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NECESSARY CONDITIONS FOR THE EXISTENCE OF SOLUTIONS OF MULTI-POINT BOUNDARY VALUE PROBLEMS

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1. **Introduction.** Recently for the existence and uniqueness of multi-point boundary value problems for ordinary differential equations several sufficient conditions have been obtained e. g. see [2, 4, 5, 9–11 and references therein]. In this paper we shall provide necessary conditions for the existence of solutions. The obtained results are sharper and generalize the results of Beesack [6], Das *et. al.* [7].

2. **Preliminary results.**

LEMMA 1. [3, 8]. *Let $g(t, s)$ be the Green's function for the boundary value problem*

$$x^{(n)}(t) = 0 \tag{2.1}$$

$$\begin{gathered} x(a_i) = x'(a_i) = \cdots = x^{(k_i)}(a_i) = 0 \qquad (1 \le i \le r) \\ a_1 < a_2 < \cdots < a_r, \quad 0 \le k_i, \quad \sum_{i=1}^{r} k_i + r = n. \end{gathered} \tag{2.2}$$

Then,

$$\int_{a_1}^{a_r} |g(t, s)|\, ds = \frac{1}{n!} \prod_{i=1}^{r} |t - a_i|^{k_i+1}. \tag{2.3}$$

(*For a particular case $r = n$ also see* [7].)

LEMMA 2. [6]. *Let $g(t,s)$ be as in lemma* 1. *Then, for $a_1 \le t, s \le a_r$,*

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$$|g(t,s)| \leq \frac{1}{(n-1)!(a_r-a_1)} \prod_{i=1}^{r} |t-a_i|^{k_i+1}. \tag{2.4}$$

LEMMA 3. *Let $\alpha = \min\{k_1, k_r\}$; then for $a_1 \leq t \leq a_r$,*

$$\prod_{i=1}^{r} |t-a_i|^{k_i+1} \leq \frac{(n-\alpha-1)^{n-\alpha-1}}{n^n} (\alpha+1)^{\alpha+1}(a_r-a_1)^n. \tag{25}$$

The proof of lemma 3 is contained in the proof of theorem 2.1 [1].

LEMMA 4. *Let $g(t, s)$ be as in lemma 1. Then,*

$$\int_{a_1}^{a_r} |g(t,s)|\, ds \leq \frac{1}{n!} \frac{(n-\alpha-1)^{n-\alpha-1}}{n^n} \cdot (\alpha+1)^{\alpha+1}(a_r-a_1)^n,\ a_1 \leq t \leq a_r \tag{2.6}$$

and

$$|g(t,s)| \leq \frac{1}{(n-1)!} \frac{(n-\alpha-1)^{n-\alpha-1}}{n^n} \cdot (\alpha+1)^{\alpha+1}(a_r-a_1)^{n-1},\ a_1 \leq t,\ s \leq a_r \tag{2.7}$$

where $\alpha = \min\{k_1, k_r\}$.

The proof follows from lemmas 1–3.

REMARK. Since $(n-x-1)^{n-x-1}(x+1)^{x+1}$ where $0 \leq x \leq \alpha = \min\{k_1, k_r\}$ is decreasing function of x, we find that $(n-\alpha-1)^{n-\alpha-1}(\alpha+1)^{\alpha+1} \leq (n-1)^{n-1}$ with equality only if $\alpha = 0$. Thus, the inequality (2.5) is an improvement over (2.13) of [6], and (2.7) over the second half of (1.4) of [6] if $\alpha \neq 0$, and reduces to same if $\alpha = 0$.

LEMMA 5. *Let $g(t, s)$ be as in lemma 1. Then,*

$$\left(\int_{a_1}^{a_r} |g(t,s)|^2\, ds\right)^{1/2} \leq \frac{1}{(n-1)!} \frac{(n-\alpha-1)^{n-\alpha-1}}{\sqrt{n}\, n^n} \cdot (\alpha+1)^{\alpha+1}(a_r-a_1)^{n-1/2},\ a_1 \leq t \leq a_r \tag{2.8}$$

where $\alpha = \min\{k_1, k_r\}$.

