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DIMENSION FORMULAS FOR THE VECTOR SPACES OF SIEGEL'S MODULAR CUSP FORMS OF DEGREE TWO AND DEGREE THREE

BY

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1. Introduction. Let H_n be the Siegel upper-half space of degree n:

$$H_n = \{Z \in M_n(\boldsymbol{C}) \mid Z = {}^tZ, \operatorname{Im} Z > 0\}.$$

Here $M_n(\boldsymbol{C})$ is the ring of $n \times n$ matrices over $\boldsymbol{C}$. The real symplectic group of degree $2n$, $\mathrm{Sp}(n, \boldsymbol{R})$, acts transtively on H_n as a group of holomorphic automorphisms by the action,

$$M(Z) = (AZ+B)(CZ+D)^{-1}, \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ in } \mathrm{Sp}(n, \boldsymbol{R}).$$

Let $\mathrm{Sp}(n, \boldsymbol{Z}) = \mathrm{Sp}(n, \boldsymbol{R}) \cap M_{2n}(\boldsymbol{Z})$ be the discrete modular subgroup of $\mathrm{Sp}(n, \boldsymbol{R})$. A holomorphic function f defined on H_n is called a modular form of weight k and degree n with respect to $\mathrm{Sp}(n, \boldsymbol{Z})$ if f satisfies the following condition: $(n \geq 2)$

1. $f(M(Z)) = [\det(CZ+D)]^k f(Z)$ for all $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathrm{Sp}(n, \boldsymbol{Z})$.

The modular form f is called a cusp form if it satisfies the further condition:

2. Suppose that $\sum a(T) \exp[2\pi i \sigma(TZ)]$ is the Fourier expansion of f; then $a(T) = 0$ if rank $T < n$. Here the summation is over all half integral matrices T such that $T \geq 0$ and $\sigma(TZ) =$ trace of TZ.

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Denote by $S(k; \mathrm{Sp}(n, Z))$ the vector space of holomorphic cusp forms of weight k and degree n with respect to $\mathrm{Sp}(n, Z)$. If $k \geq 2n+3$ and $n \geq 2$, the dimension of $S(k; \mathrm{Sp}(n, Z))$ over C is given by Selberg's trace formula as follows [5]:

$$\dim_C S(k; \mathrm{Sp}(n, Z)) = C(k, n) \int_F \sum_M \left[\det\left(\frac{1}{2i}(Z - \overline{M(Z)})\right)\right]^{-k} \times \det(\overline{CZ+D})^{-k} (\det Y)^{k-(n+1)} dX\, dY,$$

where

1. $C(k, n) = 2^{-n}(2\pi)^{-n(n+1)/2} \circ \prod_{i=0}^{n-1} \Gamma(k-(n-i-1)/2) \circ [\prod_{i=0}^{n-1} \Gamma(k-n+i/2)]^{-1}$,
2. F is a fundamental domain on H_n for $\mathrm{Sp}(n, Z)$.
3. In the summation M ranges over all matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathrm{Sp}(n, Z)/\{\pm 1\}$.

This paper is devoted to our evaluation of $\dim_C S(k; \mathrm{Sp}(2, Z))$ and to presenting an effective procedure for the computation of all the terms necessary in the determination of $\dim_C S(k; \mathrm{Sp}(3, Z))$ via Selberg's trace formula when k is sufficiently large.

Though the dimension formula for $\Gamma_2(N)$ had been known earlier from papers of U. Christian [2, 3], Y. Morita [14], T. Shintani [17] and T. Yamazaki [20], a dimension formula for $\mathrm{Sp}(2, Z)$ was not known until 1981 when one was supplied by K. Hashimoto. Here we obtain the dimension formula for $\mathrm{Sp}(2, Z)$ by a method different in important respect from Hashimoto's.

As for the dimension formula for $\Gamma_3 = \mathrm{Sp}(3, Z)$ and its principal congruence subgroups $\Gamma_3(N)$, R. Tsushima [19] obtained a formula for $\Gamma_3(N)$, when $N \geq 3$, by using the Riemann-Roch-Hirzebruch Theorem. In this thesis, we compute all possible nonzero contributions from conjugacy classes in $\mathrm{Sp}(3, Z)$ by selecting suitable representatives in $\mathrm{Sp}(3, Z)$ from each conjugacy classes. Once the conjugacy classes of $\mathrm{Sp}(3, Z)$ have been given explicitly, we can then write down the dimension formula with respect to $\mathrm{Sp}(3, Z)$ explicitly as we have done in the case $n=2$.

