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BINARY TRIANGLES*

BY

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Abstract. A binary triangle $T(a_1\cdots a_n)$ or T_n of order n is a double series $\{a_{ij}: 1\leq i\leq n,\ 1\leq j\leq n-i+1\}$ of $n(n+1)/2$ binary numbers satisfying $a_{1j}=a_j$ for $1\leq j\leq n$ and $a_{ij}\equiv a_{i-1,j}+a_{i-1,j+1}$ (mod 2) for $i>1$. The present paper studies the number of ones of a binary triangle, which is denoted by $\#T(a_1\cdots a_n)$ or $\#T_n$, and determines all binary triangles having the first four or the last two possible numbers of ones.

In [5] and [6], it was shown that $0\leq\#T_n\leq[(n^2+n+1)/3]$ for any binary triangle T_n, where $[x]$ denotes the greatest integer less than or equal to x. But $\#T_n$ does not cover all integers between 0 and $[(n^2+n+1)/3]$. The smallest possibility of $\#T_n$ is 0, the second smallest jumps to n, then $n-1+[n/2]$, then $n-1+[(n+1)/2]$, then $2n-4$ or $2n-3$. No other integer less than $2n-3$ can be $\#T_n$. On the other hand, the largest possibility of $\#T_n$ is $[(n^2+n+1)/3]$, then $[(n^2+n)/3]$ which equals the former or less than it by 1, then drops to $[(n^2+2)/3]$. No other integer greater than $[(n^2+2)/3]$ can be $\#T_n$. Also, all binary triangles with $\#T_n=0$, n, $n-1+[n/2]$, $n-1+[(n+1)/2]$, $[(n^2+n)/3]$ or $[(n^2+n+1)/3]$ are determined.

By computer calculation, a table of all possible $\#T_n$ for $1\leq n\leq 20$ is established. Observing the table, it is found that if $n=2^k-2$, then $\#T_n$ is always even. This is proved as Theorem 1 and was also proved in [2, p. 77]. Also, numbers of the type 2^k-2 form critical points of distribution of $\#T_n$, i.e. for each n such that $2^k-2<n<2^{k+1}-2$, $\#T_n$ always distribute certain type of numbers. It remains open to characterize all distribution of $\#T_n$.

1. **Introduction.** Given an n-digits binary number $a_1a_2\cdots a_n$, if under every two consecutive digits we write their sum mod 2 and continue this process as in Figure 1, we determine a *binary triangle of order n* with $n(n+1)/2$ digits. Denote this binary triangle by $T(a_1a_2\cdots a_n)$ or T_n for short. There are 2^n different binary triangles of order n.

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0 1 0 0 1 0 1 1 0 0 0
1 1 0 1 1 1 0 1 0 0
0 1 1 0 0 1 1 1 0
1 0 1 0 1 0 0 1
1 1 1 1 1 0 1
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0

FIGURE 1

Let a_{ij}, $1 \leq i \leq n$, $1 \leq j \leq n-i+1$, denote the j-th digit in the i-th row of $T(a_1 a_2 \cdots a_n)$, then

$$a_{1j} = a_j, \qquad 1 \leq j \leq n, \tag{1.1}$$

and

$$a_{ij} \equiv a_{i-1,j} + a_{i-1,j+1} \pmod 2, \qquad i > 1. \tag{1.2}$$

The number of ones in $T(a_1 a_2 \cdots a_n)$ will be denoted by $\#T(a_1 a_2 \cdots a_n)$ or $\#T_n$ for short. Let $T^*(a_1 a_2 \cdots a_n)$ denote the binary triangle *inverse* to $T(a_1 a_2 \cdots a_n)$, i. e. $T(a_n a_{n-1} \cdots a_1)$. Then we have $\#T_n^* = \#T_n$.

Harborth [5] proved, in answer to a question of Steinhaus [8, p. 47–48], that for $n \equiv 0$ or 3 (mod 4) there exist at least four binary triangles of order n in which the number of ones is equal to the number of zeros. Graphs formed from binary triangles are extensively studied in [2], [3], [4], [7]. In this paper we study the possible number of ones in a binary triangle of order n and determine all possible binary triangles having the first four and the last two possible numbers of ones.

2. **Some special binary triangles.** To indicate periodicity properties we use the overbar in a manner suggested by its use for circulating decimals. Thus $T(\overline{110})$ denotes the binary triangles with $a_{3i+1} = a_{3i+2} = 1$ and $a_{3i} = 0$, where it is not necessary that $n = 3k$ for some k. Again, $T(11\bar{0})$ denotes the binary triangle

with $a_1 = a_2 = 1$ and $a_i = 0$ for all other i; and $T(\bar{0}11)$ denotes $T^*(11\bar{0}) \cdots$ etc.

Suppose $[x]$ denotes the greatest integer not exceeding x. Let $f_{in} = 1$ if $n \equiv i \pmod 4$, else $f_{in} = 0$. Then the reader may verify the following equalities.

