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ON THE TOTAL CURVATURE OF IMMERSED MANIFOLDS, VI: SUBMANIFOLDS OF FINITE TYPE AND THEIR APPLICATIONS

BY

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Dedicated to Professor Chen-Jung Hsu on his 65th birthday,

Abstract. The order of submanifolds was introduced in Part IV of this series several years ago. In that paper, this notion was used to obtain a best possible lower bound of total mean curvature. In this paper, we will use this concept of order of submanifolds to study submanifolds of finite type. Such submanifolds can be regarded as a natural generalization of minimal submanifolds of spheres. Many fundamental results on submanifolds of finite type will be obtained in this paper. By applying our results, we will also obtain a best possible upper bound of total mean curvature. Some other related inequalities and applications will also be given.

1. **Introduction.** Let $x : M \to E^m$ be an isometric immersion of an n-dimensional closed Riemannian manifold M into a Euclidean m-space E^m. Then the mean curvature α of M satisfies the following inequality [3, I]

$$\int_M \alpha^n \, dV \geq c_n, \tag{1.1}$$

where c_n is the volume of a unit n-sphere. The equality sign of (1.1) holds if and only if M is imbedded as a convex plane curve when $n=1$ and as an ordinary n-sphere when $n>1$. This result generalized the famous inequality of Fenchel-Borsuk and also Willmore [8].

In the 1973 AMS Symposium on Differential Geometry held at Stanford University (see, p. 17 of [5], for instance) the author proposed the following.

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PROBLEM. Let M be a closed Riemannian manifold and $x: M \to E^m$ an isometric immersion from M into E^m. What can we say about the total mean curvature $\int \alpha^n dV$ of x and the Riemannian manifold M?

In Part IV of this series [3, IV], the author used the Laplace-Beltrami operator Δ of M to introduce the concept of *order of immersion.* In that paper, he used this concept to prove that if $x: M \to E^m$ is of order $\geq p$, then the total mean curvature of x satisfies

$$\int_M \alpha^n dV \geq \left(\frac{\lambda_p}{n}\right)^{n/2} \mathrm{vol}(M), \tag{1.2}$$

with equality holding when and only when x is of order p. It should be remarked that every immersion can be assumed to be of order ≥ 1. By applying (1.2) and the conformal invariance of $\int \alpha^2 dV$, the author obtained in 1979 [4] a best possible estimate on λ_1 of Δ. In particular, he showed that *if M is a conformal square torus* (i.e., a conformal Clifford torus in E^m), *then*

$$\lambda_1 \mathrm{vol}(M) \leq 4\pi^2, \tag{1.3}$$

with equality holding when and only when M is a square torus.

In this paper, we shall use the idea of order of immersions to introduce the *notion of submanifolds of finite type.* This notion can be regarded as a natural generalization of the notion of minimal submanifolds in both E^m and S^m.

In §3, we shall give the general theory of immersions of finite type. A characterization of such immersions will be obtained. Using this we can relate *Fourier series* to immersions of finite type (Theorem 3.).

In §4, we shall use the idea of immersion of finite type to obtain a best possible upper bound of total mean curvature. Some of its applications will also be given here.

In the last section, several related integral inequalities will be obtained.

REMARK. A portion of this paper was done while the author was a visiting professor at University of Notre Dame. The author would like to express his many thanks to his colleagues there for their hospitality. Especially, he would like to express his hearty thanks to Professor T. Nagano for his constant encouragement and guidance through many years.

2. **The concept of order of submanifolds.** It is well-known that an algebraic manifold or variety is defined by algebraic equations. Thus, one may define the notion of degree of an algebraic manfiold (which can also be defined by using homology). The concept of degree is both important and fundamental in algebraic geometry. On the contrary, one cannot talk about the degree for arbitrary closed submanifolds in E^m. However, by using the Riemannian structure of a closed submanifold M (induced by E^m), the author had introduced in Part IV of this series the notion of *order of immersion* for arbitrary closed submanifolds in E^m to correspond to the notion of degree for algebraic manifolds. In fact, for an isometric immersion of a compact rank one symmetric space *the notion of order of immersion coincides with the notion of degree of polynomials.* (see, the Remark after Theorem 5). Moreover, for closed curves in E^3, our notion of order of immersions coincides with the order of covering. (Theorem 5).

In this paper, we will use our notion of order of immersions to introduce submanifolds of finite type. By applying the theory, we may obtain a best possible upper bound of total mean curvature.

Our definition of immersions of finite type is based on the spectral decomposition of immersions which we already considered in Part IV of this series (also in [5]). We recall such concept as follows.

Let M be a closed Riemannian manifold and Δ the Laplace-Beltrami operator acting on differentiable functions in $C^\infty(M)$. It is well-known that Δ is an elliptic operator and it has an infinite discrete sequence of eigenvalues;

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \uparrow \infty. \tag{2.1}$$

Let $V_k = \{f \in C^\infty(M) \mid \Delta f = \lambda_k f\}$ be the eigenspace of Δ with the eigenvalue λ_k. Then the dimension of each V_k is finite. If we define an inner product on $C^\infty(M)$ by

$$(2.2) \qquad (f, g) = \int_M fg\, dV,$$

for $f, g \in C^\infty(M)$, then the *decomposition* $\sum_{k=0}^\infty V_k$ *is orthogonal* with respect to this structure. Moreover, $\sum_{k=0}^\infty V_k$ is dense in $C^\infty(M)$ (in L^2-sense). Since M is closed, V_0 is one-dimensional and it consists only of constant functions.