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2. **Main results of the case $n=2$.** We identify the group of unitary matrices of size n, $U(n)$, with the maximal compact subgroup of $\mathrm{Sp}(n, \boldsymbol{R})$ via the mapping

$$A+Bi \longrightarrow \begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

In Chapter I, we add the conjugacy classes of finite order elements in $\mathrm{Sp}(2, \boldsymbol{Z})$ and their combinations with parabolic elements of $\mathrm{Sp}(2, \boldsymbol{Z})$ to the conjugacy classes of $\Gamma_2(N)$ as already determined in [14] and obtain all conjugacy classes of $\mathrm{Sp}(2, \boldsymbol{Z})$. Contributions from conjugacy classes of $\mathrm{Sp}(2, \boldsymbol{Z})$ are calculated in Chapter II. The main results are Theorem 1 to Theorem 10 as follows.

THEOREM 1. *Suppose* $M \in \mathrm{Sp}(n, \boldsymbol{Z})$ *is conjugate in* $\mathrm{Sp}(n, \boldsymbol{R})$ *to a unitary matrix* $U=\mathrm{diag}[\lambda_1, \lambda_2, \cdots, \lambda_n]$, *where* $|\lambda_i|=1$ *and* $\lambda_i \lambda_j \neq 1$ *for all* i, j; *then the contribution of elements in* $\mathrm{Sp}(n, \boldsymbol{Z})$ *which are conjugate in* $\mathrm{Sp}(n, \boldsymbol{Z})/\{\pm 1\}$ *to* M *is given by*

$$N_{\{M\}} = |C_{M,\boldsymbol{Z}}|^{-1} \prod_{i=1}^{n} \bar{\lambda}_i^k \prod_{1 \le i \le j \le n} (1-\bar{\lambda}_i \bar{\lambda}_j)^{-1}. \tag{1}$$

Here $C_{M,\boldsymbol{Z}}$ *is the centralizer of* M *in* $\mathrm{Sp}(n, \boldsymbol{Z})/\{\pm 1\}$ *and* $|C_{M,\boldsymbol{Z}}|$ *denotes its order.*

For an element as in Theorem 1, it has an isolated fixed point in the half space H_n. It was pointed out in [6] that the possible isolated fixed points of finite order elements in $\mathrm{Sp}(2, \boldsymbol{Z})$ are $\mathrm{Sp}(2, \boldsymbol{Z})$-equivalent to one of the following:

(1) $Z_1 = iE_2$, (2) $Z_2 = \rho E_2$,

(3) $Z_3 = \begin{bmatrix} \zeta^2 & \zeta^2+\bar{\zeta}^4 \\ \zeta^2+\bar{\zeta}^4 & \zeta^3 \end{bmatrix}$, (4) $Z_4 = \begin{bmatrix} \xi & (\xi-1)/2 \\ (\xi-1)/2 & \xi \end{bmatrix}$,

(5) $Z_5 = \dfrac{i}{\sqrt{3}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, (6) $Z_6 = \mathrm{diag}[i, \rho]$.

Here

$$\rho = e^{\pi i/3}, \quad \zeta = e^{\pi i/5} \quad \text{and} \quad \xi = \frac{1+2\sqrt{2}\,i}{3}.$$

Let G_i $(i=1, 2, 3, 4, 5, 6)$ denote the isotropy group in $\mathrm{Sp}(2, \boldsymbol{Z})/\{\pm 1\}$ at Z_i $(i=1, 2, 3, 4, 5, 6)$ respectively. Then their order are 16, 36, 5, 12, 12. From these groups, we obtain 22 conjugacy classes of elliptic elements in $\mathrm{Sp}(2, \boldsymbol{Z})$ and hence Theorem 1 applies.

The total contribution from conjugacy classes of finite order elements is $N_1 + N_2$ with

$$N_1 = \begin{cases} 2^{-7}3^{-3} \times [1131,\ 229,\ -229,\ -1131,\ 427,\ -571,\ 123, \\ \qquad -203,\ 203,\ -123,\ 571,\ -427] \\ \text{if } \ k \equiv [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] \pmod{12}. \end{cases}$$

$$N_2 = \begin{cases} 5^{-1} \ \text{ if } \ k \equiv 0 \pmod 5, \quad -5^{-1} \ \text{ if } \ k \equiv 3 \pmod 5 \\ 0 \ \text{ otherwise}. \end{cases}$$

Here N_2 is the total contribution from elements of order 5.

Elements $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ of $\mathrm{SL}_2(\boldsymbol{Z})$ are considered as elements of $\mathrm{Sp}(2, \boldsymbol{Z})$ through the embedding

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

For conjugacy classes of such kind, the contribution are computed by the following theorem.