Proof. From lemma 2, we have

$$\left(\int_{a_1}^{a_r} |g(t, s)|^2 ds\right)^{1/2}$$

$$\leq \left(\frac{1}{(n-1)!(a_r - a_1)} \prod_{i=1}^{r} |t - a_i|^{k_i+1} \int_{a_1}^{a_r} |g(t, s)| ds\right)^{1/2}. \tag{2.9}$$

Using lemma 1 in (2.9), we find

$$\left(\int_{a_1}^{a_r} |g(t, s)|^2 ds\right)^{1/2}$$

$$\leq \left(\frac{1}{(n-1)!(a_r - a_1)} \times \frac{1}{n!} \prod_{i=1}^{r} |(t - a_i)|^{2k_i+2}\right)^{1/2}$$

$$\leq \frac{1}{(n-1)! \sqrt{n} \sqrt{(a_r - a_1)}} \prod_{i=1}^{r} |t - a_i|^{k_i+1}$$

and now the result (2.8) follows from lemma 3.

LEMMA 6. [1. Theorem 2.1]. *Let* $x(t) \in C^{(n)}[a_1, a_r]$ *satisfy* (2.2). *Then,*

$$|x^{(k)}(t)| \leq C_{n,k}\, m(a_r - a_1)^{n-k}, \qquad 0 \leq k \leq n-1 \tag{2.10}$$

where $m = \max_{a_1 \leq t \leq a_r} |x^{(n)}(t)|$, *and*

$$C_{n,k} = \frac{1}{(n-k)!} \frac{(n-\alpha-1)^{n-\alpha-1}}{(n-k)^{n-k}} (\alpha - k + 1)^{\alpha-k+1},$$

$$0 \leq k \leq \alpha$$

$$C_{n,\alpha+k} = \frac{k}{(n-\alpha)(n-\alpha-k)!}, \qquad 1 \leq k \leq n-\alpha-1$$

where $\alpha = \min\{k_1, k_r\}$.

3. Main results.

THEOREM 3.1. *Let the boundary value problem*

$$x^{(n)} = f(t, x, x', \cdots, x^{(n-1)}) \tag{3.1}$$

with (2.2), *have a solution where* f *is continuous on* $[a_1, a_r] \times R^n$ *and satisfies*

$$|f(t, x, x', \cdots, x^{(n-1)})| \leq b(t)|x| \tag{3.2}$$

where $b(t)$ *is a nonnegative continuous function with* $b(t) \not\equiv 0$ *on* $a_1 \leq t \leq a_r$. *Then,*

$$1<\frac{1}{(n-1)!}\frac{(n-\alpha-1)^{n-\alpha-1}}{n^n}\cdot(\alpha+1)^{\alpha+1}(a_r-a_1)^{n-1}\int_{a_1}^{a_r}b(s)\,ds \tag{3.3}$$

where $\alpha=\min\{k_1, k_r\}$.

Proof. Following Beesack [6], $x(t)$ satisfies the integral equation

$$x(t)=\int_{a_1}^{a_r}g(t,s)\,f(s,x(s),x'(s),\cdots,x^{(n-1)}(s))\,ds \tag{3.4}$$

where $g(t, s)$ is as in lemma 1. Using (3.2) in (3.4), we find

$$|x(t)|\leq\int_{a_1}^{a_r}|g(t,s)|\,b(s)\,|x(s)|\,ds. \tag{3.5}$$

Let $t\in[a_1, a_r]$ be a point where $|x(t)|$ attains its maximum, then from (3.5)

$$1<\int_{a_1}^{a_r}|g(t,s)|b(s)\,ds \tag{3.6}$$

and now (3.3) follows from (2.7).

REMARK. Inequality (3.3) is an improvement over (3.6) of [6] if $\alpha\neq 0$ and reduces to same if $\alpha=0$.

REMARK. As in [6], we take $f=x^{(n)}(t)\,x^{-1}(t)x$ in (3.1) and $b(t)=|x^{(n)}(t)\,x^{-1}(t)|$ in (3.2) and obtain from (3.3) the following extension of the Liapounoff oscillation criterion: if $x^{(n)}(t)$ and $x^{(n)}(t)\,x^{-1}(t)$ are continuous on $a_1\leq t\leq a_r$ and $x(t)$ has n zeros (counting multiplicity) including a_1 and a_r on $a_1\leq t\leq a_r$, then

$$\int_{a_1}^{a_r}|x^{(n)}(t)\,x^{-1}(t)|\,dt>\frac{(n-1)!\,n^n}{(n-\alpha-1)^{n-\alpha-1}(\alpha+1)^{\alpha+1}(a_r-a_1)^{n-1}}. \tag{3.7}$$

Inequality (3.7) is sharper than (3.10) of [6] if $\alpha\neq 0$ and if $\alpha=0$ it reduces to same. If $n=2m$, $r=2$, $\alpha=m-1$ then, (3.7) is same as (3.12) of [6].