For each function $f \in C^\infty(M)$, let f_t be the projection of f onto the subspace V_t ($t = 0, 1, 2, \cdots$). Then we have the following spectral decomposition;

$$(2.3) \qquad f = \sum_{t=0}^{\infty} f_t \qquad (\text{in } L^2\text{-sense}).$$

Because V_0 is 1-dimensional, for any non-constant function $f \in C^\infty(M)$, there is a positive integer $p \geq 1$ such that $f_p \neq 0$ and

$$(2.4) \qquad f - f_0 = \sum_{t \geq p} f_t .$$

If there are infinite f_t which are nonzero, we put $q = \infty$. Otherwise, there is an integer $q \geq p$ such that $f_q \neq 0$ and $f - f_0 = \sum_{t=p}^{q} f_t$. In both cases, we may write

$$(2.5) \qquad f - f_0 = \sum_{t=p}^{q} f_t$$

where q is either an integer or ∞. If the first case occurs, f is said to be of finite order.

For an isometric immersion $x : M \to E^m$, we put

$$(2.6) \qquad x = (x_1, \cdots, x_m),$$

where x_i is the i-th coordinate function of x. For each x_i, we have as (2.5),

$$(2.7) \qquad x_i - (x_i)_0 = \sum_{t=p_i}^{q_i} (x_i)_t \qquad i = 1, 2, \cdots, m .$$

We put

$$(2.8) \qquad p = \inf_i \{p_i\}, \qquad q = \sup_i \{q_i\}$$

where i ranges among all $i = 1, 2, \cdots, m$ such that $x_i - (x_i)_0 \neq 0$. Then p is an integer ≥ 1 and q is either an integer or ∞. It is easy to check that p and q are independent of the choice of Euclidean coordinate system $(x_1, \cdots, x_m)$. Thus *p and q are well-defined.* And we say that the immersion x (or *the submanifold M*) *is of order* $[p, q]$. In particular, if q is finite, we say that x or *M is of finite type.* Otherwise, x or M is said to be of *infinite type.*

For an immersion x of order $[p, q]$, we sometime simply say that x is of order $\geq p$ if q is not considered. (Or we say that x is *of order* $\leq q$ if p is not considered.)

By using (2.7), we have the following *spectral decomposition* (in vector form);

$$x = x_0 + \sum_{t=p}^{q} x_t . \tag{2.9}$$

An immersion x is sometime said to be of *mono-order* (*bi-order*, tri-order, $\cdots$,) if there are only $1(2, 3, \cdots)$ of x_t which is (are) non-zero. If $p = q$, we just say that *x is order p.*

If one chooses the center of gravity of x as the origin of E^m, then $x_0 = 0$. In [3, IV], the author showed that the total mean curvature is very closely related to the order of immersion. In fact he proved that if x is of order $\geq p$, then the total mean curvature admits the following best possible lower bound;

$$\int_M \alpha^n \, dV \geq \left(\frac{\lambda_p}{n}\right)^{n/2} \mathrm{vol}(M), \tag{2.10}$$

with equality holding when and only when x is of order p. (See also [2,6]).

3. Fourier series expansion and immersion of finite type.

First, we rephrase a result of Takahashi [7] in the following form.

THEOREM A. *Let $x : M \to E^m$ be an isometric immersion of an n-dimensional (not necessarily closed) Riemannian manifold M into E^m. Then x is of mono-order if and only if M is either a minimal submanifold of E^m or a minimal submanifold of a hypersphere S^{m-1} of E^m.*

From this theorem, we see that if M is a minimal submanifold of S^{m-1} centered at the origin, then we have

$$\Delta x = \lambda x, \qquad \lambda \in \mathbf{R}. \tag{3.1}$$

Because $\Delta x = -nH$ (H is the *mean curvature vector* of x), (3.1) implies

$$\Delta H = \lambda H. \tag{3.2}$$

In view of this, we give the following ***characterization*** of immersions of finite type.

THEOREM 1. *Let $x : M \to E^m$ be an isometric immersion of a closed Riemannian manifold M into E^m. Then x is of finite type if and only if H satisfies*

$$\Delta^k H + c_1 \Delta^{k-1} H + \cdots + c_{k-1} \Delta H + c_k H = 0, \tag{3.3}$$

for some integer $k \geq 1$ and some real numbers $c_1, c_2, \cdots, c_k$.

Proof. If the isometric immersion $x : M \to E^m$ is of finite type, then we have the following spectral decomposition;

$$x - x_0 = \sum_{t=p}^{q} x_t, \qquad \Delta x_t = \lambda_t x_t, \tag{3.4}$$

for some integers p and q. Thus, we find

$$-n\Delta^i H = \sum_{t=p}^{q} \lambda_t^{i+1} x_t, \qquad i = 0, 1, 2, \cdots. \tag{3.5}$$

By a straight-forward computation, we may obtain from (3.5) that

$$\Delta^k H - \sigma_1 \Delta^{k-1} H + \sigma_2 \Delta^{k-2} H - \cdots + (-1)^k \sigma_k H = 0, \quad k = q - p. \tag{3.6}$$

where $\sigma_1 = \lambda_p + \cdots + \lambda_q$, $\sigma_2 = \sum_{t<s} \lambda_t \lambda_s$, $\cdots$, $\sigma_k = \lambda_p \cdots \lambda_q$ are the elementary symmetric functions of $\lambda_p, \cdots, \lambda_q$.