THEOREM 2. *Let M be an element of the form* (2) *in* $\mathrm{Sp}(2, \boldsymbol{Z})$. *Suppose that M is conjugate in* $\mathrm{Sp}(2, \boldsymbol{R})$ *to* $\mathrm{diag}[1, \lambda]$, $\lambda \neq \pm 1$, *then the contribution of elements in* $\mathrm{Sp}(2, \boldsymbol{Z})$ *which are conjugate in* $\mathrm{Sp}(2, \boldsymbol{Z})/\{\pm 1\}$ *to M is*

$$\frac{2^{-4}3^{-1}\bar{\lambda}^k}{|G|}\left\{\frac{2k-3}{(1-\bar{\lambda}^2)(1-\bar{\lambda})} + \frac{1}{(1-\bar{\lambda})^3}\right\}, \tag{3}$$

where G is the centralizer of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *in* $\mathrm{PSL}_2(\boldsymbol{Z}) = \mathrm{SL}_2(\boldsymbol{Z})/\{\pm 1\}$ *with $|G|$ as its order.*

We also get

THEOREM 3. *Let*

$$M=\begin{bmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & s \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad a,\ b,\ c,\ d \ \textit{integers and} \ ab-cd=1 .$$

Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is conjugate in* $\mathrm{SL}_2(\boldsymbol{R})$ *to* $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ *with* $\lambda=e^{i\theta}\neq\pm 1$, *then the contribution of elements in* $\mathrm{Sp}(2, \boldsymbol{Z})$ *which are conjugate in* $\mathrm{Sp}(2, \boldsymbol{R})$ *to* M*'s as* s *ranges over a discrete subset* Ω *of* $\boldsymbol{R}^1-\{0\}$ *is*

$$(4) \qquad \frac{2^{-2}\,\bar{\lambda}^k\,\pi^{-1}}{|G|\,(1-\bar{\lambda}^2)(1-\bar{\lambda})}\lim_{\varepsilon\to 0}\cdot\sum_{s\in\Omega}(is)^{-(1+\varepsilon)} \qquad i=\sqrt{-1}\,.$$

With Theorem 2 and Theorem 3, we are able to compute contributions from comjugacy classes in Γ_1^∞ which is the semi-direct product of

$$\begin{bmatrix} 1 & 0 & 0 & s_2 \\ p & 1 & s_2 & s_3 \\ 0 & 0 & 1 & -p \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & 0 & b & 0 \\ 0 & \pm 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix}$$

with p, s_2, s_3, a, b, c, d integers and $ad-bc=1$. The total contribution is

$$N_3=\begin{cases} 2^{-5}\,3^{-3}\times[17k-294,\ -25k+325,\ -25k+254,\ 17k-261, \\ \qquad 17k-86,\ -k+53,\ -k-42,\ -7k+91, \\ \qquad -7k+2,\ -k-27,\ -k+166,\ 17k-181] \\ \text{if} \quad k\equiv[0,1,2,3,4,5,6,7,8,9,10,11] \pmod{12}\,. \end{cases}$$

The rest of contributions are from conjugacy classes in Γ_0^∞ which consists of elements of the form

$$\begin{bmatrix} E & S \\ 0 & E \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{bmatrix}=[S,\ U]\,,$$

with $S={}^tS$ in $M_2(\boldsymbol{Z})$ and U in $\mathrm{GL}_2(\boldsymbol{Z})$.

THEOREM 4. *The contribution of elements in* $\mathrm{Sp}(2, \boldsymbol{Z})$ *which are conjugate in* $\mathrm{Sp}(2, \boldsymbol{Z})/\{\pm 1\}$ *to* $\begin{bmatrix} E & S \\ 0 & E \end{bmatrix}$, $S={}^tS$ *in* $M_2(\boldsymbol{Z})$, *is*

$$2^{-9}3^{-3}5^{-1}(2k-2)(2k-3)(2k-4) \qquad (5)$$
$$- 2^{-5}3^{-2}(2k-3) + 2^{-4}3^{-1}.$$

THEOREM 5. *The contribution of elements in* Sp (2, Z) *which are conjugate in* Sp (2, Z)/{±1} *to* [S, U] *with* $S = \mathrm{diag}\,[s_1, s_2]$ *in* $M_2(Z)$ *and* $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *to the dimension formula is*

$$(-1)^k [2^{-9}3^{-2}(2k-2)(2k-4) - 2^{-6}3^{-1}(2k-3) + 2^{-5}]. \qquad (6)$$

THEOREM 6. *The contribution of elements in* Sp (2, Z) *which are conjugate in* Sp (2, Z)/{±1} *to* [S, U] *with*

$$S = \begin{bmatrix} s_1 & 1 \\ 1 & s_1 \end{bmatrix} \quad \text{and} \quad U = \mathrm{diag}\,[1, -1]$$

to the dimension formula is

$$(-1)^k [2^{-8}3^{-1}(2k-2)(2k-4) - 2^{-6}(2k-3) + 2^{-5}]. \qquad (7)$$

THEOREM 7. *The contribution of elements in* Sp (2, Z) *which are conjugate in* Sp (2, Z)/{±1} *to* [S, U] *with* $S = {}^tS$ *in* $M_2(Z)$, $\det S \neq 0$ *and* $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *to the dimension formula is*

$$(-1)^k 2^{-5}. \qquad (8)$$

THEOREM 8. *The contribution of elements in* Sp (2, Z) *which are conjugate in* Sp (2, Z)/{±1} *to* [S, U] *with* $S = \mathrm{diag}\,[s, 0]$, s *integer and* $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *to the dimension formula is*