(2.1) $\#T(\bar{0}) = 0$

(2.2) $\#T(\bar{1}) = \#T(1\bar{0}) = n$

(2.3) $\#T(\overline{01}) = \#T(01\bar{0}) = n - 1 + [n/2]$

(2.4) $\#T(\overline{10}) = \#T(11\bar{0}) = n - 1 + [(n + 1)/2]$

(2.5) $\#T(\overline{1100}) = \#T(101\bar{0}) = 2n - 2 - f_{0n}$

(2.6) $\#T(\overline{0110}) = \#T(011\bar{0}) = 2n - 2 - f_{1n}$

(2.7) $\#T(\overline{1001}) = \#T(111\bar{0}) = 2n - 2 - f_{3n}$

(2.8) $\#T(\overline{0011}) = \#T(001\bar{0}) = 2n - 3 - f_{2n}$

(2.9) $\#T(0\bar{1}) = \#T(1\bar{0}1) = 2n - 2$

(2.10) $\#T(0\overline{01}) = \#T(00\bar{1}) = 2n - 4 + [(n - 1)/2]$

(2.11) $\#T(1\overline{10}) = \#T(10\bar{1}) = 2n - 3 + [n/2]$

(2.12) $\#T(\overline{011}) = [(n^2 + n)/3]$

(2.13) $\#T(\overline{110}) = \#T(\overline{101}) = [(n^2 + n + 1)/3]$

If a certain row is known, then all rows below it are completely determined. Conversely, there are only two possibilities for the row immediately above. So there are two possible binary triangles with a given second row, e. g. if the second row is $\bar{1}$, then the binary triangle may be $T(\overline{01})$ or $T(\overline{10})$. Similarly, there are four possible binary triangles with a given third row, e. g. $T(\overline{1100})$, $T(\overline{0110})$, $T(\overline{1001})$ and $T(\overline{0011})$ have the same third row $\bar{1}$, and 2^{i-1} possibilities with a given i-th row.

Binary triangles of order $n = 2^k - 2$ are very special. They always have an even number of ones.

THEOREM 1.

$$\#T(a_1 a_2 \cdots a_n) \equiv \sum_{m=1}^{n} \left\{ \binom{n+1}{m} - 1 \right\} a_m \pmod 2.$$

Every $\#T_n$ is even if and only if $n = 2^k - 2$ for some k.

Proof. From (1.1) and (1.2) it is easy to get

$$a_{ij} \equiv \sum_{m=0}^{i-1} \binom{i-1}{m} a_{m+j} \pmod 2, \tag{2.14}$$

so that

$$\# T_n \equiv \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} \sum_{m=j}^{i+j-1} \binom{i-1}{m-j} a_m \pmod 2.$$

Let $r = m - j + 1$ and $s = i + j - m - 1$ then

$$1 \le i \le n, \quad 1 \le j \le n - i + 1, \quad j \le m \le i + j - 1$$

is equivalent to

$$1 \le m \le n, \quad 1 \le r \le m, \quad 0 \le s \le n - m.$$

Hence

$$\# T_n \equiv \sum_{m=1}^{n} \sum_{r=1}^{m} \sum_{s=0}^{n-m} \binom{r-1+s}{r-1} a_m \pmod 2.$$

Using the equality $\sum_{u=0}^{v} \binom{u+w}{w} = \binom{w+1+v}{w+1}$ twice, we get

$$\# T_n \equiv \sum_{m=1}^{n} \left\{ \binom{n+1}{m} - 1 \right\} a_m \pmod 2. \tag{2.15}$$

Suppose $n = 2^k - 2$ for some k. For each $1 \le m \le n$

$$\binom{n+1}{m} = \prod_{i=1}^{m} \frac{n+2-i}{i}. \tag{2.16}$$

For each $1 \le i \le m$ let $i = 2^s j$ with $s < k$ and j odd, then the equality $n + 2 - i = 2^s(2^{k-s} - j)$ implies that i and $n + 2 - i$ are divisible by the same power of 2. So $\binom{n+1}{m}$ is odd by (2.16) and then $\# T_n$ is always even by (2.15). On the other hand if n is not of the form $2^k - 2$, there exists at least one r such that $\binom{n+1}{r}$ is even. If we set $a_r = 1$ and the other $a_m = 0$, then $\# T_n$ is odd by (2.15).

REMARK. Theorem 1 was also proved in [2, p. 77].

3. **Binary triangles with a small number of ones.** Let r_i denote the number of ones in the i-th row of a binary triangle T_n. The binary triangles T'_{n-1}, T''_{n-2} and T'''_{n-3} are the binary triangles obtained by rejecting the first row, the first two rows, and the

first three rows of T_n respectively. Let a'_{ij} denote the j-th digit of the i-th row of T'_{n-1} and set $a'_j = a'_{1j}$. Then we have $a'_{ij} = a_{i+1,j}$ and $a'_j = a_{2j}\cdots$ etc., and similarly for a''_{ij}, a''_j, a'''_{ij} and a'''_j.

The j-th right column of T_n is the column containing all the digits a_{ij} with $1 \le i \le n+1-j$ and the j-th left column containing all $a_{i,j+1-i}$ with $1 \le i \le j$. If $a_{j+1} = a_{j+2} = \cdots = a_{j+i}$, then we write $a_1 \cdots a_j\, a^i_{j+1}\, a_{j+i+1} \cdots a_n$ as an abbreviation of $a_1 \cdots a_j\, a_{j+1} \cdots a_{j+i}\, a_{j+i+1} \cdots a_n$. So $T(0^r 10^s)$ denotes the binary triangle of order $r+s+1$ such that $a_{r+1} = 1$ and all other $a_i = 0$.

LEMMA 1. *If* $n = r+s+1 \ge 8$, $r \ge 3$ *and* $s \ge 3$, *then* $\#T(0^r 10^s) \ge 2n-3$ *except that* $\#T(0^3 10^7) = \#T(0^7 10^3) = 18 = 2n-4$.

Proof. For the case of $r = 3$, then $s \ge 4$. Denote c_i the number of ones in the i-th left column, then it is observed that $c_{4i} = 4$, $c_{4i+1} = c_{4i+2} = 2$ and $c_{4i+3} = 1$ for $i \ge 1$. But $\#T(0^3 10^s) = \sum_{j=4}^{n} c_j$. It is easy to check inductively that $\#T(0^3 10^s) \ge 2n-3 = 2s+5$ except for the case of $s = 7$; in that case, $\#T(0^3 10^7) = 18 = 2n-4 = 2s+4$.