Conversely, if H satisfies equation (3.3) for some k and $c_1, c_2, \cdots, c_k$, then by using the following spectral decomposition of x;

$$x = \sum_{t=0}^{\infty} x_t, \tag{3.7}$$

we obtain, from (3.3), that

$$\sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \cdots + c_{k-1} \lambda_t + c_k) x_t = 0. \tag{3.8}$$

For each positive integer s, (3.8) implies

$$\sum_{t=1}^{\infty} \lambda_t(\lambda_t^k + c_1 \lambda_t^{k-1} + \cdots + c_{k-1} \lambda_t + c_k)(x_t, x_s) = 0. \tag{3.9}$$

Since $(x_t, x_s) = \int \langle x_t, x_s \rangle dV = 0$ for $t \neq s$, we obtain

$$(\lambda_s^k + c_1 \lambda_s^{k-1} + \cdots + c_{k-1} \lambda_s + c_k) a_s^2 = 0, \tag{3.10}$$

where

$$a_s^2 = (x_s, x_s) = \int |x_s|^2 dV. \tag{3.11}$$

If $x_s \neq 0$, (3.10) implies

$$\lambda_s^k + c_1 \lambda_s^{k-1} + \cdots + c_{k-1} \lambda_s + c_k = 0. \tag{3.12}$$

Because (3.12) has at most k solutions and (3.10) holds for any $s \geq 1$, we see that at most k of x_t's are nonzero. This shows that the spectral decomposition (3.7) is in fact a finite sum. Thus the immersion is of finite type.

By using a similar argument, we can also characterize immersions of *mono-order*, bi-order, tri-order,..., and so on. For example, we have the following.

PROPOSITION 2. *Let $x : M \to E^m$ be an isometric immersion from a closed Riemannian manifold M into E^m. Then x is of mono-order (respectively, of bi-order,$\cdots$, etc.) if and only if H satisfies the following equation;*

$$\Delta H + c_1 H = 0 \quad \textit{(respectively, } \Delta^2 H + c_1 \Delta H + c_2 H = 0, \cdots, \textit{etc.)} \tag{3.13}$$

for some non-zero constants $c_1, c_2 \cdots$.

Let $f(s)$ be a periodic continuous function with period $2\pi r$. Then it is well-known that $f(s)$ has a *Fourier series expansion* given by

$$f(s) = \frac{b_0}{2} + b_1 \cos\left(\frac{s}{r}\right) + c_1 \sin\left(\frac{s}{r}\right) + b_2 \cos\left(\frac{2s}{r}\right) + c_2 \sin\left(\frac{2s}{r}\right) + \cdots, \tag{3.14}$$

where b_k and c_k are the Fourier coefficients defined by

(3.15) $$b_k = \frac{1}{\pi r}\int_{-\pi r}^{\pi r} f(s)\cos\left(\frac{ks}{r}\right)ds, \qquad k = 0, 1, 2, \cdots,$$

(3.16) $$c_k = \frac{1}{\pi r}\int_{-\pi r}^{\pi r} f(s)\sin\left(\frac{ks}{r}\right)ds, \qquad k = 1, 2, \cdots.$$

By using the concept of Fourier series expansion, we obtain the following classification of immersions of finite type for 1-dimensional Riemannian manifold in E^m.

THEOREM 3. *Let C be a closed curve of length $2\pi r$. Then an isometric immersion $x : C \to E^m$ is of finite type if and only if the Fourier series expansion of each coordinate function x_i has only finite non-zero terms.*

Proof. We put

(3.17) $$x^{(j)} = \frac{d^j x}{ds^j}.$$

Because $\Delta = -d^2/ds^2$ for $n = 1$, we have

(3.18) $$\Delta^j H = (-1)^j x^{(2j+2)}, \qquad j = 0, 1, 2, \cdots.$$

If x is of finite type, Theorem 1 implies that each coordinate function x_i satisfies the following homogeneous ordinary differential equation with constant coefficients;

(3.19) $$x_i^{(2k+2)} + c_1 x_i^{(2k)} + \cdots + c_{k-1} x_i^{(4)} + c_k x_i^{(2)} = 0, \qquad i = 1, 2, \cdots, m,$$

for some integer $k \geq 1$ and constants $c_1, \cdots, c_k$. Because our solutions x_i of (3.19) are periodic solutions with period $2\pi r$, each x_i is a finite linear combination of the following particular solutions;

(3.20) $$1, \quad \cos\left(\frac{n_i s}{r}\right), \quad \sin\left(\frac{m_i s}{r}\right); \qquad n_i, m_i \in Z.$$

Therefore, each x_i is of the following form

(3.21) $$c + \sum_{A=p}^{q}\left\{b_A \cos\frac{As}{r} + c_A \sin\frac{As}{r}\right\}$$

for some suitable constants c, b_A, c_A, $A = p, \cdots, q$, and integers

p, q. Thus each x_i has a Fourier series expansion of finite sum. The converse is easy to verify.

By using Theorem 3, we obtain the following.