$$2^{-6}3^{-1}(2k-3) - 2^{-4}. \qquad (9)$$

THEOREM 9. *The contribution of elements in* Sp (2, Z) *which are conjugate in* Sp (2, Z)/{±1} *to* [S, U] *with*

$$S = \begin{bmatrix} s & 1 \\ 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad s \text{ integer};$$

to the dimension formula is

$$2^{-7}(2k-3) - 2^{-4}. \qquad (10)$$

THEOREN 10. *The contribution of elements in* Sp (2, Z) *which are conjugate in* Sp (2, Z)/{±1} *to* [S, U] *with* $S = {}^tS$ *in* $M_2(Z)$ *and* $U = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ *to the dimension formula is*

$$(11) \qquad 2^{-1}3^{-3}(2k-3)-2^{-1}3^{-1}.$$

Note that

$$N_4 = (5)+(6)+(7)+(8)+(9)+(10)+(11)$$

$$(12) \qquad = \begin{cases} 2^{-7}3^{-3}5^{-1}(2k^3+96k^2-52k-3231) & \text{if } k \text{ is even}, \\ 2^{-7}3^{-3}5^{-1}(2k^3-114k^2+2018k-9051) & \text{if } k \text{ is odd}. \end{cases}$$

MAIN THEOREM I. *The dimension of the vector space of Siegel's cusp forms of degree* 2 *and weight* $k \geq 7$ *with respect to* $\mathrm{Sp}(2, \boldsymbol{Z})$ *is*

$$(13) \qquad \dim_{\boldsymbol{C}} S(k;\ \mathrm{Sp}(2, \boldsymbol{Z})) = N_1+N_2+N_3+N_4.$$

REMARK. The above formula is also true for $k=4$, 5 and 6. Here is a table of $\dim_{\boldsymbol{C}} S(k;\ \mathrm{Sp}(2, \boldsymbol{Z}))$ when $k \leq 50$.

k :	4	5	6	7	8	9	10	11	12	13	14	15
$\dim_{\boldsymbol{C}} S$:	0	0	0	0	0	0	1	0	1	0	1	0
k :	16	17	18	19	20	21	22	23	24	25	26	27
$\dim_{\boldsymbol{C}} S$:	2	0	2	0	3	0	4	0	5	0	5	0
k :	28	29	30	31	32	33	34	35	36	37	38	39
$\dim_{\boldsymbol{C}} S$:	7	0	8	0	9	0	11	1	13	0	13	1
k :	40	41	42	43	44	45	46	47	48	49	50	
$\dim_{\boldsymbol{C}} S$:	17	1	18	1	20	2	23	3	26	3	27	

3. **Main results of the case $n=3$.** To save our space, we identify $SL_2(\boldsymbol{R}) \times \mathrm{Sp}(2, \boldsymbol{R})$ with a subgroup of $\mathrm{Sp}(3, \boldsymbol{R})$ via the embedding

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \longrightarrow \begin{bmatrix} a & 0 & b & 0 \\ 0 & P & 0 & Q \\ c & 0 & d & 0 \\ 0 & R & 0 & S \end{bmatrix}.$$

Also we identify $\mathrm{Sp}(2, \boldsymbol{R}) \times SL_2(\boldsymbol{R})$ and $SL_2(\boldsymbol{R}) \times SL_2(\boldsymbol{R}) \times SL_2(\boldsymbol{R})$ with subgroups of $\mathrm{Sp}(3, \boldsymbol{R})$ via similar embeddings.

A conjugacy class $\{M\}$ of the element M in $\mathrm{Sp}(3, \boldsymbol{Z})$ has a possible nonzero contribution to the dimension formula only if

(1) M is an element of finite order,

or

(2) M is an element of infinite order and conjugate in $\mathrm{Sp}(3, \boldsymbol{R})$ to an element of the form $M' \cdot \begin{bmatrix} E & S \\ 0 & E \end{bmatrix}$, where M' is an element of finite order which has a positive dimensional fixed subvariety.

In the first case, M is conjugate in $\mathrm{Sp}(3, \boldsymbol{R})$ to a diagonal element U of $U(3)$ and the contribution from the conjugate class $\{M\}$ is given by

$$N = a(k) \cdot \mathrm{vol}\,(C_{M,\boldsymbol{Z}} \backslash C_{M,\boldsymbol{R}}) \cdot \int_{C_{M,\boldsymbol{R}} \backslash H_3} P(M, Z)^{-k} (\det Y)^{k-4}\, dX\, dY . \tag{14}$$

Here $C_{M,\boldsymbol{Z}}$ and $C_{M,\boldsymbol{R}}$ are centralizers of M in $\mathrm{Sp}(3, \boldsymbol{Z})$ and $\mathrm{Sp}(3, \boldsymbol{R})$ respectively. And

$$P(M, Z) = \left[\det\left(\frac{1}{2i}(Z - \overline{M(Z)})\right)\right] \cdot \det(\overline{CZ + D}) \quad \text{if } M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