For the case of $s = 3$, by symmetry we have $\#T(0^r 10^3) \ge 2n-3 = 2r+5$ except $\#T(0^7 10^3) = 18 = 2n-4 = 2r+4$.

For the case of $r \ge 4$ and $s \ge 4$. The first $r+4$ left columns form $T(0^r 10^3)$ and the last $s+4$ right columns form $T(0^3 10^s)$; their intersection is $T(0^3 10^3)$ with $\#T(0^3 10^3) = 9$ as in Figure 2. So

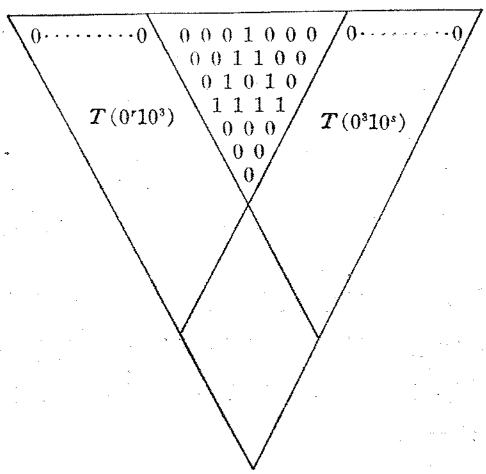

FIGURE 2

$$\begin{aligned}\#T(0^r10^s) &\geq \#T(0^3 10^s) + \#T(0^r 10^3) - \#T(0^3 10^3)\\ &\geq (2s+4) + (2r+4) - 9\\ &= 2n-3\,.\end{aligned}$$

LEMMA 2. *If* $r_1 = 1$, *then* T_n *is one of the following.*

(1) $T(1\bar{0})$ *or* $T(\bar{0}1)$ *and in this case* $\#T_n = n$.

(2) $T(01\bar{0})$ *or* $T(\bar{0}10)$ *and* $\#T_n = n - 1 + [n/2]$.

(3) $T(001\bar{0})$ *or* $T(\bar{0}100)$ *and* $\#T_n = 2n - 3 - f_{2n}$.

(4) $n = 7$, $T(0^3 10^3)$ *and* $\#T_n = 9 = n - 1 + [n/2] = 2n - 5$.

(5) $n = 11$, $T(0^3 10^7)$ *or* $T(0^7 10^3)$ *and* $\#T_n = 18 = 2n - 4$.

(6) $n \geq 8$, $T(0^r 10^s)$ *with* $r \geq 3$ *and* $s \geq 3$ *but not the one in* (5). *In this case* $\#T_n \geq 2n - 3$.

Proof. Since $r_1 = 1$, then $T_n = T(0^r 10^s)$. The lemma follows from (2.2), (2.3), (2.8) and Lemma 1.

LEMMA 3. *If* $n = r + s + 2$ *with* $r, s \geq 1$, *then* $\#T(0^r 110^s) \geq 2n - 3$ *except* $\#T(001100) = 8 = 2n - 4$. *If* $n = r + s + 3 \geq 4$, *then* $\#T(0^r 1010^s) \geq 2n - 3$ *except* $\#T(01010) = 6 = 2n - 4$.

Proof. $T(0^r 110^s)$ (vs $T(0^r 1010^s)$) is just the binary triangle obtained by rejecting the first row of $T(0^{r+1} 10^{s+1})$ (vs $T(0^{r+1} 110^{s+1})$). By (3) to (6) of Lemma 2 we have

$$\#T(0^r 110^s) = \#T(0^{r+1} 110^{s+1}) - 1 \geq 2(n+1) - 4 - 1 = 2n - 3,$$

except the case of $r = s = 2$ and in this case $\#T(001100) = 8 = 2n - 4$. And hence

$$\#T(0^r 1010^s) = \#T(0^{r+1} 110^{s+1}) - 2 \geq 2(n+1) - 3 - 2 = 2n - 3,$$

except the case of $r = s = 1$ and in this case $\#T(01010) = 6 = 2n - 4$.

LEMMA 4. *If* $r_1 = 2$, *then* $\#T_n \geq 2n - 3$ *except the following cases*:

(1) $T(11\bar{0})$ *or* $T(\bar{0}11)$; *and in this case* $\#T_n = n - 1 + [(n+1)/2]$.

(2) $T(001100)$, $T(01010)$ *or* $T(00010001000)$; *and in this case* $\#T_n = 2n - 4$.

Proof. Since $r_1 = 2$, then $T_n = T(0^r 10^s 10^t)$ with $n = r + s + t + 2$.

If $s=0$, then the case of $rt=0$ which is (1) of this lemma follows from (2.4), the case of $rt\geq 1$ follows from Lemma 3.

If $s=1$, then this lemma follows from Lemma 3.

