{D}$ on N such that*

$$R(\mathcal{D}, \mathcal{D}; \mathcal{D}^\perp, \mathcal{D}^\perp) = 0,$$

where $\mathcal{D}^\perp$ denotes the orthogonally complementary distribution of $\mathcal{D}$ in N. Here R denotes the curvature tensor of the Kaehler manifold M.

These two theorems are concerned with the maximal holomorphic subspaces $\mathcal{D}_p$ of $T_p N$.

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In this short note, we intend to supplement the corresponding theorems which are related to the maximal anti-invariant subspaces of the tangent spaces of the submanifold:

THEOREM A′. *Let N be a submanifold of a Kaehler manifold M and $\mathcal{D}_p^\perp$ the maximal anti-invariant subspace of T_pN. Suppose* $\dim \mathcal{D}_p^\perp =$ *constant. Then the anti-invariant distribution $\mathcal{D}^\perp$ is integrable if and only if the second fundamental tensor A satisfies $A_{JZ}W = A_{JW}Z$ for all vector fields Z and W belonging to $\mathcal{D}^\perp$.*

THEOREM B′. *Let N be a submanifold of a complex space form $M(c)$ with $c \neq 0$. Then N is a CR-submanifold if and only if the maximal anti-invariant subspaces $\mathcal{D}_p^\perp \subset T_pN$, $p \in N$ define a nontrivial differentiable distribution $\mathcal{D}^\perp$ on N such that*

$$R(\mathcal{D}, J\mathcal{D}; \mathcal{D}^\perp, \mathcal{D}) = 0, \qquad R(\mathcal{D}, J\mathcal{D}; \mu, \mathcal{D}) = 0,$$

where $\mathcal{D}$ denotes the orthogonally complementary distribution of $\mathcal{D}^\perp$ in N, and μ denotes the distribution for which μ_p is the orthogonal complement of $J\mathcal{D}_p^\perp$ in $T_p^\perp N$.

1. **Proof of the Theorem A′.** In the sequel, we follow the notations of the paper [2] of Blair-Chen. We denote by ∇ and $\tilde{\nabla}$ respectively the covariant differentiation with respect to the metric on N and the metric on M. Let $\nabla^\perp$ denote the covariant differentiation of the linear connection of the normal bundle of N in M. Let $Z, W \in \mathcal{D}_p^\perp$ then $JZ, JW \in T_p^\perp(N)$. Since $\tilde{\nabla}J = 0$, by Gauss and Weingarten formulas, we have

$$\begin{aligned} -A_{JW}Z + \nabla_Z^\perp JW &= \tilde{\nabla}_Z JW \\ &= J\tilde{\nabla}_Z W = J(\nabla_Z W + \sigma(Z, W)). \end{aligned}$$

That is

$$-A_{JW}Z + \nabla_Z^\perp JW = J\nabla_Z W + J\sigma(Z, W). \tag{1}$$

This implies that

$$\begin{aligned} (A_{JZ}W - A_{JW}Z) + (\nabla_Z^\perp JW - \nabla_W^\perp JZ) \\ = J(\nabla_Z W - \nabla_W Z) = J[Z, W]. \end{aligned} \tag{2}$$

Suppose that the distribution $\mathcal{D}^\perp$ is integrable, and let N' be the integral submanifold through p. Then $[Z, W] \in \mathcal{D}_p^\perp$, thus

$J[Z, W] \in T_p^\perp(N)$. Since $A_{JZ}W - A_{JW}Z$ is tangential to N and $(\nabla_Z^\perp JW - \nabla_W^\perp JZ)$ is normal to N, it follows that (see Yano-Kon [3])

$$A_{JZ}W = A_{JW}Z. \tag{3}$$

Conversely, if (3) holds, then since $(\nabla_Z^\perp JW - \nabla_W^\perp JZ)$ is normal to N, (2) implies that $J[Z, W]$ is normal to N. Assume that $[Z, W] \notin \mathcal{D}_p^\perp$, then $J[Z, W] \notin T_p(N)^\perp$ since $\mathcal{D}_p^\perp$ is the maximal anti-invariant subspace of $T_p(N)$. Thus we can conclude that $[Z, W] \in \mathcal{D}_p^\perp$ and $\mathcal{D}^\perp$ is integrable.