THEOREM 4. *Let C be a closed curve of length $2\pi r$. If $x : C \to E^m$ is an isometric immersion of finite type, then C is immersed into a hypersphere $S^{m-1}(a)$ of E^m with radius a for some a, ie., C is of spherical type.*

Proof. Let $x : C \to E^m$ be of finite type. Then Theorem 3 implies that each coordinate function takes the following form

$$x_i = c_i + \sum_{j=p}^{q} \left\{ a_i(j) \cos\left(\frac{js}{r}\right) + b_i(j) \sin\left(\frac{js}{r}\right) \right\}, \tag{3.22}$$

for some suitable constants $a_i(j)$, $b_i(j)$ and c_i, $i = 1, \cdots, m$. In particular, (3.22) gives

$$\frac{dx_i}{ds} = \sum_{j=p}^{q} j \left\{ -a_i(j) \sin\left(\frac{js}{r}\right) + b_i(j) \cos\left(\frac{js}{r}\right) \right\}. \tag{3.23}$$

Because s is the arc length of C, $\langle dx/ds, dx/ds \rangle \equiv 1$. Thus (3.23) implies

$$\begin{aligned} \sum_{i=1}^{m} a_i(j)\, a_i(k) &= \sum_{i=1}^{m} b_i(j)\, b_i(k) = \sum_{i=1}^{m} a_i(j)\, b_i(k) \\ &= \sum_{i=1}^{m} b_i(j)\, a_i(k) = 0, \qquad \text{for} \quad j \neq k, \end{aligned} \tag{3.24}$$

$$\sum_{i=1}^{m} a_i(j)^2 = \sum_{i=1}^{m} b_i(j)^2 = e(j)^2, \qquad \text{for} \quad j = 1, \cdots, q, \tag{3.25}$$

$$\sum_{i=1}^{m} a_i(j)\, b_i(j) = 0, \qquad \text{for} \quad j = 1, \cdots, q, \tag{3.26}$$

where $e(j)$, $j = 1, \cdots, q$, are constants.

From (3.23), (3.24), (3.25), and (3.26) we may conclude that

$$\sum_{i=1}^{m} (x_i - c_i)^2 = \sum_{j=p}^{q} e(j)^2 \left\{ \cos^2\left(\frac{js}{r}\right) + \sin^2\left(\frac{js}{r}\right) \right\} = \sum_{j=p}^{q} e(j)^2 .$$

This proves the theorem.

If $m = 3$, we have the following stronger result.

THEOREM 5. *Let C be a closed curve of length $2\pi r$. Then*

(a) *If an isometric immersion $x : C \to E^3$ is of finite type, then x is of mono-order, say of order p.*

(b) *The order p of the immersion x is equal to the order of natural covering map; $C \to S^1(r/p) \subset E^2$, where $S^1(r/p)$ is a plane circle of radius r/p. And*

(c) *Every isometric imbedding from C into E^3 of finite type is the standard imbedding from C into a circle $S^1(r) \subset E^2$.*

Proof. Since $x : C \to E^3$ is of finite type, x_1, x_2 and x_3 take the form (3.22) for some suitable c_i, $a_i(t)$, and $b_i(t)$. Because s is the arc length of C, $\langle dx/ds,\ dx/ds\rangle \equiv 1$. Using (3.24) – (3.26) for $m = 3$, we can in fact prove that $x_3 = c_3$ and x_1, x_2 are of the following forms;

$$x_1 = c_1 + \sqrt{\frac{1}{p^2} - a^2}\,\sin\left(\frac{ps}{r}\right) \pm a\cos\left(\frac{ps}{r}\right), \tag{3.27}$$

$$x_2 = c_2 \mp \sqrt{\frac{1}{p^2} - a^2}\,\sin\left(\frac{ps}{r}\right) + a\sin\left(\frac{ps}{r}\right), \tag{3.28}$$

for some positive integer p and real number a, $0 < a \leq 1/p^2$ for suitable x_1, x_2, x_3. This proves the theorem.

REMARK. Let $x : S^n(1) \to E^{n+1}$ be the standard imbedding from $S^n(1)$ into E^{n+1}. Then we have

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 0\,. \tag{3.29}$$

Let $y : S^n(1) \to E^m$ be an isometric immersion from $S^n(1)$ into E^m. Then we have the following equations;

$$y^A = y^A(x_1, \cdots, x_{n+1}), \qquad A = 1, \cdots, m\,. \tag{3.30}$$

It is well-known that the eigenspace V_k of S^n is spanned by homogeneous polynomials of degree k (in terms of $x_1, \cdots, x_{n+1}$) [1]. Thus, we may conclude that the immersion is of order $[p, q]$ for some integers p, q if and only if y^A are polynomials in $x_1, \cdots, x_{n+1}$ with lowest degree p and highest degree q (except constant terms). Thus, *in this case, the notion of order of immersion and the notion of degree of polynomials coincide.* In fact, similar relation also holds for other rank one symmetric spaces.

We state the following Propositions. Their proofs are trivial. So we omit them.

