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LIE IDEALS OF PRIME RINGS WITH DERIVATIONS

BY

P. H. LEE (李白飛) AND T. K. LEE (李秋坤)

Abstract. Let R be a prime ring of char $R \neq 2$ with center Z and $d \neq 0$ a derivation on R. Suppose that U is a Lie ideal of R then $U \subseteq Z$ if any one of the following conditions is satisfied: (1) $d^2(U) \subseteq Z$, (2) $[a, d(U)] \subseteq Z$ for some $a \notin Z$, (3) $[d(U), d(U)] \subseteq Z$, (4) $d\delta(U) \subseteq Z$ for some derivation $\delta \neq 0$, (5) $[u, d(u)] \in Z$ for all $u \in U$, (6) $ad(U) \subseteq Z$ for some $a \neq 0$.

The objective of this paper is to extend the results of ours proved previously. In [2] we examine the action of derivations on a prime ring—the properties which imply the commutativity of the ring. Here we shall investigate the relationship between the derivations and the Lie ideals of a prime ring. As we shall see, the same conditions imposed on the Lie ideals, instead of the ring itself, will force the Lie ideals to be central.

In everything that follows *R will be a prime ring of characteristic not* 2 and *U will always denote a Lie ideal of R.* The center of R will be denoted by Z throughout. For elements a and b in R, $[a, b]$ will be the element $ab-ba$. However, given two subsets A and B of R then $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[a, b]$ where $a \in A$, $b \in B$.

We begin the material of this paper with the

LEMMA. *Let $d \neq 0$ be a derivation of R such that $d(Z) \neq 0$. If $a \in R$ and $[a, d(U)] \subseteq Z$, then either $a \in Z$ or $U \subseteq Z$.*

Proof. Choose $\alpha \in Z$ with $d(\alpha) \neq 0$. It is easily seen that $d(\alpha) \in Z$. For $u \in U$, $x \in R$, we have $\alpha[u, x] = [u, \alpha x] \in U$ so $[a, d(\alpha[u, x])] \in Z$ by hypothesis. Now $[a, d(\alpha[u, x])] = [a, d(\alpha)[u, x] + \alpha d([u, x])] = d(\alpha)[a, [u, x]] + \alpha[a, d([u, x])]$

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and since $[a, d([u, x])] \in Z$ we obtain that $d(\alpha)[a, [u, x]] \in Z$. But $d(\alpha) \in Z$ and $d(\alpha) \neq 0$, it follows that $[a, [u, x]] \in Z$ for all $u \in U$, $x \in R$. That is, $[a, [U, R]] \subseteq Z$ and a fortiori $[a, [a, [U, R]]] = 0$. Since $[U, R]$ is a Lie ideal of R, either $a \in Z$ or $[U, R] \subseteq Z$ by [1; Theorem 1]. However, if $[U, R] \subseteq Z$ then $[U, [U, R]] = 0$ and hence $U \subseteq Z$.

Next we prove a result which generalizes simultaneously Theorem 1 in [1] and Theorem 3 in [2].

THEOREM 1. *Let $d \neq 0$ be a derivation of R such that $d^2(U) \subseteq Z$. Then $U \subseteq Z$.*

Proof. Expanding $d^2([U, U]) \subseteq Z$ and using $d^2(U) \subseteq Z$ we have $[d(U), d(U)] \subseteq Z$. Suppose that $d(Z) \neq 0$; by the preceding lemma $[d(U), d(U)] \subseteq Z$ implies $d(U) \subseteq Z$ which yields $U \subseteq Z$ by [1; Lemma 6]. Therefore we may assume, in addition, that $d(Z) = 0$. Then $d^3(U) = d(d^2(U)) \subseteq d(Z) = 0$. In view of [1; Lemma 11] we obtain that $d^3 = 0$.

