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DELAYED SUMS AND BOREL SUMMABILITY OF INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM VARIABLES

BY

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Abstract. In this paper, the following sufficient condition for Euler summability is established: for a sequence A_n of real numbers, if $\max\{|A_{n+1}+A_{n+2}+\cdots+A_{n+j}|/\max\{j, \sqrt{n}\}, 0\leq j\leq n^\alpha\}\to 0$ for some $\frac{1}{2}<\alpha<\frac{2}{3}$, then $A_n\to 0\,(E, q)$ for every $q>0$. Using this sufficient condition, we show: if $X_1, X_2, \cdots$ are independent, identically distributed random variables, then the following three conditions (E refers to Euler summability and B to Borel summability) are all equivalent: 1) $EX_1=0$, $EX_1^2<\infty$; 2) $X_n\to 0\,(E, q)$ a.e. for every $q>0$; 3) $X_n\to 0\,(B)$ a.e.

1. **Introduction.** Let $(Z_n, n\geq 0)$ be a sequence of random variables with $Z_0=0$. For nonnegative real numbers s and t, put $Z_t=Z_{[t]}$, $Z_{s,t}=Z_{[s]+[t]}-Z_{[s]}$, $\bar{Z}_t=\max\{Z_j, 0\leq j\leq t\}$, $Z_t^*=\max\{|Z_j|, 0\leq j\leq t\}$, $\bar{Z}_{s,t}=\max\{Z_{s,j}, 0\leq j\leq t\}$ and $Z_{s,t}^*=\max\{|Z_{s,j}|, 0\leq j\leq t\}$.

If $S_n=\sum_{1\leq j\leq n}X_n$, $n\geq 0$, for some random variables, then $S_{m,n}$ are called delayed sums of $(X_n, n\geq 1)$. Delayed sums of real numbers have been utilized in [11, p. 80] in proving a Tauberian theorem of Hardy, while delayed sums of independent random variables in the form of $S_{2^n,2^n}$ have been employed in [4] to discuss the strong law of large numbers.

We shall always assume that $(S_n, n\geq 1)$ are the partial sums of independent, identically distributed random variables $(X_n, n\geq 1)$ with $S_0=0$. In §2, we will prove that if $EX_1=0$ and $EX_1^2<\infty$, then $n^{-1/2}S^*_{n,n^\alpha}\to 0$ a.e. for every $0>\alpha>1$ (Theorem 1). In [7] and [5], Hsu, Robbins and Erdös establish that $EX_1=0$ and $EX_1^2<\infty$, if and only if $\sum P[|S_n|>n\varepsilon]<\infty$ for every $\varepsilon>0$. By the Borel-

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Cantelli lemma, the last condition implies that $S_{n,n}/n \to 0$ a.e. It is easy to see that the converse is false. In §3, we will prove, as an application of delayed sums, that $\sum P[|S_n| > n\varepsilon] < \infty$ for every $\varepsilon > 0$ if and only if $n^{-1/2} S_{n,n^{1/2}} \to 0$ a.e. (Theorem 2). For another application, in §4, we will prove that for a sequence A_n of real numbers, if for some $\frac{1}{2} < \alpha < \frac{2}{3}$, $\max\{|A_{n+1} + \cdots + A_{n+j}|/\max(j, \sqrt{n}),\ 0 \le j \le n^{\alpha}\} \to 0$, then for every $q > 0$, $A_n \to 0$ (E, q), i.e., A_n is summable to 0 by the Euler method (E, q) (for definition see [6, p. 180]). In §5, we will prove that the conditions $EX_1 = 0$ and $EX_1^2 < \infty$, $X_n \to 0$ (E, q) a.e. for every (or any) $q > 0$, and that $X_n \to 0$ (B) a.e. (for definition, see [6, p. 182]) are all equivalent.

2. **Delayed sums.** Modifying the method used in [5], we have the following result about delayed sums as Theorem 1, which will be applied to obtain some results about tail probabilities (Theorem 2) and Borel summability (Theorem 4). For the proof of Theorem 1, we need the following lemmas.

LEMMA 1. *Let Y be a random variable with $P[X \le 1] = 1$ and $E|Y|^q < \infty$ for some $1 \le q \le 2$. Then*

$$Ee^Y \le 1 + EY + E|X|^q \le \exp\{EX + E|X|^q\}.$$

Proof. The proof is easy, since

$$Ee^Y \le \int_{[Y \le -1]} e^{-1} + \int_{[Y > -1]} (1 + Y + Y^2)$$
$$\le 1 + \int_{[Y > -1]} (Y + |Y|^q) \le 1 + EY + E|Y|^q.$$

LEMMA 2. *Let $(Y_n, n \ge 1)$ and $(Z_n, n \ge 1)$ be two sequences of random variables such that $P[Y_n + Z_n \to 0] = 1$* and *$Y_n \xrightarrow{P} 0$. If for each $n \ge 1$, $(Y_1, \cdots, Y_n)$ is independent of Z_n, then $P[Y_n \to 0] = 1$.*

Lemma 2 is known, see [3].