PROPOSITION 6. *Let* $x : M \to E^m$ *and* $\bar{x} : \bar{M} \to E^{\bar{m}}$ *be two isometric immersions of closed manifolds. Then the product immersion* $(x, \bar{x}) : M \times \bar{M} \to E^{m+\bar{m}}$ *is of finite type if and only if both* x *and* $\bar{x}$ *are of finite type.*

PROPOSITION 7. *Let* $x : M \to E^m$ *and* $\bar{x} : M \to E^{\bar{m}}$ *be two isometric immersions of a closed Riemannian manifold* M *into* E^m *and* $E^{\bar{m}}$, *respectively. Then the normalized diagonal immersion* $\tilde{x} : M \to E^{m+\bar{m}}$; $y \to (1/\sqrt{2})\,(x(y), \bar{x}(y))$ *is of finite type if and only if both* x *and* $\bar{x}$ *are of finite type.*

By using these two Propositions, one can construct infinitely many submanifolds of finite type. For example, the standard imbedding from $S^1(a) \times S^1(b)$ in E^4 is of bi-order if $a \neq b$. Of course, there exist many submanifolds of finite type which are not of the types given by Propositions 6 and 7. For example, one can isometrically imbed $S^1(a) \times S^1(b)$, $a \neq b$, in E^6 as a surface of bi-order which is not of either type.

REMARK. Theorem 1 and Proposition 2 also hold if one replaces H by the position vector of the submanifold. Their proofs are given in the same way.

REMARK. Submanifolds of finite type will be treated in more details in a forthcoming book of the author.

4. **An upper bound of total mean curvature.** Throughout this paper, we assume that M is an n-dimensional closed Riemannian manifold and $x : M \to E^m$ an isometric immersion from M into E^m. We denote by ∇ and $\tilde{\nabla}$ the Riemannian connections on M and E^m, respectively. And denote by A, h and H the Weingarten map, the second fundamental form and the mean curvature vector of x. The mean curvature α is given by $\alpha = \|H\| = \langle H, H\rangle^{1/2}$, where $\langle\, ,\rangle$ is the inner product on both M and E^m.

By using the spectral decomposition, we have the following.

THEOREM 8. *Let $x : M \to E^m$ be an isometric imbedding of order $\le q$. Then we have*

$$\int_M \alpha^k \, dV \le \left(\frac{\lambda_g}{n}\right)^{k/2} \mathrm{vol}(M), \tag{4.1}$$

for $k = 1, 2, 3, 4$. The equality sign holds if and only if x is of order q.

Proof. Assume $x : M \to E^m$ is an isometric imbedding of order $\le q$. For any fixed vector a in E^m, we put

$$f = \langle H, a \rangle . \tag{4.2}$$

Then, for any tangent vector X in TM, we have

$$Xf = \langle \tilde{\nabla}_X H, a \rangle = -\langle A_H X, a \rangle + \langle D_X H, a \rangle, \tag{4.3}$$

where D is the normal connection on the normal bundle $T^\perp M$. Thus, for any $Y \in TM$, we have

$$\begin{aligned} YXf &= -\langle \tilde{\nabla}_Y (A_H X), a \rangle + \langle \tilde{\nabla}_Y D_X H, a \rangle \\ &= -\langle \nabla_Y (A_H Y), a \rangle - \langle h(Y, A_H X), a \rangle \\ &\quad - \langle A_{D_X H} Y, a \rangle + \langle D_Y D_X H, a \rangle. \end{aligned}$$

Therefore, the Laplace-Bletrami operator Δ of M satisfies

$$\begin{aligned} \Delta \langle H, a \rangle &= -\sum_i E_i E_i \langle H, a \rangle + \sum_i (\nabla_{E_i} E_i) \langle H, a \rangle \\ &= \sum_i \langle (\nabla_{E_i} A_H) E_i + A_{D_{E_i} H} E_i + h(E_i, A_H E_i), a \rangle \\ &\quad + \langle \Delta^D H, a \rangle, \end{aligned} \tag{4.4}$$

where $E_1, \cdots, E_n$ form an orthonormal frame in TM and

$$\Delta^D H = \sum_i \{ D_{\nabla_{E_i} E_i} H - D_{E_i} D_{E_i} H \} . \tag{4.5}$$

Δ^D is nothing but the Laplacian of the normal bundle.

Since (4.4) holds for any constant vector a in E^m, we find

$$\Delta H = \Delta^D H + \sum_i \{ h(E_i, A_H E_i) + A_{D_{E_i} H} E_i + (\nabla_{E_i} A_H) E_i \} . \tag{4.5}$$

Regard ∇A_H and A_{DH} as (1, 2)-tensors in $T^*M \otimes T^*M \otimes TM$ which are defined by

$$(\nabla A_H)(X, Y) = (\nabla_X A_H) Y, \tag{4.6}$$

$$A_{DH}(X, Y) = A_{D_X H} Y. \tag{4.7}$$

If we put

$$\bar{\nabla} A_H = \nabla A_H + A_{DH}, \tag{4.8}$$

then we obtain

$$\mathrm{tr}(\bar{\nabla} A_H) = \sum_i \{A_{D_{E_i} H} E_i + (\nabla_{E_i} A_H) E_i\}. \tag{4.9}$$