If $s\geq 2$, then the first $r+3$ left columns form $T(0^r100)$ and the last $t+3$ right colums form $T(0010^t)$; and the intersection of the first $r+s+2$ left columns and the last $s+t+2$ right columns is $T(10^s1)$. But by (2.8) and (2.9) we have

$$\#T(0^r100)=2(r+3)-3-f_{2,r+3}=2r+3-f_{3r}, \tag{3.1}$$

$$\#T(0010^t)=2(t+3)-3-f_{2,t+3}=2t+3-f_{3t}, \tag{3.2}$$

$$\#T(10^s1)=2(s+2)-2=2s+2. \tag{3.3}$$

There are 6 ones in T_n counted twice in the above three binary triangles, see Figure 3, so

$$\begin{aligned}\#T_n &\geq (2r+3-f_{3r})+(2t+3-f_{3t})+(2s+2)-6+\#R\\ &=2n-2-f_{3r}-f_{3t}+\#R\end{aligned}$$

where $\#R$ is the number of ones in

$$R=T_n\backslash T(0^r100)\backslash T(0010^t)\backslash T(10^s1).$$

So $\#T_n\geq 2n-3$ except for the case of $r\equiv t\equiv 3 \pmod 4$, i.e. $f_{3r}=f_{3t}=1$. For the case of $r\equiv t\equiv 3 \pmod 4$, we will find some a_{ij} in R such that $a_{ij}=1$ except for the case of $r=s=t=3$; and so $\#T_n\geq 2n-3$ except for the case of $\#T(00010001000)=2n-4$. If $s=2$, choose a_{5r} in R, then by (2.14) in the proof of Theorem 1 we have

$$a_{5r}\equiv\sum_{m=0}^{4}\binom{4}{m}a_{m+r}=a_r+a_{r+4}=0+1=1 \pmod 2.$$

If $s=3$ and $r\neq 3$ (or $t\neq 3$ up to symmetric), then $r\geq 7$. Choose $a_{9,r-3}$ in R, then

$$a_{9,r-3}\equiv\sum_{m=0}^{8}\binom{8}{m}a_{m+r-3}=a_{r-3}+a_{r+5}=0+1=1 \pmod 2.$$

If $s\geq 4$, choose a_{6r} in R, then

$$\begin{aligned}a_{6r}&\equiv\sum_{m=0}^{5}\binom{5}{m}a_{m+r}=a_r+a_{r+1}+a_{r+4}+a_{r+5}\\ &=0+1+0+0=1 \pmod 2)\end{aligned}$$

So the proof is complete.

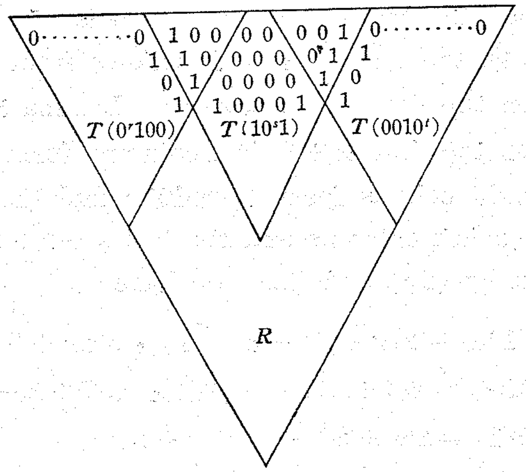

FIGURE 3

There is exactly one binary triangle of order n such that $\# T_n = 0$, i.e. $T(\bar{0})$ itself, which is called the *zero triangle.*

THEOREM 2. *If* $\# T_n > 0$, *then* $\# T_n \geq n$. $\# T_n = n$ *if and only if* T_n *is one of the following*: (1) $T(\bar{1})$, (2) $T(1\bar{0})$ *or* $T(\bar{0}1)$, (3) $n = 3$, $T(010)$.

Proof. The cases of $n = 1$, 2 are clear. Suppose $n \geq 3$ and the theorem holds for any $n' < n$.

If $\# T'_{n-1} = 0$, we know that the second row of T_n is $\bar{0}$ and then T_n is either $T(\bar{0})$ or $T(\bar{1})$. $T_n = T(\bar{0})$ is impossible since $\# T_n > 0$ and $T_n = T(\bar{1})$ implies $\# T_n = n$ by (2.2).

If $r_1 = 1$, then by Lemma 2 we know that either $\# T_n > n$ or $\# T_n = n$; and $\# T_n = n$ only when $T_n = T(1\bar{0})$ or $T(\bar{0}1)$ as in (1) of Lemma 2, or else $T_n = T(010)$ as in (2) of Lemma 2 with $n = 3$.

If otherwise $\# T'_{n-1} > 0$ and $r_1 \geq 2$, then by induction hypothesis $\# T'_{n-1} \geq n - 1$ and so $\# T_n = r_1 + \# T'_{n-1} \geq n + 1 > n$.

So, in any case, $\# T_n > 0$ implies $\# T_n \geq n$ and $\# T_n = n$ implies that T_n is one of the listed binary triangles. On the other hand, it is clear that all listed binary triangles satisfy $\# T_n = n$.

THEOREM 3. *If* $\# T_n > n \geq 4$, *then* $\# T_n \geq n - 1 + [n/2]$. *The equality holds if and only if* T_n *is one of the following*: (1) $T(\bar{0}1)$,

(2) $T(01\bar{0})$ *or* $T(\bar{0}10)$, (3) n *even*, $T(\bar{1}\bar{0})$, *or* $T(11\bar{0})$ *or* $T(\bar{0}11)$, (4) $n=6$, $T(001100)$ *or* $T(001000)$ *or* $T(000100)$, (5) $n=7$, $T(0001000)$.

Proof. If $n=4$, then $n-1+[n/2]=n+1$, so $\#T_n>n$ implies $\#T_n \geq n-1+[n/2]$. By actual computation, we can show the equality holds only when T_n is (1), (2) or (3). Suppose $n \geq 5$ and the theorem holds for any $n' < n$.

If $\#T'_{n-1}=0$, then T_n is either $T(\bar{0})$ or $T(\bar{1})$. In this case $\#T_n \leq n$.