THEOREM 1. *Let $X, X_1, X_2, \cdots$ be i.i.d. random variables with $S_n = \sum_{1 \le j \le n} X_j$ for $n \ge 0$.*

(i) *If $EX = 0$, $E|X|^r < \infty$ for some $1 \le r \le 2$, and $E(X^+)^p < \infty$ for some $p \ge r$, then for every $0 < \alpha < r/p$*

$$\limsup n^{-1/p} \bar{S}_{n,n^{\alpha}} = 0 \quad a.e. \tag{2.1}$$

(ii) (a) *If* $EX=0$ *and* $E|X|^p<\infty$ *for some* $p\geq 1$, *for every* $0<\alpha<\min(1, 2/p)$

$$n^{-1/p}S^*_{n,n^\alpha}\longrightarrow 0 \quad a.e. \tag{2.2}$$

(ii) (b) *If* $E|X|^p<\infty$ *for some* $1\leq p<2$ *and* $EX=0$, *or if* $E|X|^p<\infty$ *for some* $0<p<1$,

$$n^{-1/p}S^*_{n,n}\longrightarrow 0 \quad a.e. \tag{2.3}$$

(iii) *If* $E(X^+)^p<\infty$ *for some* $0<p<1$, *or if* $EX^p=0$ *for* $p=1$,

$$\limsup n^{-1/p}\bar{S}_{n,n}=0 \quad a.e. \tag{2.4}$$

(iv) *Conversely, if for some* $p>0$ *and some sequence* k_n *of positive integers* $n^{-1/p}S_{n,k_n}\to 0$ *a.e., then* $E|X|^p<\infty$.

(v) *If for some* $p\geq 1$

$$\limsup n^{-1/p}\bar{S}_{n,n^{1/p}}=0 \quad a.e., \tag{2.5}$$

then $E(X^+)^p<\infty$ *and* $EX\leq 0$, *and if for some* $0<p<1$

$$\limsup n^{-1/p}\bar{S}_{n,n}=0 \quad a.e., \tag{2.6}$$

then $E(X^+)^p<\infty$.

Proof. Choose $\alpha<\beta<1$ such that $\beta r>\alpha p$ and for $n\geq 1$ put $m=[n^\alpha]$ and

$$X'_n=X_nI_{[X_n\geq n^{1/p}]},\qquad X''_n=X_nI_{[X_n\leq n^{\beta/p}]},\qquad X'''_n=X_n-X'_n-X''_n,$$

$$S'_n=\sum_1^n X'_j,\quad S''_n=\sum_1^n X''_j,\quad S'''_n=\sum_1^n X'''_j,\quad S'_0=S''_0=S'''_0=0.$$

Since $E(X^+)^p<\infty$, by the Borel-Cantelli lemma $P[X'_n\neq 0 \text{ i.O.}]=0$. Hence

$$n^{-1/p}\bar{S}'_{n,m}\longrightarrow 0 \quad \text{a.e.} \tag{2.7}$$

Choose $k=1,2,\cdots$ so that $k(\beta-\alpha)>1$. Then

$$P[n^{-1/p}\bar{S}'''_{n,m}>k]\leq P[X'''_{n+j}\neq 0 \text{ for at least } k \text{ of } j\text{'s}, 1\leq j\leq m]$$
$$\leq\binom{m}{k}P^k[X\geq n^{\beta/p}]=O(n^{(\alpha-\beta)k}),$$

$$\sum P[n^{-1/p}\bar{S}'''_{n,m}>k]<\infty.$$

By the Borel-Cantelli lemma

(2.8) $$\limsup n^{-1/p} \bar{S}'''_{n,m} \leq k \quad \text{a. e.}$$

For $j = h+1, \cdots, n+m$, put $Y_j = 2^{-1} n^{-\beta/p}(X''_j - EX''_j)$. Then $Y_{n+1}, \cdots, Y_{n+m}$ are independent and $EY_j = 0$. Since

(2.9) $$n^{-\beta/p} \sum_{i=1}^{m} |EX''_{n+i}| \leq n^{-\beta/p} \sum_{i=1}^{m} \int_{[X \geq n^{\beta/p}]} X \leq n^{\alpha-\beta} E(X^+)^p = o(1),$$

for all large n and $n+1 \leq j \leq n+m$, we have $Y_j \leq 1$ and then by Lemma 1

$$E e^{Y_j} \leq \exp\{E|Y_j|^r\} \leq \exp\{n^{-r\beta/p} E|X|^r\}.$$

Hence by independence,

(2.10) $$E \exp\left\{\sum_{i=1}^{m} Y_{n+i}\right\} = \exp\{O(n^{\alpha - r\beta/p})\} = 1 + o(1),$$

and for all large n, by (2.9)