On the other hand, if $\xi_1, \cdots, \xi_{m-n}$ form an orthonormal frame in $T^\perp M$ so that $\xi_1 \| H$, then we also have

$$\|A_1\|^2 H + \mathfrak{a}(H) = \sum_i h(E_i, A_H E_i), \tag{4.10}$$

where $\|A_1\|^2 = \mathrm{tr}(A^2_{\xi_1})$ and

$$\mathfrak{a}(H) = \sum_{r=2}^{m-n} (\mathrm{tr}\, A_H A_{\xi_r})\, \xi_r \tag{4.11}$$

is the so-called allied mean curvature vector of M in E^m. By combining (4.5), (4.9), and (4.11), we obtain

$$\Delta H = \Delta^D H + \|A_1\|^2 H + \mathfrak{a}(H) + \mathrm{tr}\, \bar{\nabla} A_H. \tag{4.12}$$

Consequently, we find

$$\langle \Delta H, H \rangle = \langle \Delta^D H, H \rangle + \|A_H\|^2. \tag{4.13}$$

It is well-known that

$$\Delta x = -nH. \tag{4.14}$$

Thus, we have

$$n^2 \int \alpha^2\, dV = (\Delta x, \Delta x). \tag{4.15}$$

Assume that $x: M \to E^m$ is an isometric imbedding of order $\le q$. Then we have

$$x = \sum_{t \le q} x_t, \qquad \Delta x_t = \lambda_t x_t. \tag{4.16}$$

Thus, we get from (3.15) that

$$n^2 \int \alpha^2\, dV = \sum_{t=p}^{q} \lambda_t^2 a_t^2, \tag{4.17}$$

where $a_t^2 = \int \langle x_t, x_t \rangle dV \geq 0$ as defined in [3 IV, 5]. Similarly, we also have

$$n^3 \int \langle H, \Delta H \rangle dV = \sum_{t=p}^{q} \lambda_t^3 a_t^2, \tag{4.18}$$

$$-n \int \langle x, H \rangle dV = \sum_{t=p}^{q} \lambda_t a_t^2. \tag{4.19}$$

We put

$$\begin{aligned} A = n^2 \int \langle H, \Delta H \rangle dV \\ - n^2(\lambda_p + \lambda_q) \cdot \int \alpha^2 dV - n\lambda_p \lambda_q \int \langle x, H \rangle dV. \end{aligned} \tag{4.20}$$

From (4.17), (4.18), and (4.19), we obtain

$$A = \sum_{t=p+1}^{q-1} (\lambda_t - \lambda_p)(\lambda_t - \lambda_q) a_t^2 \leq 0, \tag{4.21}$$

with equality holding when and only when x is of bi-order $p \wedge q$. Consequently, we have

$$\begin{aligned} n \int \langle H, \Delta H \rangle dV \leq n(\lambda_p + \lambda_q) \cdot \int \alpha^2 dV \\ + \lambda_p \lambda_q \int \langle x, H \rangle dV. \end{aligned} \tag{4.22}$$

By using (4.13), (4.22), and the following Minkowski-Hsiung's formula;

$$\begin{aligned} n \int dV = (dx, dx) = (x, \delta dx) \\ = (x, \Delta x) = -n \int \langle x, H \rangle dV. \end{aligned} \tag{4.23}$$

we find

$$\begin{aligned} 0 \geq n^2 \int \langle \Delta^D H, H \rangle dV + \\ n^2 \cdot \int \|A_H\|^2 dV - n^2(\lambda_p + \lambda_q) \int \alpha^2 dV + n\lambda_p \lambda_q \int dV. \end{aligned} \tag{4.24}$$

Since M is closed, the Hopf lemma implies

$$\int \langle \Delta^D H, H \rangle dV = \int \|DH\|^2 dV. \tag{4.25}$$

Let $k_1, \cdots, k_n$ be the eigenvalues of A_H. Then it is easy to see the following identity;

$$\|A_H\|^2 = n\alpha^4 + \frac{1}{n}\sum_{i<j}(k_i - k_j)^2. \tag{4.26}$$

Combining (4.24), (4.25), (4.26) and Schwartz's inequality, we find

$$\begin{aligned} 0 &\geq n^2 \int \|DH\|^2 dV + n\int \sum_{i<j}(k_i - k_j)^2 dV + n^3 \int \alpha^4 dV \\ &\quad - n^2(\lambda_p + \lambda_q)\int \alpha^2 dV + n\lambda_p \lambda_q \int dV \\ &\geq n^2 \int \|DH\|^2 dV + n\int \sum_{i<j}(k_i - k_j)^2 dV \\ &\quad + n^3 \left(\int \alpha^2 dV\right)^2 \Big/ \int dV \\ &\quad - n^2(\lambda_p + \lambda_q)\int \alpha^2 dV + n\lambda_p \lambda_q \int dV. \end{aligned} \tag{4.27}$$

Hence, we obtain

$$\begin{aligned} 0 &\geq n \operatorname{vol}(M) \int \|DH\|^2 + \operatorname{vol}(M)\int \sum_{i<j}(k_i - k_j)^2 dV \\ &\quad + \left(n\int \alpha^2 dV - \lambda_p \operatorname{vol}(M)\right)\left(n\int \alpha^2 dV - \lambda_q \operatorname{vol}(M)\right). \end{aligned} \tag{4.28}$$

Because we have $\int \alpha^2 dV \geq \lambda_p \operatorname{vol}(M)/n$ (see [3], or (5.7)). We obtain from (4.28) that

$$\int \alpha^2 dV \leq \left(\frac{\lambda_q}{n}\right) \operatorname{vol}(M). \tag{4.29}$$

Thus, by (4.27), we get

$$n^2 \int \alpha^4 dV \leq n(\lambda_p + \lambda_q)\int \alpha^2 dV - \lambda_p \lambda_q \int dV \leq \lambda_q^2 \operatorname{vol}(M).$$

This proves

$$\int \alpha^4 dV \leq \left(\frac{\lambda_q}{n}\right)^2 \operatorname{vol}(M). \tag{4.30}$$

Thus, by using Hölder's inequality, we find

$$\int \alpha^k dV \leq \left(\int \alpha^4 dV\right)^{k/4} \left(\int dV\right)^{1-k/4}, \tag{4.31}$$

for $k<4$. In particular, by setting $k=1$, 2, or 3 and by using (4.30), we obtaininequality (4.1).