If M_1 and M_2 are conjugate in $\mathrm{Sp}(3, \boldsymbol{R})$ but not in $\mathrm{Sp}(3, \boldsymbol{Z})$, then we have

$$\int_{C_{M_1,\boldsymbol{R}} \backslash H_3} P(M_1, Z)^{-k} (\det Y)^{k-4}\, dX\, dY = \int_{C_{M_2,\boldsymbol{R}} \backslash H_3} P(M_2, Z)^{-k} (\det Y)^{k-4}\, dX\, dY \tag{15}$$

Hence the ratio of their contribution to the dimension formula is $\mathrm{vol}\,(C_{M_1,\boldsymbol{Z}} \backslash C_{M_1,\boldsymbol{R}}) : \mathrm{vol}\,(C_{M_2,\boldsymbol{Z}} \backslash C_{M_2,\boldsymbol{R}})$.

Let M be an element of finite order in $\mathrm{Sp}(3, \boldsymbol{Z})$. Suppose that M is conjugate in $\mathrm{Sp}(3, \boldsymbol{R})$ to $U = \mathrm{diag}\,[\lambda_1, \lambda_2, \lambda_3]$ of $U(3)$ and

$$\Omega = \{Z \in H_3 \mid M(Z) = Z\} .$$

Then Ω is a nonempty set and is holomorphic to

$$\Omega' = \{W \in D_3 \mid W = {}^tUWU\} .$$

According to the complex dimension of Ω, we have following cases for conjugacy classes of finite order.

(1) $U=[1, 1, 1]$, $\dim_C \Omega = 6$.

(2) $U=[1, 1, -1]$, $\dim_C \Omega = 4$.

(3) $U=[1, 1, \lambda]$, $\dim_C \Omega = 3$.

(4) $U=[1, -1, \lambda]$, $\dim_C \Omega = 2$.

(5) $U=[\lambda, \lambda, \bar\lambda]$, $\dim_C \Omega = 2$.

(6) $U=[1, \lambda, \bar\lambda]$, $\dim_C \Omega = 2$.

(7) $U=[1, \lambda_1, \lambda_2]$, $\dim_C \Omega = 1$.

(8) $U=[\lambda_1, \lambda_2, \bar\lambda_2]$, $\dim_C \Omega = 1$; $\lambda_1 \neq \lambda_2$ and $\lambda_1 \neq \bar\lambda_2$.

(9) $U=[\lambda_1, \lambda_2, \lambda_3]$, $\dim_C \Omega = 0$.

In the above, we have λ^2, $\lambda_i\lambda_j \neq 1$ for all $i \leq j$. The corresponding contribution is given by N_i $(i=0, 1, 2, 3, 4, 5, 6, 7, 8)$ as follows.

(a) $U=[1, 1, 1]$, $\dim_C \Omega = 6$ and $M=E_6$.

(16) $$N_0 = 2^{-15}3^{-6}5^{-2}7^{-1}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6).$$

(Theorem 3, 5.3; Chapter V)

(b) $U=[1, 1, -1]$, $\dim_C \Omega = 4$;

(17) $$N_1 = c \cdot 2^{-14}3^{-4}5^{-1}(2k-2)(2k-4)^2(2k-6).$$

($c=1$ if $M=\operatorname{diag}[1, 1, -1, 1, 1, -1]$, Theorem 4, 5.3; Chapter V)

(c) $U=[1, 1, \lambda]$, $\dim_C \Omega = 3$;

(18) $$N_2 = c \cdot 2^{-9}3^{-3}5^{-1}\bar\lambda^k \left\{ \frac{(2k-3)(2k-4)(2k-5)}{(1-\bar\lambda)^3(1+\bar\lambda)} + \frac{3(2k-4)(2k-5)}{(1-\bar\lambda)^4} + \frac{6(2k-4)}{(1-\bar\lambda)^5} \right\}.$$

(c^{-1}=order of the centralizer of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $PSL_2(Z)$ if $M=E_4\times\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, Theorem 4, 4.4; Chapter IV)

(d) $U=[1, -1, \lambda]$, $\dim_C \Omega = 2$;

(19) $$N_3 = c \cdot 2^{-9}3^{-2}\bar\lambda^k \left\{ \frac{(2k-4)(2k-6)}{(1-\bar\lambda^2)^2} + \frac{4(2k-4)}{(1-\bar\lambda^2)^3} \right\}.$$

(c^{-1} = order of centralizer of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $PSL_2(Z)$ if $M=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, Theorem 20, 4.5; Chapter IV)