If $\#T'_{n-1}=n-1$, we know by Theorem 2 that the second row of T_n is $\bar{1}$, $1\bar{0}$ or $\bar{0}1$. There are several possibilities for T_n, namely $T(\bar{0}\bar{1})$, $T(\bar{1}\bar{0})$, $T(1\bar{0})$, $T(0\bar{1})$ and the binary triangles inverse to them. By (2.2), (2.3), (2.4) and (2.9), then the theorem holds.

If $r_1=1$, then by Lemma 2 we know that either $\#T_n>n-1+[n/2]$ or $\#T_n=n-1+[n/2]$; and the equality holds only when T_n is $T(01\bar{0})$ or $T(\bar{0}10)$ as in (2) of Lemma 3, $T(001000)$ or $T(000100)$ as in (3) of Lemma 3 with $n=6$, or else $T(0001000)$ as in (4) of Lemma 3.

If $r_1=2$, then by Lemma 4 we know that either $\#T_n>n-1+[n/2]$ or $\#T_n=n-1+[n/2]$; and the equality holds only when T_n is $T(11\bar{0})$ or $T(\bar{0}11)$ with n even as in (1) of Lemma 4, $T(001100)$ or $T(01010)$ as in (2) of Lemma 4.

If otherwise $\#T'_{n-1}>n-1$ and $r_1 \geq 3$, then by induction hypothesis $\#T'_{n-1} \geq n-2+[(n-1)/2]$ and hence $\#T_n=\#T'_{n-1}+r_1 \geq n+1+[(n-1)/2]>n-1+[n/2]$.

So in any case, $\#T_n>n$ implies $\#T_n \geq n-1+[n/2]$ and equality holds implies that T_n is one of the listed binary triangles. On the other hand, it is clear that the listed binary triangles satisfy $\#T_n=n-1+[n/2]$.

In section 2 we saw that there are binary triangles with $\#T_n=n-1+[(n+1)/2]$. This number is equal to $n-1+[n/2]$ or greater than it by 1, depending on whether n is even or odd. Similar to Theorem 3 we have Theorem 4.

THEOREM 4. *For odd* $n>3$, $\sharp T = n-1+[(n+1)/2]$ *if and only if* T_n *is one of the following*: (1) $T(\overline{10}) = T^*(\overline{10})$, (2) $T(11\bar{0})$ *or* $T(\bar{0}11)$, (3) $n=5$, $T(00100)$, $T(01100)$ *or* $T(00110)$.

THEOREM 5. *If* $n \geq 7$ *and* $\sharp T_n > n-1+[(n+1)/2]$, *then*

$$\sharp T_n \geq \begin{cases} 2n-4 & \text{if } n \equiv 2 \pmod 4 \text{ or } n = 11, \\ 2n-3 & \text{otherwise.} \end{cases}$$

Proof. The case of $n=7$ is clear since $n-1+[(n+1)/2]=10$ and $2n-3=11$. Suppose $n \geq 8$ and the theorem holds for all $n' < n$.

If $\sharp T'_{n-1} \leq n-2+[n/2]$, then by Theorem 2, 3 and 4, the second row of T_n is $\bar{0}$, $\bar{1}$, $1\bar{0}$, $\overline{01}$, $\overline{10}$, $01\bar{0}$, $11\bar{0}$, 0001000 or their inverse. So T_n is $T(\bar{0})$, $T(\bar{1})$, $T(\overline{10})$, $T(\overline{01})$, $T(1\bar{0})$, $T(0\bar{1})$, $T(\cdots\overline{1100}\cdots)$, $T(00\bar{1})$, $T(11\bar{0})$, $T(01\bar{0})$, $T(10\bar{1})$, $T(0^4 1^4)$, $T(1^4 0^4)$, or their inverse. In any case the theorem holds.

If $r_1 = 1$, then by Lemma 2 the theorem holds.

If $r_1 = 2$, then by Lemma 4 the theorem holds.

If otherwise $r_1 \geq 3$ and $\sharp T'_{n-1} \geq 2n-6$, then $\sharp T_n = r_1 + \sharp T'_{n-1} \geq 2n-3$.

4. Binary triangles with a large number of ones.

From the above section, $T(\overline{a_1 a_2 \cdots a_n})$ denotes a binary triangle T_n having periodical first row $a_1 a_2 \cdots a_n$. We know that

$$T(\overline{a_i a_{i+1} \cdots a_n a_1 a_2 \cdots a_{i-1}}) = T(a_i a_{i+1} \cdots \overline{a_n a_1 a_2 \cdots a_n}), \qquad i = 1, 2, \cdots, n.$$

For convenience we write $T(\cdots \overline{a_1 a_2 \cdots a_n} \cdots)$ or $T(\cdots \overline{a_i a_{i+1} \cdots a_n a_1 a_2 \cdots a_{i-1}} \cdots)$ to denote any one of the n triangles.

LEMMA 5. *If* $T'''_{n-3} = T(\cdots \overline{110} \cdots)$, *then there are four possibilities for* T_n:

(1) $T(\cdots \overline{110} \cdots)$, *in this case* $r_1 + r_2 + r_3 = 2n-2$;

(2) $T(\cdots \overline{010} \cdots)$, *in this case* $\sharp T_n \leq [(n^2+2)/3]$ *and* $r_1 + r_2 + r_3 \leq (5n-2)/3$;

(3) $T(\cdots\overline{001110}\cdots)$, *in this case* $r_1+r_2+r_3\le(3n-1)/2$;

(4) $T(\cdots\overline{000101111010}\cdots)$, *in this case* $r_1+r_2+r_3\le(4n+4)/3$.