If the equality sign of (4.1) holds, then all the inequalities above become equalities, In particular, from (4.28), we see that H is parallel and M is pseudo-umbilical. Thus, by applying a result of Yano-Chen [9], we conclude that x is of order q. The converse of this is easy to verify.

As an immediate consequence of Theorem 8, we obtain the following.

THEOREM 9. *Let $x : M \to E^m$ be an isometric imbedding of order $\leq q$. If α is constant, then*

$$\int_M \alpha^n \, dV \leq \left(\frac{\lambda_q}{n}\right)^{n/2} \operatorname{vol}(M) . \tag{4.32}$$

The equality sign holds if and only if x is of order q.

REMARK. To author's knownledge, inequalities (4.1) and (4.23) give the first upper bound of total mean curvature.

By using Theorem 8, we may obtain the following best possible estimates on λ_k.

PROPOSITION 10. *Let $x : M \to S_0^m(1) \subset E^{m+1}$ be an isometric imbedding of M into the unit hypersphere S^m of E^{m+1}. If x is of order $[p, q]$, then*

$$\lambda_p \leq n \leq \lambda_q . \tag{4.33}$$

In particular, if $\lambda_p = n$ (respectively, $\lambda_q = n$), then x is of order p (respectively, of order q.)

Proof. Let $\bar{\alpha}$ be the mean curvature of M in S^m. Then we have $\alpha^2 = 1 + \bar{\alpha}^2$. Thus, from Theorem 8, we obtain

$$\operatorname{vol}(M) \leq \int \alpha^2 \, dV \leq \left(\frac{\lambda_q}{n}\right) \operatorname{vol}(M) .$$

This shows that $\lambda_q \geq n$. If $n = \lambda_q$, then $\bar{\alpha} = 0$ and x is of order q up to translations of E^{m+1}. The converse of this is clear.

If $p = 0$, then $\lambda_p \leq n$ is trivial. So we assume that $p \geq 1$. From (4.19) and (4.23), we find

$$n \int dV = -n \int \langle x, H \rangle \, dV = \sum_{t=p}^{q} \lambda_t a_t^2 \geq \lambda_p \sum_{t=p}^{q} a_t^2 . \tag{4.34}$$

Since M lies in the unit hypershpere $S_0^m(1)$ centered at 0, we have

$$n \operatorname{vol}(M) \geq \lambda_p \int \langle x, x \rangle dV = \lambda_p \operatorname{vol}(M). \tag{4.35}$$

This proves $n \geq \lambda_p$. If $n = \lambda_p$, (4.34) and (4.35) imply that x is of order p.

Proposition 10 implies immediately the following.

COROLLARY 11. *Let $x : M \to S_0^m(1) \subset E^{m+1}$ be an isometric immersion. If the center of gravity is at the center of $S_0^m(1)$, then*

$$\lambda_1 \leq n. \tag{4.36}$$

The equality sign holds if and only if x is of order 1.

Other immediate consequences of (1.2) and Theorem 8 are the following.

COROLLARY 12. *Let $x : M \to E^m$ be an isometric imbedding of order $\geq p$ (respectively, $\leq q$). If α is constant, then*

$$\lambda_p \leq n\alpha^2 \quad (\textit{respectively}, \ \lambda_q \geq n\alpha^2). \tag{4.37}$$

In particular, $\lambda_p = n\alpha^2$ (respectively, $\lambda_q = n\alpha^2$) if and only if x is of order p (respectively, of order q).

COROLLARY 13. *Let M be an n-dimensional closed Riemannian manifold with $\lambda_q \leq n$. Then every isometric imbedding $x : M \to S^m(1) \subset E^{m+1}$ of order $\leq q$ is a minimal imbedding into $S^m(1)$. Moreover, we have $\lambda_q = n$.*

5. **Some related inequalities.** In this section, we will obtain many inequalities related to inequality (4.1).

PROPOSITION 14. *Let $x : M \to E^m$ be an isometric immersion. Then we have*

$$\int \|dH\|^2 dV \geq \frac{1}{n} \int \{\lambda_1 \lambda_2 - n(\lambda_1 + \lambda_2)\alpha^2\} dV. \tag{5.1}$$

The equality sign holds if and only if x is of order [1, 2].

