BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 11, Number 2, June 1983

# ON THE ADJOINT OF SPECTRAL OPERATORS

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**Abstract.** If $T^*$ is a prespectral operator with resolution of the identity $E^*(\cdot)$ of class $X$, we prove that $T$ is a spectral operator with resolution of the identity $E(\cdot)$. An example is also given to show that if $T^*$ is a prespectral operator with resolution of the identity $F(\cdot)$ which are not predual then $T$ need not be spectral.

It is shown in [3] that if $T$ is a spectral operator on a Banach space $X$ with resolution of the identity $E(\cdot)$ then the adjoint $T^*$ of $T$ is a prespectral operator with resolution of the identity $E^*(\cdot)$ of class $X$, where $E^*(\cdot)$ is the adjoint operator of $E(\cdot)$ in $L(X)$. The converse part of this result is proved in this note. Namely, if $T^*$ is a prespectral operator with resolution of the identity $E^*(\cdot)$ of class $X$ then $T$ is a spectral operator with resolution of the identity $E(\cdot)$. An example is also given to show that if $T^*$ is a prespectral operator with resolution of the identity $F(\cdot)$ which are not the adjoint of some operators in $L(X)$ then $T$ need not be a spectral operator.

We use here the notations and definitions of [2]. Let $X$ be a complex Banach space with dual space $X^*$. Operator means bounded linear operator. The Banach algebra of operators on $X$ is denoted by $L(X)$. A family $\Gamma \subset X^*$ is called total if $y \in X$ and $f(y) = 0$ for all $f \in \Gamma$, then $y = 0$. If $\Sigma$ is a $\sigma$-algebra of subsets of an arbitrary set $A$, suppose that a mapping $E(\cdot)$ from $\Sigma$ into a Boolean algebra of projections on $X$ satisfying the following conditions:

(1) $E(\delta_1) + E(\delta_2) - E(\delta_1) E(\delta_2) = E(\delta_1 \cup \delta_2)$,

(2) $E(\delta_1) E(\delta_2) = E(\delta_1 \cap \delta_2)$,

(3) $E(A - \delta) = I - E(\delta)$,

Received by the editors November 16, 1981 and in revised form Feburary 3, 1983.

(4) $E(A) = I$,

(5) there is a $M > 0$ sueh that $\|E(\delta)\| < M$ for all $\delta \in \Sigma$,

(6) there is a total linear subspace $\Gamma$ of $X^*$ such that $(E(\cdot)x, y)$ is countably additive on $\Sigma$, for each $x$ in $X$ and $y$ in $\Gamma$. Then $E(\cdot)$ is called a spectral measure of class $(\Sigma, \Gamma)$.

An operator $T \in L(X)$ is called a prespectral operator of class $\Gamma$ if there is a spectral measure $E(\cdot)$ of class $(\Sigma_p, \Gamma)$ with value in $L(X)$ such that $TE(\delta) = E(\delta)T$, and $\sigma(T|E(\delta)X) \subset \bar{\delta}$, where $\Sigma_p$ is the $\sigma$-algebra of Borel subsets of complex plane. The spectral measure $E(\cdot)$ is called a resolution of the identity of class $\Gamma$ for $T$. If $\Gamma = X^*$, $T$ is called a spectral operator.

THEOREM. *Let $T$ be an operator on $X$ and let $E(\delta)$ be an operator on $X$ for every $\delta \in \Sigma_p$. Then $T$ is a spectral operator with resolution of the identity $E(\cdot)$ if and only if $T^*$ is a prespectral operator with resolution of the identity $E^*(\cdot)$ of class $X$.*

**Proof.** The necessity is proved by Dunford [3, Lemma 6].

Conversely, if $T^*$ is a prespectral operator with resolution of the identity $E^*(\cdot)$ of class $X$. Since $E^*(\cdot)$ is a spectral measure of class $(\Sigma_p, X)$, it follows that $E(\cdot)$ is a spectral measure of class $(\Sigma_p, X^*)$ by taking the second adjoint operator $E^{**}(\cdot)$ and restricting to $X$ as a subspace of $X^{**}$.

We shall prove that $T$ is a prespectral operator with resolution of the identity $E(\cdot)$ of calss $X^*$, and $T$ is therefore a spectral operator.

Since $T^* E^*(\delta) = E^*(\delta) T^*$, thus $TE(\delta) = E(\delta) T$.