Proof. Since the fourth row of T_n is $\cdots\overline{110}\cdots$, so the third row is either $\cdots\overline{110}\cdots$ or $\cdots\overline{010}\cdots$, the second row is $\cdots\overline{110}\cdots$, $\cdots\overline{010}\cdots$ or $\cdots\overline{001110}\cdots$, and the first row is $\cdots\overline{110}\cdots$, $\cdots\overline{010}\cdots$, $\cdots\overline{001110}\cdots$ or $\cdots\overline{000101111010}\cdots$.

The equality in (1) is clear. For (2),

$$\#T_n=r_1+\#T'_{n-1}\le[(n+2)/3]+[(n^2-n+1)/3]=[(n^2+2)/3].$$

And $r_1+r_2+r_3\le(5n+c)/3$ for some constant c since any three consecutive right columns of the first three rows have exactly 5 ones. c is determined by testing all possible cases for $n=3,4,5$ and the first row begins with $010\cdots$, $100\cdots$, $001\cdots$. Similarly for the cases of (3) and (4).

LEMMA 6. *For any binary triangle T_n we have $r_1+r_2+r_3\le 2n-2$. If $r_1+r_2+r_3=2n-2$, then $T_n=T(\cdots\overline{110}\cdots)$, or $T'''_{n-3}=T(\overline{0})$, or else there are three consecutive zeros in the first row of T'''_{n-3}. If $r_1+r_2+r_3\ge 2n-3$, then there are no three consecutive zeros in the first row of T_n.*

Proof. To prove $r_1+r_2+r_3\le 2n-2$ we want to prove that any trapezoid part of T_n as in Figure 4 has at most $2n-2$ ones by induction on n. For the case of $n=2,3$, it is obvious that $r_1+r_2+r_3\le 2n-2$. Suppose $n\ge 4$ and the lemma holds for all $n'<n$. In Figure 4, if $a=b=c=1$, then $d=e=0$. By induction hypothesis, there are at mort $2(n-2)-2$ ones in the trapezoid marked $*$'s. So

$$r_1+r_2+r_3\le 2(n-2)-2+4=2n-2.$$

If a, b, c are not all ones, there are at most $2(n-2)-2$ ones in the trapezoid marked d, e, f and $*$'s. So again $r_1+r_2+r_3\le 2n-2$. Thus $r_1+r_2+r_3\le 2n-2$ for any binary triangle.

$$\begin{array}{l} a\ d\ *\ *\ \cdots\ *\ * \\ \ \ b\ e\ *\ *\cdots *\ * \\ \ \ \ \ c\ f\ *\ *\cdots *\ * \end{array}$$

FIGURE 4

$$\begin{array}{l} *\ *\cdots *\ *\ x \\ \ \ *\ *\cdots *\ *\ y \\ \ \ \ \ *\ *\cdots *\ *\ z \end{array} \quad \text{or} \quad \begin{array}{l} x\ *\ *\cdots *\ * \\ y\ *\ *\cdots *\ * \\ z\ *\ *\cdots *\ * \end{array}$$

FIGURE 5

By a similar argument, any parallelogram part of T_n as in Figure 5 has at most $2n$ ones under the condition that x, y, z are not all ones.

Suppose $r_1 + r_2 + r_3 = 2n - 2$. If there is run of r consecutive ones in the first row of T_n as in Figure 6. There are at most $2s$

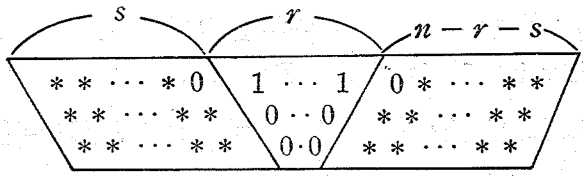

FIGURE 6

ones in the left parallelogram part and at most $2(n-r-s)$ ones in the right parallelogram part. Therefore $2n - 2 = r_1 + r_2 + r_3 \leq 2s + 2(n-r-s) + r$ which implies that $r \leq 2$. Also it is impossible the $\cdots 010 \cdots$ appears in the first row, as can be shown by replacing the trapezoid part of Figure 6 by $\begin{array}{c} 0\,1\,0 \\ 1\,1 \\ 0 \end{array}$ and get $2n - 2 = r_1 + r_2 + r_3 \leq 2s + 2(n-3-s) + 3$ which is a contradiction. So we have

(1) every run of ones in the first row of T_n has length at most 2; and exactly 2 except the beginning run and the ending run.

Similarly we can prove

(2) every run of zeros in the first row of T_n has length at most 2; exactly 1 for the first run and ending run.

(3) If $r_1 + r_2 + r_3 = 2n - 3$, then every run of zeros has length at most 2.

Now if there are not two consecutive zeros in the first row of T_n, then $T_n = T(\cdots \overline{110} \cdots)$ by (1) and (2). Otherwise 1001 appears as in Figure 7. Suppose $T'''_{n-3} \neq T(\bar{0})$. There is a one in the fourth row of T_n, assume it is in the right parallelogram part without loss generality. By (1), then $a = 1$ and so $b = 0$, $c = 1$, $x = 0$. By (1) again, then $d = 0$ and so $e = f = 1$, $y = 0$. So there are three consecutive zeros in the fourth row of T_n, i.e. the first row of T'''_{n-3}.

The last statement follows from (2) and (3).