Since it is known that for a CR-submanifold of a Kaehler manifold the anti-invariant distribution $\mathcal{D}^\perp$ is integrable [2], we have the following:

COROLLARY. *Let N be a CR-submanifold of a Kaehler manifold M, then* $A_{JZ}W = A_{JW}Z$ *holds for all vector fields Z and W belonging to* $\mathcal{D}^\perp$, *where A denotes the second fundamental form of N in M.*

2. **Proof of Theorem B′.** Let X, Y, W be any tangent vector fields of M. Since M is a complex space form $M = M(c)$, $c \neq 0$ we have

$$\begin{aligned} R(X, JY)W = \frac{c}{4}\{ & g(JY, W)X - g(X, W)JY \\ & - g(Y, W)JX + g(JX, W)Y \\ & - 2g(X, Y)JW\}. \end{aligned} \tag{4}$$

Suppose N is a CR-submanifold. Then for any vector fields X, Y, Z in $\mathcal{D}$, W in $\mathcal{D}^\perp$ and $\zeta \in \mu$, (4) gives

$$\begin{cases} R(X, JY)W = -\dfrac{c}{2}g(X, Y)JW, \\ R(X, JY)\zeta = -\dfrac{c}{2}g(X, Y)J\zeta. \end{cases} \tag{5}$$

Thus

$$(6)\qquad \begin{cases} R(X, JY;\ W, Z) = -\dfrac{c}{2} g(X, Y)\, g(JW, Z) = 0, \\ R(X, JY;\ \zeta, Z) = -\dfrac{c}{2} g(X, Y)\, g(J\zeta, Z) \\ \qquad\qquad = \dfrac{c}{2} g(X, Y)\, g(\zeta, JZ) = 0. \end{cases}$$

That is,

$$(7)\qquad R(\mathcal{D}, J\mathcal{D};\ \mathcal{D}^{\perp}, \mathcal{D}) = 0, \qquad R(\mathcal{D}, J\mathcal{D};\ \mu, \mathcal{D}) = 0.$$

Conversely, if the maximal anti-invariant subspaces $\mathcal{D}_p^{\perp}$ define a nontrivial differentiable distribution such that (7) holds, then

$$R(X, JX)\, W = -\frac{c}{2} g(X, X)\, JW,$$

since

$$(8)\qquad g(JX, W) = g(J^2 X, JW) = -g(X, JW) = 0.$$

Hence

$$(9)\qquad \begin{aligned} 0 = R(X, JX;\ W, Z) &= -\frac{c}{2} g(X, X)\, g(JW, Z) \\ &= \frac{c}{2} g(X, X)\, g(W, JZ). \end{aligned}$$

Thus

$$(10)\qquad g(W, JZ) = 0$$

for all $W \in \mathcal{D}^{\perp}$ and $Z \in \mathcal{D}$. This means that

$$(11)\qquad J\mathcal{D} \perp \mathcal{D}^{\perp}.$$

Furthermore, from (4) we also have

$$(12)\qquad \begin{aligned} & g(R(X, JY)\,\zeta, Z) \\ & \quad = \frac{c}{4} \{ g(JY, \zeta)\, g(X, Z) + g(JX, \zeta)\, g(Y, Z) \\ & \qquad\qquad - 2g(X, Y)\, g(J\zeta, Z) \}. \end{aligned}$$

Thus the second equation of (7) implies that

$$(13)\qquad 0 = R(X, JX;\ \zeta, X) = cg(X, X)\, g(JX, \zeta).$$

This implies that

$$J\mathscr{D} \perp \mu. \tag{14}$$

Obviously $J\mathscr{D} \perp J\mathscr{D}^{\perp}$. Therefore $J\mathscr{D} \subset TN$ and $J\mathscr{D} = \mathscr{D}$. Thus N is a CR-submanifold.

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