Proof. By using the spectral decomposition of x, we obtain

$$n^2(\Delta H, H) = (\Delta^2 x, \Delta x) = \sum_{t \geq 1} \lambda_t^3 a_t^2, \tag{5.2}$$

(5.3) $$n^2(H, H) = (\Delta x, \Delta x) = \sum_{t\geq 1} \lambda_t^2 a_t^2,$$

(5.3) $$-n(x, H) = (x, \Delta x) = \sum_{t\geq 1} \lambda_t a_t^2 .$$

Thus, we find

(5.4) $$\begin{aligned} &n^2(\Delta H, H) - n^2(\lambda_1 + \lambda_2)(H, H) - n\lambda_1 \lambda_2(x, H) \\ &\quad = \sum_{t\geq 3} \lambda_t(\lambda_t - \lambda_1)(\lambda_t - \lambda_2) a_t^2 \geq 0 . \end{aligned}$$

On the other hand, we also have

(5.5) $$(\Delta H, H) = (\delta dH, H) = (dH, dH) .$$

Thus, by using (4.23), (5.4), and (5.5), we obtain (5.1). If the equality sign of (5.1) holds, then, from (5.4), we obtain $a_t = 0$ for $t \geq 3$. Thus x is of order ≤ 2. The converse of this is trivial.

PROPOSITION 15. *Let* $x: M \to E^m$ *be an isometric immersion of order* $\geq p$. *Then*

(5.6) $$\int \|\Delta^k H\|^{2r}\, dV \geq \left(\frac{\lambda_p^{2k+1}}{n}\right)^r \operatorname{vol}(M),$$

(5.7) $$\int \|d\Delta^k H\|^{2r}\, dV \geq \left(\frac{\lambda_p^{2k+2}}{n}\right)^r \operatorname{vol}(M),$$

for $r\geq 1$ *and* $k = 0, 1, 2, \cdots$, *where* $\Delta^0 H = H$. *The equality sign of* (5.6) *or* (5.7) *holds for some* r *and* k *if and only if* x *is an imbedding of order* p.

Proof. Because $\Delta x = -nH$, the spectral decomposition of x gives

(5.8) $$n^2 \int \|\Delta^k H\|^2\, dV = n^2(\Delta^k H, \Delta^k H) = \sum_{t\geq p} \lambda_t^{2k+2} a_2^t,$$

(5.9) $$n^2 \int dV = -n(x, H) = \sum_{t\geq p} \lambda_t a_t^2 .$$

Thus, we find

(5.10) $$\begin{aligned} &n^2 \int \|\Delta^k H\|^2\, dV - n\lambda_p^{2k+1} \operatorname{vol}(M) \\ &\quad = \sum_{t\geq p+1} (\lambda_t^{2k+1} - \lambda_p^{2k+1})\, \lambda_t a_t^2 \geq 0 . \end{aligned}$$

This shows that

$$\int \|\Delta^k H\|^2 \, dV \geq \left(\frac{\lambda_p^{2k+1}}{n}\right) \operatorname{vol}(M) . \tag{5.11}$$

By applying Hölder's inequality, we obtain (5.6).

For (5.7), we consider

$$\begin{aligned} n^2(d\Delta^k H, d\Delta^k H) &= n^2(\Delta^k H, \delta d\Delta^k H) \\ &= n^2(\Delta^k H, \Delta^{k+1} H) = (\Delta^{k+1} x, \Delta^{k+2} x) \\ &= \sum_{t \geq p} \lambda_t^{2k+3} a_t^2 . \end{aligned} \tag{5.12}$$

By using (5.9) and (5.12), we obtain (5.7) for $r = 1$. Then, by using Hölder's inequality, we obtain (5.7) for any $r \geq 1$. The equality sign can be verified in a similar way as before.

Since every immersion can be chosen to be of order ≥ 1 and both sides of (5.6) and (5.7) are independent of the choice of the origin of E^m, Proposition 15 implies immediately the following.

PROPOSITION 16. *Let $x : M \to E^m$ be an isometric immersion. Then we have*

$$\int \|\Delta^k H\|^{2r} \, dV \geq \left(\frac{\lambda_1^{2k+1}}{n}\right)^r \operatorname{vol}(M), \tag{5.13}$$

$$\int \|d\Delta^k H\|^{2r} \, dV \geq \left(\frac{\lambda_1^{2k+2}}{n}\right)^r \operatorname{vol}(M), \tag{5.14}$$

for $r \geq 1$ and $k = 0, 1, 2, \cdots$. The equality sign of (5.13) or (5.14) holds for some r and k if and only if x is an imbedding of order 1.

REMARK. If $k = 0$ and $r = 1$, (5.13) is due to [2,6].

By similar argument as given in the proof of Proposition 15, we may also obtain the following.

PROPOSITION 17. *Let $x : M \to E^m$ be an isometric imbedding of order $\leq q$. Then we have*

$$\int \|\Delta^k H\|^2 \, dV \leq \left(\frac{\lambda_q^{2k+1}}{n}\right) \operatorname{vol}(M), \tag{5.15}$$

$$\int \|d\Delta^k H\|^2 \, dV \leq \left(\frac{\lambda_q^{2k+2}}{n}\right) \operatorname{vol}(M), \tag{5.16}$$

for $k=0, 1, 2, \cdots$. *The equality sign of* (5.15) *or* (5.16) *for some* k *holds if and only if* x *is of order* q.

From Proposition 15 we obtain immediately the following.

COROLLARY 18. *If* M *is a closed submanifold of* E^m, *then* $\Delta^k H \neq 0$, $k=0, 1, 2, \cdots$.

If $k=0$, this says that there is no closed minimal submanifolds in E^m which is well-known.

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