For $u \in U$ we claim that if $d^2(u) = 0$ then $d(u) \in Z$. If u, v are in U, $u[u, d(v)] = [u, ud(v)] \in U$ so $d^2(u[u, d(v)]) \in Z$. Since $d^2(u)$, $d^2(v)$ are in Z, $d([u, d(v)]) = [d(u), d(v)]$ and $d^2([u, d(v)]) = 0$. Therefore $d^2(u[u, d(v)]) = d^2(u)[u, d(v)] + 2d(u)[d(u), d(v)] \in Z$. Now if $d^2(u) = 0$ then $d(u)[d(u), d(v)] \in Z$ for all $v \in U$. Since $[d(u), d(v)] \in Z$ it follows that $[d(u), d(v)] = 0$ for all $v \in U$ which leads to $d(u) \in Z$ by [1; Theorem 2].

Let $u \in U$; expanding $d^2([u, d^2(R)])$ and using $d^2(u) \in Z$ and $d^3 = 0$ we have $d^2([u, d^2(R)]) = 0$. Since $[u, d^2(R)] \subseteq U$, by the previous paragraph $d([u, d^2(R)]) \subseteq Z$, that is, $[d(u), d^2(R)] \subseteq Z$. In particular, $[d(u), d^2(ud(x))] \in Z$ for all $x \in R$. Therefore $d(u)$ commutes with $[d(u), d^2(ud(x))] = [d(u), d^2(u)d(x) + 2d(u)d^2(x)] = d^2(u)[d(u), d(x)] + 2d(u)[d(u), d^2(x)]$. Both $d^2(u)$ and $[d(u), d^2(x)]$ are central, so we arrive at $d^2(u)[d(u), [d(u), d(x)]] = 0$. If $d^2(u) \neq 0$, it cannot be a zero-divisor and hence $[d(u), [d(u), d(x)]] = 0$. However, if $d^2(u) = 0$ then $d(u) \in Z$ and $[d(u), [d(u), d(x)]] = 0$ trivially. At any rate, $[d(u), [d(u), d(x)]] = 0$ for all $u \in U$ and $x \in R$. Replacing x by

$d(u)x$ and expanding $[d(u), [d(u), d(d(u)x)]]$ we get $d^2(u)[d(u), [d(u), x]] = 0$ and so $[d(u), [d(u), x]] = 0$ for all $x \in R$. Thus $d(u) \in Z$ for all $u \in U$ and consequently $U \subseteq Z$.

Now we remove the restriction $d(Z) \neq 0$ in the Lemma and prove the following

THEOREM 2. *Let $d \neq 0$ be a derivation of R and let $a \in R$ be such that $[a, d(U)] \subseteq Z$. Then either $a \in Z$ or $U \subseteq Z$.*

Proof. In light of the Lemma it suffices to consider only the case when $d(Z) = 0$. We assume that $U \not\subseteq Z$ and proceed to show that $a \in Z$.

For $u \in U$ we have that $[a, d([a, u])] \in Z$. Since $[a, d(u)] \in Z$, $[a, d([a, u])] = [a, [d(a), u] + [a, d(u)]] = [a, [d(a), u]] \in Z$ and hence $d([a, [d(a), u]]) = 0$. That is, $[d(a), [d(a), u]] + [a, d([d(a), u])] = 0$ and consequently $[d(a), [d(a), u]] \in Z$ for all $u \in U$. By Theorem 1 we have that $d(a) \in Z$. As a matter of fact $d(a) = 0$. Expanding $[a, d([a^2, u])] \in Z$ and making use of $d(a) \in Z$ and $[a, d(u)] \in Z$, we obtain that $d(a)[a, [a, u]] \in Z$ and so either $d(a) = 0$ or $[a, [a, u]] \in Z$ for all $u \in U$. However, if $[a, [a, U]] \subseteq Z$ it follows from Theorem 1 that $a \in Z$ and hence $d(a) = 0$ as well. Thus $[a, d^2(u)] = d([a, d(u)]) \in d(Z) = 0$ for all $u \in U$.