(e) $U=[\lambda, \lambda, \bar{\lambda}]$, $\dim_C \Omega = 2$;

$$N_4 = c \cdot 2^{-3}\, \bar{\lambda}^k \frac{(2k-3)(2k-5)}{(1-\bar{\lambda}^2)^3(1-\lambda^2)}. \tag{20}$$

(Theorem 23, 4.7; Chapter IV)

(f) $U=[1, \lambda, \bar{\lambda}]$, $\dim_C \Omega = 2$;

$$N_5 = c \cdot 2^{-6} \frac{(2k-3)(2k-5)}{(1-\bar{\lambda})(1-\lambda)(1-\bar{\lambda}^2)(1-\lambda^2)}. \tag{21}$$

(Theorem 14, 5.5; Chapter IV)

(g) $U=[1, \lambda_1, \lambda_2]$, $\dim_C \Omega = 1$;

$$\begin{aligned} N_6 = & \frac{c \cdot 2^{-4} 3^{-1} (\bar{\lambda}_1 \bar{\lambda}_2)^k}{(1-\bar{\lambda}_1^2)(1-\bar{\lambda}_1\bar{\lambda}_2)(1-\bar{\lambda}_2^2)} \\ & \cdot \left\{ \frac{2k-4}{(1-\bar{\lambda}_1)(1-\bar{\lambda}_2)} + \frac{2(1-\bar{\lambda}_1\bar{\lambda}_2)}{(1-\bar{\lambda}_1)^2(1-\bar{\lambda}_2)^2} \right\}. \end{aligned} \tag{22}$$

(c^{-1} = order of centralizer of $\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ in $\mathrm{Sp}(2, \boldsymbol{Z})$ if $M = E_2 \times \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, Theorem 2, 4.3; Chapter IV)

(h) $U=[\lambda_1, \lambda_2, \bar{\lambda}_2]$, $\dim_C \Omega = 1$;

$$\begin{aligned} N_7 = & \frac{c \cdot 2^{-2}\, \bar{\lambda}_1^k}{(1-\bar{\lambda}_1^2)(1-\bar{\lambda}_2^2)(1-\lambda_2^2)} \\ & \cdot \left\{ \frac{2k-4}{(1-\bar{\lambda}_1\bar{\lambda}_2)(1-\bar{\lambda}_1\lambda_2)} + \frac{1-\bar{\lambda}_1^2}{(1-\bar{\lambda}_1\bar{\lambda}_2)^2(1-\bar{\lambda}_1\lambda_2)^2} \right\}. \end{aligned} \tag{23}$$

(Theorem 21, 4.6; Chapter IV)

(i) $U=[\lambda_1, \lambda_2, \lambda_3]$, $\dim_C \Omega = 0$;

$$N_8 = |C_{M,\boldsymbol{Z}}|^{-1} (\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3)^k \prod_{1 \le i \le j \le 3} (1-\bar{\lambda}_i \bar{\lambda}_j)^{-1}. \tag{24}$$

(Theorem 1, 4.2; Chapter IV)

In the above formulas, c is a rational number depends only on $\mathrm{vol}\,(C_{M,\boldsymbol{Z}} \backslash C_{M,\boldsymbol{R}})$. This gives a complete treatment of computation of contributions from conjugacy classes of finite order elements in $\mathrm{Sp}(3, \boldsymbol{Z})$.

For the second case, we have to choose a suitable family of S for each fixed M' so that the total contributions from such family of conjugacy classes is given by

$$N' = a(k) \lim_{\varepsilon \to 0} \sum_{S} \int_{C_{M,Z} \backslash H_3} \cdot P(M, Z, S)^{-k} (\det Y)^{k-4} \underset{\text{factor}}{(\text{convergence})^{-\varepsilon}} \, dX \, dY . \tag{25}$$

Here are some typical examples which appear in our calculation.

(j) $M = [S, E_3]$. The total contributions is

(26)
(1) 0 if $\operatorname{rank} S = 1$,
(2) $-2^{-9} 3^{-2} 5^{-1} (2k-4)$ if $\operatorname{rank} S = 2$,
(3) $2^{-7} 3^{-3} (\operatorname{mod} 2^{-4} 3^{-4} 5^{-1} 7^{-1})$ if $\operatorname{rank} S = 3$.

(Theorem 5, 6, 12; Chapter V)

(k) $M = [S, U]$ with

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} s_1 & s_{12} & 0 \\ s_{12} & s_2 & 0 \\ 0 & 0 & s \end{bmatrix} \text{ and } U = [1, 1, -1] .$$

The total contributions is

(27)
(1) $-2^{-13} 3^{-3} 5^{-1} (2k-3)(2k-4)(2k-5)$ if $S_1 = 0$ and s runs over all nonzero integers,
(2) $-2^{-11} 3^{-3} (2k-3)(2k-5)$ if $\operatorname{rank} S_1 = 1$ and $s = 0$;
(3) $2^{-9} 3^{-2} (2k-4)$ if $\operatorname{rank} S_1 = 1$ and s runs over all nonzero integers,
(4) $2^{-10} 3^{-2} (2k-4)$ if $\operatorname{rank} S_1 = 2$ and $s = 0$,
(5) $-2^{-8} 3^{-1}$ if $\operatorname{rank} S_1 = 2$ and s runs over all nonzero integers.

