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EMBEDDING DISTRIBUTIVE LATTICES IN VECTOR LATTICES*

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Dedicated to Professor Chen-Jung Hsu on his 65-th birthday

Abstract. Embedding of distributive lattices with smallest element into vector lattices is considered together with some observations of the embedding.

1. Let L be a distributive lattice with the smallest element θ. We shall prove without recourse to Stone Representation Theorem that L can be embedded in a conditionally complete vector lattice which is the order dual of the vector lattice of all valuations of bounded variation on L vanishing at θ. In particular, when L also has a largest element we infer that L can be embedded in $C(S)$ for some compact Hausdorff space S. This fact does not seem to have been observed before. Following Birkhoff [1] a real-valued function μ defined on L is called a valuation if $\mu(x)+\mu(y) = \mu(x\wedge y)+\mu(x\vee y)$ for all x, y of L. But for our purpose, by a valuation μ we also understand that $\mu(\theta)=0$. If μ is a valuation and F a generic finite chain $x_1\leq\cdots\leq x_n$ in L, let

$$(\mu^+;\ F)=\sum_{k=1}^{n-1}\{\mu(x_{k+1})-\mu(x_k)\}^+,$$

$$(|\mu|;\ F)=\sum_{k=1}^{n-1}|\mu(x_{k+1})-\mu(x_k)|,$$

where for a real number r, $r^+=r$ if $r\geq 0$ and $r^+=0$ if $r<0$. Then we define $|\mu|$, μ^+, μ^- on L by

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$$|\mu|(x) = \operatorname*{Sup}_{F\subset[\theta,x]} (|\mu|;\ F);$$

$$\mu^+(x) = \operatorname*{Sup}_{F\subset[\theta,x]} (\mu^+;\ F);$$

$$\mu^-(x) = (-\mu)^+(x),\ x \in L.$$

A valuation μ is said to be of bounded variation if $|\mu|(x) < +\infty$ for all $x \in L$. If a valuation μ is of bounded variation, then $|\mu|$, μ^+ and μ^- are monotone nondecreasing valuations on L and the following hold: $|\mu| = \mu^+ + \mu^-$, $\mu = \mu^+ - \mu^-$. Obviously monotone nondecreasing valuations are nonnegative and of bounded variation on L.

Now let X be the space of all valuations of bounded variation on L, then X is a real vector space and the set X_+ of all monotone nondecreasing valuations on L is a convex cone with tip θ. If we order X using X_+ as the the positive cone, then X becomes a vector lattice such that for $\mu \in X$, $\mu^+ = \mu \vee 0$ and $-\mu^- = \mu \wedge 0$. For these facts we refer to [1]. We shall denote by X^π the order dual of X. Now let $\tau : L \to X^\pi$ be defined by $\tau(x)(\mu) = \mu(x)$ for $x \in L$, $\mu \in X$. We shall prove the following theorem:

THEOREM 1. *τ is a lattice isomorphism of L into X^π.*

We note that since the order dual of a vector lattice is a conditionally complete vector lattice, theorem 1 implies that every distributive lattice with a smallest element can be embedded in a conditionally complete vector lattice. If L has also a largest element U, then X becomes a Banach lattice with the norm defined by $\|\mu\| = |\mu|(U)$, $\mu \in X$. In this case the topological dual $X^* = X^\pi$ (see, for instance, [5]). X is then obviously an abstract (L)-space in the sene of Kakutani [2] and hence X^* is an abstract (M)-space whose unit ball X_1^* is $[-\tau(U), \tau(U)]$. Thus X^* is isometric and lattice isomorphic as Banach lattice with the space $C(S)$ of all continuous real-valued functions on a compact Hausdorff space S [3]. Hence we have the following corollary to Theorem 1:

COROLLARY. *If L is a distributive lattice with smallest and largest elements, then L can be embedded in $C(S)$ for some compact Hausdorff space S.*

2. **To prove Theorem 1,** we note first that τ is obviously order preserving and $\tau(x\vee y)\geq\tau(x)\vee\tau(y)$ for $x, y\in L$. For $\mu\in X$ and $x\in L$, let μ_x be defined by $\mu_x(z)=\mu(x\wedge z)$ for $z\in L$. μ_x is easily seen to be in X. Now for $\mu\in X_+$,

$$\begin{aligned}(\tau(x)\vee\tau(y))(\mu)&\\ &=\sup\{\tau(x)(\mu_1)+\tau(y)(\mu_2):\mu_1,\ \mu_2\geq 0 \text{ and } \mu_1+\mu_2=\mu\}\\ &\geq\tau(x)(\mu_x)+\tau(y)(\mu-\mu_x)=\mu_x(x)+\mu(y)-\mu_x(y)\\ &=\mu(x)+\mu(y)-\mu(x\wedge y)=\mu(x\vee y)=\tau(x\vee y)(\mu)\,.\end{aligned}$$

Thus $\tau(x)\vee\tau(y)\geq\tau(x\vee y)$ for $x, y\in L$. Hence $\tau(x)\vee\tau(y)=\tau(x\vee y)$. Similarly, $\tau(x)\wedge\tau(y)=\tau(x\wedge y)$ for $x, y\in L$. We have shown that τ is an order preserving map which preserves lattice operations. It remains to prove that τ is an injection. Let $x, y\in L$ and $x\neq y$. Obviously one of the ideals $[\theta, x]$ and $[\theta, y]$ contains only one of x and y, say $y\notin[\theta, x]$. Then there is a prime ideal P which contains $[\theta, x]$ but not y [4]. Let $\mu: L\rightarrow R$ be defined by $\mu(z)=0$ if $z\in P$ and $\mu(z)=1$ otherwise. Then since P is prime, $z_1\wedge z_2\in P$ implies that either $z_1\in P$ or $z_2\in P$, from which we infer that μ is a valuation. Since μ is nonnegative and monotone nondecreasing, $\mu\in X$. But $\tau(x)(\mu)=\mu(x)=0\neq 1=\mu(y)=\tau(y)(\mu)$. Thus $\tau(x)\neq\tau(y)$, which shows that τ is injective. Theorem 1 is proved.

3. **We give an observation to conclude our note.** We have remarked in section 1 that there is an isometry and lattice isomorphism T from X^* onto $C(S)$ for some compact Hausdorff space S. Using T we can transfer each $\mu\in X$ to be a bounded linear functional l_μ on $C(S)$ by

$$l_\mu(f)=\langle\mu, T^{-1}(f)\rangle,\quad f\in C(S)$$

where $\langle\cdot,\cdot\rangle$ is the pairring between X and X^*. Consequently there is a regular Radon measure $\hat{\mu}$ defined on all Borel subsets of S such that $l_\mu(f)=\int_S f d\hat{\mu}$ for $f\in C(S)$. In particular

$$\mu(x)=\int_S (T\circ\tau)(x)\,d\hat{\mu}\tag{1}$$

for all $x\in L$. Thus to every valuation μ of bounded variation on L corresponds a regular Radon measure $\hat{\mu}$ on S such that (1) holds.

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