$$\begin{aligned} P[n^{-1/p} S''_{n,m} \geq 1] &\leq P\left[n^{-1/p} \max_{1 \leq j \leq m} (S''_{n,j} - ES''_{n,j}) \geq 1/2\right] \\ &\leq P\left[\max_{1 \leq j \leq m} \sum_{i=1}^{j} Y_{n+i} \geq 4^{-1} n^{(1-\beta)/2}\right] \\ &= P\left[\max_j \exp\left\{\sum_{i=1}^{j} Y_{n+i}\right\} \geq \exp\{4^{-1} n^{(1-\beta)/n}\}\right] \\ &\leq \exp\{-4^{-1} n^{(1-\beta)/n}\} E\left(\exp\left\{\sum_{i=1}^{m} Y_{n+i}\right\}\right), \end{aligned}$$

where the last inequality is due to the submartingale property. By (2.10)

$$\sum_{1}^{\infty} P[n^{-1/p} \bar{S}''_{n,m} \geq 1] = O\left(\sum_{1}^{\infty} \exp\{-4^{-1} n^{(1-\beta)/n}\}\right) < \infty,$$

and hence

(2.11) $$\limsup n^{-1/p} \bar{S}''_{n,m} \leq 1 \quad \text{a. e.}$$

From (2.7), (2.8) and (2.11),

(2.12) $$\limsup n^{-1/p} \bar{S}_{n,m} \leq k+1 \quad \text{a. e.}$$

Since k is independent of $E|X|^r$ and $E(X^+)^p$, we see that (2.12) yields (2.1).

(ii) (a) Follows immediately from (i).

(ii) (b) For $0<p<2$ by SLLN of Kolmogorov (for $p=1$) and of Marcinkiewicz and Zygmund (for $p\neq 1$) (see [9, pp. 242–243])

$$n^{-1/p} S_n \longrightarrow 0 \quad \text{a. e.}$$

Hence $n^{-1/p} S_n^* \to 0$ a. e. and

$$n^{-1/p} S_{n,n}^* \leq n^{-1/p}(S_n^* + S_{2n}^*) \longrightarrow 0 \quad \text{a. e.}$$

(iii) For $p=1$, let $\varepsilon>0$ and $0<M<\infty$ such that $EXI_{[X\leq -M]} > -\varepsilon$. Set $Y_n = X_n I_{[X_n>-M]}$. By (2.3)

$$n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} (Y_{n+i} - EY_{n+i}) \right| \longrightarrow 0 \quad \text{a. e.}$$

Hence

$$\limsup n^{-1} \bar{S}_{n,n} \leq \limsup n^{-1} \max_{1\leq j\leq n} \sum_{i=1}^{j} Y_{n+i} \leq EY_1 \leq \varepsilon \quad \text{a. e.}$$

For $0<p<1$, by (2.3)

$$n^{-1/p} \bar{S}_{n,n} \leq n^{-1/p} \sum_{i=n+1}^{2n} X_i^+ \longrightarrow 0 \quad \text{a. e.}$$

(iv) For each $n\geq 1$, let $Z_n = X_{n+1}$ and $Y_n = \sum_{2\leq j\leq k_n} X_{n+j}$. Then $(Z_1,\cdots, Z_n)$ and Y_n are independent for each $n\geq 1$. Since

$$n^{-1/p}(Z_n + Y_n) \longrightarrow 0 \quad \text{a. e.}, \qquad n^{-1/p} Z_n = n^{-1/p} X_{n+1} \xrightarrow{P} 0,$$

by Lemma 2 we have $n^{-1/p} Z_n \to 0$ a. e. and by the Borel-Cantelli lemma, $E|X|^p < \infty$.

(v) If (2.5) holds for some $p\geq 1$ or if (2.6) holds for some $0<p<1$,

$$\limsup n^{-1/p} X_{n+1} \leq 0 \quad \text{a. e.}$$

Hence $E(X^+)^p<\infty$. To prove that $EX\leq 0$ for the case when $p\geq 1$, let $EX=\delta>0$. Then $\delta<\infty$ and by WLLN, $n^{-1/p} S_{n^{1/p}} \xrightarrow{P} \delta$ and $n^{-1/p} S_{n,n^{1/p}} \xrightarrow{P} 0$. Hence

$$\limsup n^{-1/p} \bar{S}_{n,n^{1/p}} \geq \delta \quad \text{a. e.},$$

a contradiction. Therefore $EX\leq 0$.

3. **Convergence rate of tail probabilities.** Let $X, X_1, X_2 \cdots$ be i. i. d. random variables with $S_n = \sum_{1\le j\le n} X_j$ for $n \ge 0$. As an application of delayed sums $S_{n,m}$, we will prove Theorem 2 in this section, which yields some results of [7] ($p=2$ and $\alpha=1$), [5] ($p=2$ and $\alpha=1$), [10] ($p=\alpha=1$), and [8]. The main point of Theorem 2, however, is the equivalence of (3.5), a global property, and (3.7), a local property. The proof of Theorem 2 also gives a simple derivation of the following result of Spitzer [10]: $\sum n^{-1} P[|S_n| > \varepsilon n] < \infty$ for every $\varepsilon > 0$ if and only if $EX = 0$ by applying SLLN and Lemma 2.