(Theorem 7, 8, 9, 10, 5.4; Chapter V)

(l) $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} E & S \\ 0 & E \end{bmatrix}$ with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is conjugate in $SL_2(\boldsymbol{R})$ to $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $\lambda = e^{i\theta}$ $(\sin\theta \neq 0)$. The total contributions is

(28)
(1) $\dfrac{-2^{-7} 3^{-2} \bar{\lambda}^k}{|G|} \left\{ \dfrac{2k-4}{(1-\bar{\lambda})^3 (1+\bar{\lambda})} + \dfrac{1}{(1-\bar{\lambda})^4} \right\}$ if $\operatorname{rank} S = 1$,
(2) $\dfrac{2^{-6} 3^{-1} \bar{\lambda}^k}{|G|} \dfrac{1}{(1-\bar{\lambda})^3 (1+\bar{\lambda})}$ if $\operatorname{rank} S = 2$.

(Theorem 11, 4.3, Chapter IV)

(m) $M=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\times\begin{bmatrix} 1 & s_1 \\ 0 & 1\end{bmatrix}\times\begin{bmatrix} -1 & s_2 \\ 0 & -1\end{bmatrix}$ with $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ as in previous case. The contribution of conjugacy classes represented by M's is

$$(29)\quad \begin{cases} (1) & \dfrac{-2^{-7}3^{-1}\bar{\lambda}^k}{|G|}\left\{\dfrac{2k-4}{(1-\bar{\lambda}^2)^2}+\dfrac{1}{(1-\bar{\lambda})^3(1+\bar{\lambda})}\right\} \text{ if } s_2=0 \\ & \text{and } s_1 \text{ runs over all nonzero integers,} \\ (2) & \dfrac{-2^{-7}3^{-1}\bar{\lambda}^k}{|G|}\left\{\dfrac{2k-4}{(1-\bar{\lambda}^2)^2}+\dfrac{1}{(1-\bar{\lambda})(1+\bar{\lambda})^3}\right\} \text{ if } s_1=0 \\ & \text{and } s_2 \text{ runs over all nonzero integers,} \\ (3) & \dfrac{2^{-5}\bar{\lambda}^k}{|G|\,(1-\bar{\lambda}^2)^2} \text{ if } s_1 \text{ and } s_2 \text{ run over all nonzero} \\ & \text{integers,} \end{cases}$$

(Theorem 20, 4.5; Chapter IV)

(n) $M=\begin{bmatrix} 1 & s \\ 0 & 1\end{bmatrix}\times\begin{bmatrix} P & Q \\ R & S\end{bmatrix}$ with $\begin{bmatrix} P & Q \\ R & S\end{bmatrix}$ is conjugate in $\mathrm{Sp}(2,\boldsymbol{R})$ to $[\lambda_1, \lambda_2]$, λ_1^2, $\lambda_1\lambda_2$, $\lambda_2^2\neq 1$ and has cetralizer G in $\mathrm{Sp}(2,\boldsymbol{Z})$. The total contribution as s runs over all nonzero integers is

$$(30)\qquad \frac{-2^{-2}(\bar{\lambda}_1\bar{\lambda}_2)^k}{|G|\,(1-\bar{\lambda}_1^2)(1-\bar{\lambda}_1\bar{\lambda}_2)(1-\bar{\lambda}_2^2)(1-\bar{\lambda}_1)(1-\bar{\lambda}_2)}.$$

Combining results in (16), (17), (26) and (27); we obtain

MAIN THEOREM II. *The dimension formula for the principal congruence subgroup* $\Gamma_3(2)$ *of* $\Gamma_3=\mathrm{Sp}(3,\boldsymbol{Z})$ *is*

$$\begin{aligned} &\dim_C S(k;\ \Gamma_3(2)) \\ &\quad=[\Gamma_3:\Gamma_3(2)] \\ &\qquad\times[2^{-15}3^{-6}5^{-2}7^{-1}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6) \\ &\qquad+2^{-14}3^{-4}5^{-1}(2k-2)(2k-4)^2(2k-6) \\ &\qquad-2^{-14}3^{-4}5^{-1}(2k-3)(2k-4)(2k-5) \\ &\qquad-2^{-13}3^{-3}(2k-3)(2k-5)-2^{-14}3^{-2}5^{-1}(2k-4) \\ &\qquad+2^{-13}3^{-1}(2k-4)-2^{-12}3^{-1}+2^{-13}3^{-3}*] \end{aligned}$$

for an even integer $k\geq 9$, *where* $[\Gamma_3:\Gamma_3(2)]=2^9 3^4\cdot 35$ *and the final term* $*$ *is determined modulo an integral multiple of* $2^{-9}3^{-4}5^{-1}7^{-1}$.