If $\delta \in \Sigma_p$, and $\lambda \in C - \bar{\delta}$. Since $(T^*|E^*(\delta) X^*) \subset \bar{\delta}$ so that $(\lambda I^* - T^*|E^*(\delta) X^*)^{-1}$ exists, and is denoted by $R_\delta$, then $R_\delta \in L(E^*(\delta)X^*)$. Set $P_\delta = R_\delta(E^*(\delta))$ which is in $L(X^*)$. Then $E^*(\delta)P_\delta = E^*(\delta) R_\delta E^*(\delta) = R_\delta E^*(\delta) = P_\delta$ and

$$P_\delta E^*(\delta) = R_\delta E^*(\delta) E^*(\delta) = R_\delta E^*(\delta) = P_\delta .$$

Therefore, $E^*(\delta) P_\delta = P_\delta E^*(\delta)$, and thus $E^{**}(\delta) P_\delta^* = P_\delta^* E^{**}(\delta)$. Hence $P_\delta^*$ maps $E^{**}(\delta) X^{**}$ into $E^{**}(\delta) X^{**}$. Since

$$(\lambda I^* - T^*) P_\delta = E^*(\delta) = P_\delta(\lambda I^* - T^*),$$

it follows that $(\lambda I^{**} - T^{**}) P_\delta^* = P_\delta^* (\lambda I^{**} - T^{**}) = E^{**}(\delta)$.

Therefore, $(\lambda I^{**} - T^{**}|E^{**}(\delta)X^{**})^{-1}$ exists and equals $P_\delta^*|E^{**}(\delta)X^{**}$. By regarding $X$ as subspace of $X^{**}$, there obtains

$$\begin{aligned}((\lambda I^{**} - T^{**})|E^{**}(\delta)X)^{-1} &= ((\lambda I^{**} - T^{**})|E(\delta)X)^{-1} \\ &= ((\lambda I - T)|E(\delta)X)^{-1}.\end{aligned}$$

Hence $\lambda \notin \sigma(T|E(\delta)X)$, thus $\sigma(T|E(\delta)X) \subset \bar{\delta}$.

Therefore $T$ is a prespectral operator with resolution of the identity $E(\cdot)$ of class $X^*$, and the proof is complete.

LEMMA. *Let $K$ be a compact Hausdorff space and let $\phi$ be a continuous algebra homomorphism of $C(K)$ into $L(X)$ with $\phi(I) = I$. Then for every $S$ in $\phi(C(K))$, $S^*$ is a prespectral operator with a resolution of the identity of class $X$.*

(Cf. [1] and [2; Th. 5.21.]).

EXAMPLE. Let $X = C([0, 1])$, define $(Tf)(t) = tf(t)$, $t \in [0, 1]$ and $f \in X$. Then $\sigma(T) = [0, 1]$, define

$$\phi : X = C(\sigma(T)) \longrightarrow L(X)$$

by $\phi(g)f = gf$. Then $\phi$ is a bicontinuous algebra isomorphism from $X$ into $L(X)$ such that $\phi(g_0) = I$ and $\phi(g_1) = T$, where $g_0(t) = 1$, and $g_1(t) = t$. By the Lemma above, $T^*$ is therefore a prespectral operator with resolution of the identity of class $X$.

Suppose that $E^2 = E$ in $L(X)$ and $TE = ET$. Since for very $h \in X$, $(TE)(h)(t) = (ET)(h)(t)$, it follows that

$$\begin{aligned}&(*) \quad t(Eh)(t) = E(Th)(t) \qquad t \in [0, 1], \\ &(**) \ I \cdot (Eh) = E(I \cdot h).\end{aligned}$$

Claim that $Ef = (Eg_0) \cdot f$ for $f(t) = t^n$, $n = 0, 1, \cdots$. By induction, for $n = 0$, $f = g_0$, thus $Ef = (Eg_0) \cdot f$.

If $g(t) = t^{n+1}$, and $f(t) = t^n$, then $g = I \cdot f$, and

$$\begin{aligned}(Eg)(t) = (E(I \cdot f))(t) &= (I \cdot (Ef))(t), \text{ by } (**) \\ &= I(t)(Ef)(t) \\ &= t(Ef)(t) \\ &= t(Eg_0)(t)\, f(t), \text{ by induction} \\ &= (Eg_0)(t)\, g(t).\end{aligned}$$

Thus $Ef = (Eg_0) \cdot f$. This proves the claim.

By Stone-Weierstrass theorem, $Ef = (Eg_0) \cdot f$, for all $f \in X$. Choose $f = Eg_0$, then $(Eg_0)^2 = Eg_0$, and thus $Eg_0 = 0$ or $I$. It follows that $E = 0$ or $I$.

This shows that $T$ is not a prespectral operator of any class, and therefore $T$ is not spectral, but $T^*$ is a prespectral operator with resolution of the identity of class $X$, and provides an example of the kind required.

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