COROLLARY 3.2. *In theorem* 3.1, *inequality* (3.3) *can be replaced by*

$$1 < \frac{1}{(n-1)!} \frac{(n-\alpha-1)^{n-\alpha-1}}{\sqrt{n}\; n^n} \cdot (\alpha+1)^{\alpha+1}\left(\int_{a_1}^{a_r} b^2(t)\,dt\right)^{1/2}(a_r-a_1)^{n-1/2}. \tag{3.8}$$

Proof. Using Schwarz's inequality in the right side of (3.6), we find

$$1 < \left(\int_{a_1}^{a_r} |g(t,s)|^2\,ds\right)^{1/2}\left(\int_{a_1}^{a_r} b^2(s)\,ds\right)^{1/2}$$

and (3.8) follows from lemma 5.

REMARK. It is easy to construct examples to show that in general (3.8) is non-comparable with (3.3), however if $b(t) = k$(positive constant) (3.8) is better than (3.3) by a factor $1/\sqrt{n}$ and hence better than Beesack's inequality (3.6) [6] by a factor $((n-\alpha-1)^{n-\alpha-1}(\alpha+1)^{\alpha+1})/(\sqrt{n}\,(n-1)^{n-1})$.

THEOREM 3.3. *Let the boundary value problem* (3.1),(2.2) *have a solution where* f *is continuous on* $[a_1, a_r] \times R^n$ *and satisfies*

$$|f(t, x, x', \cdots, x^{(n-1)})| \le \sum_{i=0}^{n-1} L_i |x^{(i)}| \tag{3.9}$$

where L_i, $0 \le i \le n-1$ *are nonnegative constants.*
Then,

$$1 \le \sum_{k=0}^{n-1} L_k C_{n,k}(a_r-a_1)^{n-k} = \Theta \tag{3.10}$$

where the $C_{n,k}$ *are the same as in lemma* 6.

Proof. Let $x(t)$ be a solution of (3.1), (2.2). Then from (3.9) and lemma 6, we find

$$|x^{(n)}(t)| \le \sum_{k=0}^{n-1} L_k C_{n,k}(a_r-a_1)^{n-k} \times \max_{a_1 \le t \le a_r} |x^{(n)}(t)|$$

and hence

$$m \le \Theta m \tag{3.11}$$

where $m = \max_{a_1 \le t \le a_r} |x^{(n)}(t)|$. Evidently $m > 0$, since otherwise $x(t)$ would coincide on $[a_1, a_r]$ with a polynomial of degree $p < n$ and will not have n zeros. Thus, $\Theta \ge 1$.

REMARK. In (3.9) at least one of the L_i, $0 \le i \le n-1$ will not be zero, otherwise $x(t)$ will coincide on $[a_1, a_r]$ with a polynomial of degree $p < n$ and will not be a solution of (3.1), (2.2).

REMARK. If in (3.9), $L_0 > 0$ and $L_i = 0$; $1 \le i \le n-1$ then (3.10) can be replaced by strict inequality

$$1 < \frac{1}{n!} \frac{(n-\alpha-1)^{n-\alpha-1}(\alpha+1)^{\alpha+1}}{n^n} (a_r - a_1)^n L_0 \qquad (3.12)$$
$$(= C_{n,0}(a_r - a_1)^n L_0)$$

which follows from (3.6) and lemma 4. It would be desirable to know whether less than or equal in (3.10) can be replaced by strict inequality, in the general case.

REMARK. If $b(t) = L_0$ in (3.2), then (3.12) is better than (3.8) by a factor $1/\sqrt{n}$ and than (3.3) by a factor $1/n$ and than (3.6) of Beesack [6] by a factor $((n-\alpha-1)^{n-\alpha-1}(\alpha+1)^{\alpha+1}/n(n-1)^{n-1})$. If $r = n (\alpha = 0)$ then (3.12) is the same as one of the conclusions (4.5) of [7].

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