MAIN THEOREM III. *The dimension formula for the principal congruence subgroup* $\Gamma_3(N)$ $(N\geq 3)$ *of* Γ_3 *is given by*

$$\begin{aligned}\dim_C S(k;\ \Gamma_3(N)) &= [\Gamma_3 : \Gamma_3(N)] \\ &\quad\times [2^{-15}3^{-6}5^{-2}7^{-1}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6) \\ &\quad - 2^{-9}3^{-2}5^{-1}(2k-4)N^{-5} + 2^{-7}3^{-3}N^{-6}*],\end{aligned}$$

where k *is an even integer* ≥ 9, $[\Gamma_3 : \Gamma_3(N)] = \frac{1}{2}N^{21}\cdot \prod_{p|N}(1-p^{-2})(1-p^{-4})(1-p^{-6})$ (p : *prime*) *and the final term* $*$ *is determined modulo an integral multiple of* $2^{-3}3^{-4}5^{-1}7^{-1}N^{-6}$.

REMARK. Here we are unable to give precise formulas for these two Theorems directly since it is difficult to compute the contribution $\xi_3(0)$ (as defined in [17]) coming from conjugacy classes of the form $[S, E]$ with rank $S = 3$. In our calculation, we obtain only that $\xi_3(0) = 2^{-7}3^{-3} + l \cdot 2^{-3}3^{-4}5^{-1}7^{-1}$ (l an integer). Main Theorem III is less precise that given in [19] where R. Tsushima gave the dimension formula for the principal congruence subgroup $\Gamma_3(N)$ in the form

$$\begin{aligned}\dim_C S(k;\ \Gamma_3(N)) &= [\Gamma_3 : \Gamma_3(N)] \\ &\quad\times [2^{-15}3^{-6}5^{-2}7^{-1}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6) \\ &\quad - 2^{-9}3^{-2}5^{-1}(2k-4)N^{-5} + 2^{-7}3^{-3}N^{-6}].\end{aligned}$$

However, we may compare this formula with the formula in Main Theorem III. This allows us to infer that $\xi_3(0) = 2^{-7}3^{-3}$ (a result that hitherto defied direct verification), and therefore to eliminate the integral multiples indicated by our asterisk in Main Theorem II and III.

For the remaining case $[S, U]$ with U is conjugate in $GL_3(\mathbf{R})$ to

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix} \qquad (\sin\theta \neq 0),$$

we have

THEOREM 11. *The contribution of elements in* $\mathrm{Sp}(3, \mathbf{Z})$ *which are conjugate in* $\mathrm{Sp}(3, \mathbf{Z})/\{\pm 1\}$ *to* $[S, U]$ *with*

$$S = \operatorname{diag}[s_1, s_2, s_2] \quad and \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

s_1 and s_2 are integers, is

$$2^{-10}3^{-2}(2k-3)(2k-5) - 2^{-10}3^{-1}(2k-4) \\ - 2^{-9}3^{-1}(2k-4) + 2^{-8}.$$

THEOREM 12. *The contribution of elements in* Sp (3, $\boldsymbol{Z}$) *which are conjugate in* Sp (3, $\boldsymbol{Z}$)/{±1} *to* $[S, U]$ *with* $S = {}^tS$ *in* $M_3(\boldsymbol{Z})$ *and*

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

is

$$2^{-5}3^{-4}(2k-3)(2k-5) - 2^{-3}3^{-3}(2k-4) \\ - 2^{-4}3^{-2}(2k-4) + 2^{-3}3^{-1}.$$

4. **Remark.** To get an explicit dimension formula for the modular group Sp (3, $\boldsymbol{Z}$), it remains

(1) to find all elliptic conjugacy classes of Sp (3, $\boldsymbol{Z}$) and determine the order of each conjugacy class.

(2) to find all conjugacy classes of finite order elements in Sp (3, $\boldsymbol{Z}$) which have a positive dimensional set of fixed points and determine vol $(C_{M,\boldsymbol{Z}} \backslash C_{M,\boldsymbol{R}})$ for each such conjugacy class $\{M\}$.

A recent communication form Dr. K. Hashimoto informed the author that it is unnecessary to classify the elliptic conjugacy classes of Sp (3, $\boldsymbol{Z}$) in order to compute the total contributions from them. Hence it is hopeful to solved (1) in this way.

For the case $n = 2$, elements of finite order in Sp (2, $\boldsymbol{Z}$) which have a positive dimensional set of fixed points are conjugate in Sp (2, $\boldsymbol{Z}$) to elements in the stablizers of cup

$$\begin{bmatrix} z_1 & * \\ * & i\infty \end{bmatrix} \text{ and } \begin{bmatrix} i\infty & * \\ * & i\infty \end{bmatrix}$$

It is optimistic to expect this property holds for the case $n=3$ so that (2) can be solved.

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