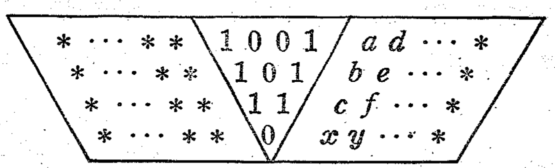

FIGURE 7

THEOREM 6. $\# T_n \leq [(n^2+n+1)/3]$ *for any binary triangle* T_n. *The equality holds if and only if* T_n *is* $T(\overline{110})$, $T(\overline{101})$ *or* $T(\overline{011})$ *with* $n \not\equiv 1 \pmod 3$.

Proof. The theorem is obvious for $n = 1, 2, 3$. Suppose $n \geq 4$ and the theorem holds for all $n' < n$. By the induction hypothesis we have

$$\begin{aligned} T'''_{n-3} &\leq [((n-3)^2 + n - 3 + 1)/3] \\ &= [(n^2+n+1)/3] - (2n-2). \end{aligned} \tag{4.1}$$

By Lemma 6 we have $r_1 + r_2 + r_3 \leq 2n - 2$. So

$$\# T_n = r_1 + r_2 + r_3 + \# T'''_{n-3} \leq [(n^2+n+1)/3].$$

The equality holds only when $r_1 + r_2 + r_3 = 2n - 2$ and the equality holds in (4.1). By Lemma 6, $T_n = T(\cdots \overline{110} \cdots)$ or $T'''_{n-3} = T(\overline{0})$ which is impossible, or else there are three consecutive zeros in the first row of T'''_{n-3} which contradicts to the induction hypothesis. Thus $T_n = T(\cdots \overline{110} \cdots)$. But from (2.12) and (2.13), we have

$$\# T(\overline{011}) = \# T(\overline{101}) = [(n^2+n+1)/3]$$

and

$$\# T(011) = [(n^2+n)/3] = [(n^2+n+1)/3] \quad \text{whenever } n \not\equiv 1 \pmod 3.$$

So the proof of this theorem is complete.

The first statement of Theorem 6 was also proved in [5] and [6]. In section 2, we have $\# T(\overline{011}) = [(n^2+n)/3]$, which is equal to $[(n^2+n+1)/3]$ or less than it by 1 when $n \equiv 1 \pmod 3$. Here we have the following theorem.

THEOREM 7. *If* $\# T_n = [(n^2+n+1)/3] - 1$, *then* T_n *is one of the following*: (1) $n = 1$, $T(0)$, (2) $n = 3$, $T(111)$, $T(010)$, $T(100)$, $T(001)$, (3) $n = 4$, $T(0110)$, $T(1001)$, $T(1110)$, $T(0111)$, (4) $n = 5$,

$T(01110)$, $T(01011)$, $T(11101)$, $T(01001)$ *and the binary triangles inverse to them,* (5) $n \equiv 1 \pmod 3$, $T(\overline{011}) = T^*(\overline{011})$.

Proof. The cases of $n \leq 5$ hold by examining all possible binary triangles. For the case of $n = 6$, since $[(n^2+n+1)/3] - 1 = 13$, by Theorem 1, it is impossible to have a triangle such that $\#T_n = 13$. Suppose $n \geq 7$ and the theorem holds for all $n' < n$.

By Lemma 6 and Theorem 6, either $r_1 + r_2 + r_3 = 2n - 3$ and $\#T'''_{n-3} = [((n-3)^2 + (n-3)+1)/3]$ or else $r_1 + r_2 + r_3 = 2n-2$ and $\#T'''_{n-3} = [((n-3)^2 + (n-3)+1)/3] - 1$. In the first case, the fourth row of T_n is $\cdots\overline{110}\cdots$, so by Lemma 5 we have $r_1 + r_2 + r_3 \neq 2n-3$ except for the case of $n = 7$ and $T_n = T(\cdots\overline{010}\cdots)$, in this case $\#T_n \leq 17 < 18 = [(n^2+n+1)/3] - 1$. In the second case, $T_n = T(\cdots\overline{110}\cdots)$ or $T'''_{n-3} = T(\bar{0})$ or else there are three consecutive zeros in the fourth row of T_n. But $\#T(\cdots\overline{110}\cdots) = [(n^2+n)/3] = [(n^2+n+1)/3] - 1$ only when $n \equiv 1 \pmod 3$ and $T_n = T(\overline{011}) = T^*(\overline{011})$; and if $T'''_{n-3} = T(\bar{0})$ or the first row of T'''_{n-3} has three consecutive zeros, then it contradicts the induction hypothesis. Thus the proof of this theorem is complete.

THEOREM 8. *If* $n \geq 6$ *and* $T_n \neq T(\cdots\overline{110}\cdots)$, *i.e.* $\#T_n < [(n^2+n+1)/3] - 1$, *then* $\#T_n \leq [(n^2+2)/3]$. *If* $\#T_n = [(n^2+2)/3]$, *then there are no three consecutive zeros in the first row of* T_n *except* $T(110001)$, $T(110001110)$ *and their inverse triangles.*

Proof. For the case of $n = 6, 7, 8$, since $[(n^2+n+1)/3] = [(n^2+2)/3] + 2$, so $\#T_n < [(n^2+n+1)/3] - 1$ implies $\#T_n \leq [(n^2+2)/3]$. If $\#T_n = [(n^2+2)/3]$, then by examining all triangles we know that there are no three consecutive zeros in the first row of T_n except $T(110001)$ and $T(100011)$. Suppose $n \geq 9$ and the theorem holds for all n' with $6 \leq n' \leq n$.