LEMMA 3. *For* $\alpha > 0$ *and* $\alpha p \ge 1$,

(3.1) $$\sum n^{\alpha p-2} P[\bar{S}_n > \varepsilon n^{\alpha}] < \infty \quad \textit{for every} \quad \varepsilon > 0,$$

(3.2) $$\limsup n^{-\alpha/(\alpha p-1)} \bar{S}_{n^{\alpha p/(\alpha p-1)}, n^{1/(\alpha p-1)}} = 0 \quad a.e. \quad \textit{if} \quad \alpha p > 1,$$

and

(3.2)′ $$\limsup 2^{-n/p} \bar{S}_{2^n, 2^n} = 0 \quad a.e. \quad \textit{if} \quad \alpha p = 1$$

are equivalent.

Proof. (i) Let $\alpha p > 1$. Then (3.1) $\Leftrightarrow$

$$\int_1^{\infty} t^{\alpha p-2} P[\bar{S}_t > \varepsilon t^{\alpha}]\, dt < \infty \quad \text{for every} \quad \varepsilon > 0$$

$\Leftrightarrow$ for any constant $k > 0$

$$\int_1^{\infty} P[\bar{S}_{k t^{1/(\alpha p-1)}} > \varepsilon t^{\alpha/(\alpha p-1)}]\, dt < \infty \quad \text{for every} \quad \varepsilon > 0$$

$\Leftrightarrow$

(3.3) $$\sum_1^{\infty} P[\bar{S}_{n^{\alpha p/(\alpha p-1)}, k n^{1/(\alpha p-1)}} > \varepsilon n^{\alpha/(\alpha p-1)}] < \infty \quad \text{for every} \quad \varepsilon > 0.$$

Since $\bar{S}_{n^{\alpha p/(\alpha p-1)}, n^{1/(\alpha p-1)}}$ are independent, by the Borel-Cantelli lemma (3.2) $\Leftrightarrow$ (3.3) with $k=1 \Leftrightarrow$ (3.1).

(ii) Let $\alpha p = 1$. Then (3.1) $\Leftrightarrow$

$$\int_1^{\infty} t^{-1} \bar{P}[S_t > \varepsilon t^{\alpha}]\, dt < \infty \quad \text{for every} \quad \varepsilon > 0$$

$\Leftrightarrow$ for any $k > 0$

$$\int_1^{\infty} P[\bar{S}_{k2^t} > \varepsilon\, 2^{\alpha t}]\, dt > \infty \quad \text{for every} \quad \varepsilon > 0$$

$\Leftrightarrow$

$$(3.4)\qquad \sum_{1}^{\infty} P[\bar{S}_{2^n,k2^n} > \varepsilon\, 2^{\alpha n}] < \infty \quad \text{for every } \varepsilon > 0 .$$

Since $\bar{S}_{2^n,2^n}$ are independent, $(3.2)' \Leftrightarrow (3.4)$ with $k=1 \Leftrightarrow (3.1)$.

COROLLARY 1. *Let $E|X|^r < \infty$ for some $1 \leq r \leq 2$, $E(X^+)^p < \infty$ for some $p \geq r$ and $EX=0$. If $\beta r > 1$ or $\beta r > 1$ when $1 \leq r \leq 2$, then*

$$(3.5)\qquad \sum n^{\beta p-2} P[\bar{S}_n > n^\beta] < \infty .$$

Proof. (i) Let $\beta r > 1$. Then $(\beta p)^{-1} < r/p$ and by (2.1)

$$\limsup m^{-1/p}\, \bar{S}_{m,\, m^{1/(\beta p)}} = 0 \quad \text{a. e.}$$

Hence

$$(3.6)\qquad \begin{aligned} &\limsup m^{-1/p}\, \bar{S}_{m,\,(m+1)^{1/(\beta p)}} \\ &\quad \leq \limsup m^{-1/p} (S_{m,1} + S_{m+1,\,(m+1)^{1/(\beta p)}}) = 0 \quad \text{a. e.} \end{aligned}$$

Put $m_n = [n^{\beta p/(\beta p-1)}]$. Then

$$\begin{aligned} &\limsup n^{-\beta/(\beta p-1)}\, \bar{S}_{n^{\beta p/(\beta p-1)},\, n^{1/(\beta p-1)}} \\ &\quad \leq \limsup m_n^{-1/p}\, \bar{S}_{m_n,\,(m_n+1)^{1/(\beta p)}} = 0 \quad \text{a. e.} \end{aligned}$$

By Lemma 3, (3.5) holds.

(ii) Let $1 \leq r < 2$ and $\beta r = 1$. Then by (2.3)

$$n^{-\beta} S^*_{n,n} \longrightarrow 0 \quad \text{a. e.}$$

Hence $2^{-n\beta} S^*_{2^n,2^n} \to 0$ a. e. and by Lemma 3, (3.5) holds.

Corollary 1 extends some results of [8] to the one-sided case. The condition $E|X|^r < \infty$ for some $1 \leq r \leq 2$ in Corollary 1 cannot be dropped in general; for a counterexample, see [12].

THEOREM 2. *For $\alpha > 0$ and $\alpha p \geq 1$, all of the following conditions are equivalent:*

$$(3.7)\qquad \sum n^{\alpha p-2} P[S^*_n > \varepsilon n^\alpha] < \infty \quad \textit{for every } \varepsilon > 0 ,$$

$$(3.8)\qquad t^{-1/p} S^*_{t,\, t^{1/(\alpha p)}} \longrightarrow 0 \quad \textit{a. e.} \quad \textit{as } t \to \infty ,$$

$$(3.9)\qquad n^{-1/p} S^*_{n,\, n^{1/(\alpha p)}} \longrightarrow 0 \quad \textit{a. e.} \quad \textit{as } n = 1, 2, \cdots \to \infty ,$$

(3.10) $$n^{-\alpha/(\alpha p-1)} S_{n^{\alpha p/(\alpha p-1)}, n^{1/(\alpha p-1)}} \longrightarrow 0 \quad a.e. \quad as \quad n=1,2,\cdots\to\infty, \quad if \quad \alpha p>1,$$

(3.10)′ $$2^{-n/p} S_{2^n, 2^n} \longrightarrow 0 \quad a.e. \quad as \quad n=1,2,\cdots\to\infty, \quad if \quad \alpha p=1,$$

(3.11) $$E|X|^p<\infty, \quad \alpha>\frac{1}{2} \quad if \quad E|X|^p>0, \qquad EX=0 \quad if \quad 0<\alpha\leq 1.$$

Proof. (i) (3.7) ⇒ (3.8). Let $\alpha p>1$. Then by (3.3)

$$n^{-\alpha/(\alpha p-1)} S^*_{n^{\alpha p/(\alpha p-1)}, kn^{1/(\alpha p-1)}} \longrightarrow 0 \quad \text{a. e.}$$

where $k>3\alpha p/(\alpha p-1)$. Since for all large n and $n\leq t<n+1$,

$$t^{\alpha p/(\alpha p-1)} + t^{1/(\alpha p-1)} \leq (n+2)^{\alpha p/(\alpha p-1)} \leq n^{\alpha p/(\alpha p-1)} + kn^{1/(\alpha p-1)},$$

$$t^{-\alpha/(\alpha p-1)} S^*_{t^{\alpha p/(\alpha p-1)}, t^{1/(\alpha p-1)}} \leq 2n^{-\alpha/(\alpha p-1)} S^*_{n^{\alpha p/(\alpha p-1)}, kn^{1/(\alpha p-1)}} \longrightarrow 0 \quad \text{a. e.}$$

implying (3.8).

Now let $\alpha p=1$. Then by (3.4)

$$2^{-\alpha n} S^*_{2^n, k2^n} \longrightarrow 0 \quad \text{a. e.},$$

where $k>3$. Since for all large n and $n\leq t<n+1$,

$$2^t + 2^t \leq 2^{n+2} \leq 2^n + k2^n,$$

$$2^{-\alpha t} S^*_{2^t, 2^t} \leq 2^{-\alpha n} S^*_{2^n, k2^n} \longrightarrow 0 \quad \text{a. e.}$$

implying (3.8).

(ii) (3.8) ⇒ (3.9) is trivial.

(iii) (3.7) ⇐ (3.9) ⇒ (3.10) and (3.10)′.

Assume that (3.9) holds. Then

(3.12) $$n^{-1/p} S^*_{n, (n+1)^{1/p}} \leq n^{-1/p} (S^*_{n, n+1} + S^*_{n+1, (n+1)^{1/(\alpha p)}}) \longrightarrow 0 \quad \text{a. e.}$$

If $\alpha p>1$, by putting $n=[m^{\alpha p/(\alpha p-1)}]$ we have by (3.12)

(3.13) $$m^{-\alpha/(\alpha p-1)} S^*_{m^{\alpha p/(\alpha p-1)}, m^{1/(\alpha p-1)}} \longrightarrow 0 \quad \text{a. e.}$$

yielding (3.10). If $\alpha p=1$, by putting $n=2^m$ we have, by (3.9)

(3.13)′ $$2^{-m/p} S^*_{2^m, 2^m} \longrightarrow 0 \quad \text{a. e.}$$

yielding (3.10)′. By independence, from (3.13) and (3.13)′

(3.14) $$\sum_1^\infty P[S^*_{m^{1/(\alpha p-1)}} > \varepsilon m^{\alpha/(\alpha p-1)}] < \infty \quad \text{for every } \varepsilon>0 \quad \text{if } \alpha p>1,$$

(3.14)′ $\sum_1^\infty P[S^*_{2^m} > \varepsilon 2^{m/p}] < \infty$ for every $\varepsilon > 0$ if $\alpha p = 1$.