If $\#T'''_{n-3} \geq [((n-3)^2 + (n-3)+1)/3] - 1$, then the first row of T'''_{n-3} is $\cdots\overline{110}\cdots$ by Theorem 6 and 7. By Theorem 6

$$\#T'''_{n-3} \leq [((n-3) + (n-3) + 1)/3] = [(n^2+n+1)/3] - 2n + 2$$
$$= [(n^2+2)/3] + [n/3] - 2n + 2,$$

and then

$$\#T_n - [(n^2+2)/3] \leq r_1 + r_2 + r_3 + [n/3] - 2n + 2. \tag{3.4}$$

By Lemma 5, there are only three possibilities for T_n since $T_n \neq T(\cdots\overline{110}\cdots)$. For the case of $T_n = T(\cdots\overline{010}\cdots)$, $\#T_n \leq [(n^2+2)/3]$ and there are no three consecutive zeros in the first row of T_n. For the case of $T_n = T(\cdots\overline{001110}\cdots)$, $r_1 + r_2 + r_3 \leq (3n-1)/2$. So by (3.4) and the fact that $(3n-1)/2 + [n/3] - 2n + 2 \leq 0$ for any $n \geq 9$, then $\#T_n \leq [(n^2+2)/3]$. If $\#T_n = [(n^2+2)/3]$, then $(3n-1)/2 + [n/3] - 2n + 2 = 0$ and hence $n = 9$ and $r_1 + r_2 + r_3 = 13$. It is easy to check that T_n or $T_n^* = T(110001110)$. For the case of $T_n = T(\cdots\overline{000101111010}\cdots)$, $r_1 + r_2 + r_3 \leq (4n+4)/3$. So by (3.4) and the fact that $(4n+4)/3 + [n/3] - 2n + 2 \leq 1/3$ for any $n \geq 9$, then $\#T_n \leq [(n^2+2)/3]$. If $\#T_n = [(n^2+2)/3]$, then $0 \leq (4n+4)/3 + [n/3] - 2n + 2 \leq 1/3$ and hence $n = 9$ and $r_1 + r_2 + r_3 = 13$. It is easy to check that T_n or $T_n^* = T(010111101)$ and so there are no three consecutive zeros in the first row of T_n.

If $\#T'''_{n-3} < [((n-3)^2 + (n-3) + 1] - 1$, then by induction hypothesis $\#T'''_{n-3} \leq [((n-3)^2+2)/3] = [(n^2+2)/3] - 2n + 3$. If $r_1 + r_2 + r_3 \leq 2n - 3$, then $\#T_n = r_1 + r_2 + r_3 + \#T'''_{n-3} \leq [(n^2+2)/3]$. If $r_1 + r_2 + r_3 = 2n - 2$, then by Lemma 6 either $T'''_{n-3} = T(\bar{0})$ and hence $\#T_n = 2n - 2 < [(n^2+2)/3]$, or else there are three consecutive zeros in the first row of T'''_{n-3}, in this case by induction hypothesis $\#T'''_{n-3} < [((n-3)^2+2)/3]$ and so still have $\#T_n \leq [(n^2+2)/3]$. If $\#T_n = [(n^2+2)/3]$, then $r_1 + r_2 + r_3 \geq 2n - 3$. By Lemma 6, there are no three consecutive zeros in the first row of T_n.

The proof of this theorem is complete.

5. **Conclusion.** From the above results we know that $0 \leq \#T_n \leq [(n^2+n+1)/3]$ for any binary triangle T_n, but not any number between 0 and $[(n^2+n+1)/3]$ is a value of some $\#T_n$. The smallest possibility of $\#T_n$ is 0, next jumps to n, then $n - 1 + [n/2]$, then $n - 1 + [(n+1)/2]$, and then $2n - 4$ or $2n - 3$. There is no integer other than the above numbers which can be $\#T_n$ such that $\#T_n \leq 2n - 3$. On the other hand, the greatest possibility of $\#T_n$ is $[(n^2+n+1)/3]$, then $[(n^2+n)/3]$, and then drops to $[(n^2+2)/3]$. There is no integer other than these three numbers which can be $\#T_n$ such that $\#T_n \geq [(n^2+2)/3]$. The possible values of $\#T_n$ between $2n - 3$ and $[(n^2+2)/3]$ are more

complicated, and left as open problems. It is interesting that any integer between n and $[(n^2+n+1)/3]$ is a value of some $\#T_n$ when $1 \le n \le 5$. Any integer between $2n-3$ and $[(n^2+2)/3]$ is a value of some $\#T_n$ when $7 \le n \le 13$. By using a computer, we get a table for all possible values of $\#T_n$ up to $n=20$. It seems that any integer of the form 2^m-2 is a critical point of a certain kind of distribution of $\#T_n$. Now we have the following table. (In [2, 85], a table for $n \le 12$ is given, where the possible numbers of ones so as the numbers of triangles achieving these numbers of ones are listed.)

n	all possible $\#T_n$
1	0 1
2	0 2
3	0 3 4
4	0 4 5 6 7
5	0 5 6 7 8 9 10
6	0 6 8 10 12 14
7	0 7 9 10···18 19
8	0 8 11 13 14···21 22 24
9	0 9 12 13 15 16···26 27 30
10	0 10 14 16 17···33 34 36 37
11	0 11 15 16 18 19···40 41 44
12	0 12 17 21 22···47 48 52
13	0 13 18 19 23 24···56 57 60 61
14	0 14 20 24 26···even numbers···64 66 70
15	0 15 21 22 27 28 30 33 34···74 75 80
16	0 16 23 29 30 31 32 35 37 38···85 86 90 91
17	0 17 24 25 31 32 33 34 35 38 39···94 95 97 102
18	0 18 26 32 34 35 37 40 41···103 104 106 108 114
19	0 19 27 28 35 36 39 43 44···117 118 120 121 126 127
20	0 20 29 37 38 40 41 44 45 47 48···132 134 140

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