Since (3.3) with $k=1$ implies (3.1) if $\alpha p > 1$ and (3.4) with $k=1$ implies (3.1) if $\alpha p = 1$, (3.7) holds.

(iv) (3.10) $\Rightarrow$ (3.11) if $\alpha p > 1$ and (3.10)′ $\Rightarrow$ (3.11) if $\alpha p = 1$.

Let $(X'_n, n \geq 1)$ and $(X_n, n \geq 1)$ be i. i. d. random vectors. Put $Y_n = X_n - X'_n$ and $T_n = \sum_{1 \leq j \leq n} Y_j$ for $n \geq 0$. Since $S_{m^{\alpha p/(\alpha p-1)}, m^{1/(\alpha p-1)}}$ are independent if $\alpha p > 1$ and $S_{2^m, 2^m}$ are independent if $\alpha p = 1$, by (3.10)

$$\sum_1^\infty P[|S_{m^{1/(\alpha p-1)}}| > \varepsilon m^{\alpha/(\alpha p-1)}] < \infty \quad \text{for every } \varepsilon > 0 \quad \text{if } \alpha p > 1$$

and by (3.10)′

$$\sum_1^\infty P[|S_{2^m}| > \varepsilon 2^{m/p}] < \infty \quad \text{for every } \varepsilon > 0 \quad \text{if } \alpha p = 1.$$

By the Levy inequality (see [9, p. 247]),

(3.15) $\sum_1^\infty P[T^*_{m^{1/(\alpha p-1)}} > \varepsilon m^{\alpha/(\alpha p-1)}] < \infty$ for every $\varepsilon > 0$ if $\alpha p > 1$.

(3.15)′ $\sum_1^\infty P[T^*_{2^m} > \varepsilon 2^m] < \infty$ for every $\varepsilon > 0$ if $\alpha p = 1$.

Hence, as in (iii)

(3.16) $\sum n^{\alpha p-2} P[T^*_n > \varepsilon n^\alpha] < \infty$ for every $\varepsilon > 0$.

By (i), $t^{-1/p} S^*_{t, t^{1/(\alpha p)}} \to 0$ a. e. and therefore $n^{-1/p} X_n \to 0$ a. e. By the Borel-Cantelli lemma $E|X|^p < \infty$. Since $\alpha p \geq 1$, by the central limit theorem, either one of (3.10) and (3.10)′ implies $\alpha > \frac{1}{2}$ if $E|X|^p > 0$. If $\frac{1}{2} < \alpha \leq 1$, then $p \geq 1$ and by SLLN, either one of (3.10) and (3.10)′ implies $EX = 0$.

(v) (3.11) $\Rightarrow$ (3.9). If $E|X|^p = 0$, then (3.9) holds trivially. Hence we can assume that $\alpha > \frac{1}{2}$. If $0 < p < 1$, (3.9) follows from (2.3). If $\frac{1}{2} < \alpha \leq 1$, then $p \geq 1$ and $EX = 0$ and (3.9) follows from (2.2) and (2.3). If $\alpha > 1$ and $p \geq 1$, by (2.2)

$$\limsup n^{-1/p} S^*_{n, n^{1/(\alpha p)}} = \limsup n^{-1/p} Z_{n, n^{1/(\alpha p)}} = 0 \quad \text{a. e.}$$

where $Z_n = \sum_{1\le j\le n}(X_j - EX_j)$.

4. **Euler and Borel summability methods for real numbers.** Let $A_n = \sum_1^n a_j$ be a sequence of real numbers. Put

$$e_n = e_{m,n} = (q+1)^{-m}\binom{m}{n} q^{m-n} \quad \text{for } q>0 \text{ and } 0\le n\le m,$$

$$b_n = b_{\lambda,n} = e^{-\lambda}\lambda^n/n! \quad \text{for } \lambda>0 \text{ and } n=0,1,2,\cdots.$$

If for $m=1,2,\cdots$,

$$E_m = \sum_0^m e_n A_n = (q+1)^{-m}\sum_0^m \binom{m}{n} q^{m-n} A_n \longrightarrow A$$

as $m\to\infty$, then A_n is said to be Euler summable to A, denoted by $A_n \to A(\mathrm{E}, q)$. If for each $\lambda>0$,

$$B(\lambda) = \sum_0^\infty b_n A_n = e^{-\lambda}\sum_0^\infty \lambda^n A_n/n!$$

converges and $B(\lambda)\to A$ as $\lambda\to\infty$, then A_n is said to be Borel summable to A, denoted by $A_n\to A(\mathrm{B})$.

The summability methods (E, q) and B have been discussed in detail in [6, Chapters 8 and 9]. In this section we will establish some sufficient conditions for the methods (E, q) and B, which seem to be new.

For e_n, put $M=[(m+1)/(q+1)]$, $c=(q+1)/2q$ and $h=n-M$. From [6, p. 201] we have

LEMMA 4. *If* $\frac{1}{2}<\alpha<\frac{2}{3}$, *then*

$$\sum_{|h|>m^\alpha} e_n = O(e^{-m^\eta}), \tag{4.1}$$

where η *is any number less than* $2\alpha-1$, *and for* $|h|\le m^\alpha$

$$e_n = \left(\frac{c}{\pi M}\right)^{1/2} e^{-ch^2/M}\left\{1 + O\left(\frac{|h|+1}{m}\right) + O\left(\frac{|h|^3}{m^2}\right)\right\}. \tag{4.2}$$

THEOREM 3. *If for some* $\frac{1}{2}<\alpha<\frac{2}{3}$,

$$\max_{0\le j\le n^\alpha} |A_n + A_{n+1} + \cdots + A_{n+j}|/\max(j, \sqrt{n}) \longrightarrow 0, \tag{4.3}$$

then for every $q>0$,

$$A_n \longrightarrow 0(\mathrm{E}, q).$$

Proof. By (4.3), $A_n = o(n^{1/2})$. Put

$$E_m = \sum_0^m e_n A_n = \sum_{|h|>m^\alpha} + \sum_{|h|\le m^\alpha} = E' + E'', \tag{4.4}$$

say. Then for all large m,

$$E' = o(1) \sum_{|h|>m^\alpha} e_n n^{1/2} = o(m^{1/2} e^{-m^\eta}) = o(1) \tag{4.5}$$

by (4.1), where $0 < \eta < 2\alpha - 1$. By (4.2)

$$\begin{aligned} E'' &= \sum_{|h|\le m^\alpha} e_n A_n \\ &= \left(\frac{c}{\pi M}\right)^{1/2} \sum_{|h|\le m^\alpha} e^{-ch^2/M}\left\{1 + O\left(\frac{|h|+1}{m}\right) + O\left(\frac{|h|^3}{m^2}\right)\right\} A_n . \end{aligned} \tag{4.6}$$

Since $n = M+h = M(1+o(1))$ for $|h| \le m^\alpha$, $A_n = o(M^{1/2})$ for $|h| \le m^\alpha$ and

$$\begin{aligned} M^{-1/2} &\sum_{|h|\le m^\alpha} e^{-ch^2/M}\left\{O\left(\frac{|h|+1}{m}\right) + O\left(\frac{|h|^3}{m^2}\right)\right\} |A_n| \\ &= o(1) \sum_{|h|\le m^\alpha} e^{-ch^2/M}\left\{\frac{|h|+1}{m} + \frac{|h|^3}{m^2}\right\} \\ &= o(1) \int_{-\infty}^{\infty} e^{-ct^2/M}(|t| + 1 + |t|^3 m^{-1})\, m^{-1}\, dt \\ &= o(1) \int_{-\infty}^{\infty} e^{-ct^2}\left(|t|\frac{M}{m} + \frac{M^{1/2}}{m} + \left(\frac{M}{m}\right)^2 |t|^3\right) dt \\ &= o(1) \int_{-\infty}^{\infty} e^{-ct^2}(|t| q^{-1} + o(1) + |t|^3 q^{-2})\, dt = o(1) \end{aligned}$$

By (4.4), (4.5) and (4.6) as $m \to \infty$

$$E_m = \left(\frac{c}{\pi M}\right)^{1/2} \sum_{|h|\le m^\alpha} e^{-ch^2/M} A_n + o(1) . \tag{4.7}$$

Put $T_{M,j} = \sum_{|h|<j} A_n$. Then

$$\begin{aligned} \sum_{|h|\le m^\alpha} e^{-ch^2/M} A_n &= \sum_{0\le j\le m^\alpha} e^{-cj^2/M}(T_{M,j} - T_{M,j-1}) \\ &= \sum_{0\le j<[m^\alpha]} (e^{-cj^2/M} - e^{-c(j+1)^2/M})\, T_{M,j} + e^{-c[m^\alpha]^2/M}\, T_{M,[m^\alpha]} . \end{aligned}$$

To simplify the notation, we set $x v y = \max(x, y)$ for real numbers x and y. Since (4.3) implies

$$\max_{0 \leq j \leq 2n^\alpha} \frac{|A_n + A_{n+1} + \cdots + A_{n+j}|}{jvn^{1/2}} \leq \max_{0 \leq j \leq n^\alpha} \frac{|A_n + \cdots + A_{n+j}|}{jvn^{1/2}} + \max_{n^\alpha < j \leq 2n^\alpha} \frac{|A_{n+[n^\alpha]+1} + \cdots + A_{n+j}|}{jvn^{1/2}} = o(1)\,, \tag{4.8}$$

we have from (4.3) and (4.8) for $0 \leq j \leq m^\alpha$

$$\begin{aligned} |T_{M,j}| &\leq |A_M + A_{M+1} + \cdots + A_{M+j}| + |A_{M-j} + A_{M-j+1} + \cdots + A_{M-1}| \\ &\leq o(jvM^{1/2}) + o(jv(M-j)^{1/2}) = o(jvM^{1/2}) \end{aligned}$$

uniformly as $m \to \infty$. Hence

$$\begin{aligned} \sum_{|h| \leq m^\alpha} e^{-ch^2/M} A_n &= o(1) \sum_{0 \leq j \leq M^\alpha} (e^{-cj^2/M} - e^{-c(j+1)^2/M})(jvM^{1/2}) + o(1) \\ &= o(1) \int_0^{m^\alpha} (e^{-ct^2/M} - e^{-c(t+1)^2/M})(tvM^{1/2})\,dt + o(1) \\ &= o(M) \int_0^{m^\alpha M^{-1/2}} (e^{-ct^2} - e^{-ct^2 - 2ctM^{-1/2} - cM^{-1}})(tv1)\,dt + o(1) \\ &= o(M) \int_0^1 e^{-ct^2} \left(\frac{2ct}{\sqrt{M}} + \frac{c}{M}\right) dt \\ &\quad + o(M) \int_1^{m^\alpha M^{-1/2}} te^{-ct^2} \left(\frac{2ct}{\sqrt{M}} + \frac{c}{M}\right) dt + o(1) \\ &= o(M^{1/2}) + o(M^{1/2}) \int_1^\infty t^2 e^{-ct^2}\,dt + o(1) = o(M^{1/2})\,. \end{aligned}$$

Hence

$$\sum_{|h| \leq m^\alpha} e^{-ch^2/M} A_n = o(M^{1/2}) + o(1)\,. \tag{4.9}$$

From (4.7) and (4.9), $E_m = o(1)$ and therefore $A_n \to 0(\mathrm{E}, q)$. Since $A_n \to A(\mathrm{E}, q)$ implies $A_n \to A(\mathrm{B})$ (see [6, p. 183]), we have:

COROLLARY 2. *If* (4.3) *holds for some* $\frac{1}{2} < \alpha < \frac{2}{3}$, *then* $A_n \to 0(\mathrm{B})$.

5. Euler and Borel summabilities of i. i. d. random variables. From Theorems 1 and 3 and Lemma 2, we can prove the following theorem, which implies that for i. i. d. random variables the Euler summability method (E, q) is equivalent to the Borel summability method (B) and is more restrictive than the Cesaro summability (C, 1).

THEOREM 4. *Let* $X, X_1, X_2, \cdots$ *be i. i. d. random variables. Then the following conditions are equivalent.*

(5.1) $$EX=0, \qquad EX^2<\infty,$$

(5.2) $$X_n \longrightarrow 0\ (\mathrm{E},\ q) \quad a.e. \qquad \textit{for every (or any)} \quad q>0,$$

(5.3) $$X_m \longrightarrow 0\ (\mathrm{B}) \quad a.e.$$

Proof. By Theorems 1 (ii) (a) and 3, (5.2) holds if (5.1) does. It is well known that (5.3) follows from (5.2) (see [6, p. 183]). Now assume that (5.3) holds. Then

(5.4) $$X_n^s \longrightarrow 0\ (\mathrm{B}) \quad \text{a.e.}$$

where $X_n^s = X_n - X_n'$ with i.i.d. random vectors $(X_n',\ n \geq 1)$ and $(X_n,\ n \geq 1)$. Put $Y_m = \sum_{n=1}^m b_{m,n} X_n^s$ and $Z_m = \sum_{n=m+1}^{\infty} b_{m,n} X_n^s$, $m=1, 2, \cdots$. Then $(Y_1, \cdots, Y_m)$ and Z_m are independent for each $m \geq 1$. By (5.4)

(5.5) $$Y_m + Z_m \longrightarrow 0 \quad \text{a.e.}$$

Hence

$$Y_m + Z_m \xrightarrow{P} 0.$$

Since Y_m and Z_m are independent and symmetric, by the Levy inequality [9, p. 247], $Y_m \xrightarrow{P} 0$. By Lemma 2,

(5.6) $$Y_m \longrightarrow 0 \quad \text{a.e.}, \qquad \sum_{n=1}^m b_{m,n} X_n^s \longrightarrow 0 \quad \text{a.e.}$$

Repeating the previous argument, we have

$$b_{m,n} X_m^s \longrightarrow 0 \quad \text{a.e.}$$

By the Stirling formula, $b_{m,m} = (2\pi m)^{-1/2}(1+o(1))$. Therefore by the Borel-Cantelli lemma, $E(X_1^s)^2 < \infty$ and then $EX^2 < \infty$. Since (5.1) implies (5.3),

(5.7) $$X_n - EX \longrightarrow 0\ (\mathrm{B}) \quad \text{a.e.}$$

From (5.3) and (5.7), we obtain that $EX=0$.

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