Let $u \in U$ and $v \in [U, U]$; then $d(v)$, $[d(v), u]$ and $[d(v), au]$ are all in U. Thus both $[a, d([d(v), u])]$ and $[a, d([d(v), au])]$ are central. Now $[a, d([d(v), u])] = [a, [d^2(v), u] + [d(v), d(u)]] = [a, [d^2(v), u]] \in Z$ because both $[a, d(v)]$ and $[a, d(u)]$ are in Z. And, $[a, d([d(v), au])] = [a, [d^2(v), au] + [d(v), ad(u)]] = a[a, [d^2(v), u]] + [d(v), a]\,[a, d(u)] \in Z$ so $a[a, [d^2(v), u]] \in Z$. From $[a, [d^2(v), u]] \in Z$ and $a[a, [d^2(v), u]] \in Z$ it follows that either $a \in Z$ or $[a, [d^2(v), u]] = 0$ for all $u \in U$ and $v \in [U, U]$. Since $U \not\subseteq Z$ it follows from [1; Lemma 3] that $[U, U] \not\subseteq Z$ and hence $d^2([U, U]) \not\subseteq Z$ by Theorem 1. So if $[a, [d^2([U, U]), U]] = 0$ then $a \in Z$ as a consequece of [1; Theorem 4]. This completes the proof.

As shown at the beginning of the proof of Theorem 1 one can easily verify the following generalization of [2; Theorem 2].

THEOREM 3. *Let* $d \neq 0$ *be a derivation of* R *such that* $[d(U), d(U)] \subseteq Z$. *Then* $U \subseteq Z$.

Now we extend Theorem 1 to a more general situation.

THEOREM 4. *Let* d *and* δ *be nonzero derivations of* R *such that* $d\delta(U) \subseteq Z$. *Then* $U \subseteq Z$.

Proof. For $u \in U$ and $v \in [U, U]$ we have $d\delta([u, \delta(v)]) \in Z$. That is, $[d(u), \delta^2(v)] \in Z$. If $U \nsubseteq Z$ then by Theorem 2 we have that $\delta^2([U, U]) \subseteq Z$ and hence $[U, U] \subseteq Z$ by Theorem 1 and so $U \subseteq Z$, a contradiction. Hence $U \subseteq Z$.

Next we sharpen Posner's theorem [3; Theorem 2] to

THEOREM 5. *Let* $d \neq 0$ *be a derivation of* R *such that* $[u, d(u)] \in Z$ *for all* $u \in U$. *Then* $U \subseteq Z$.

Proof. By linearizing the relation $[u, d(u)] \in Z$ we have that $[u, d(v)] + [v, d(u)] \in Z$ for all $u, v \in U$. Replace v by $[u, v]$ and expand explicitly; we obtain that $[u, [d(u), v]] + [u, [u, d(v)]] + [[u, v], d(u)] \in Z$. The Jacobi's identity yields $[u, [d(u), v]] + [[u, v], d(u)] = [[v, d(u)], u] + [[u, v], d(u)] = -[[d(u), u], v] = 0$. Hence $[u, [u, d(v)]] \in Z$ for all $u, v \in U$. Let $v \in [U, U]$ and replace u by $u + d(v)$; we obtain that $[d(v), [d(v), u]] \in Z$ for all $u \in U$ and $v \in [U, U]$. By Theorem 1 either $d([U, U]) \subseteq Z$ or $U \subseteq Z$. However, as we have seen before, $d([U, U]) \subseteq Z$ leads to $U \subseteq Z$ as well. Thus the theorem is proved.

We conclude this paper with a result of the same flavor as Theorem 2.

THEOREM 6. *Let* $d \neq 0$ *be a derivation of* R *and let* $a \in R$ *be such that* $ad(U) \subseteq Z$. *Then* $a = 0$ *or* $U \subseteq Z$.

Proof. If $a \in Z$ then either $a = 0$ or $d(U) \subseteq Z$ (and hence $U \subseteq Z$). So we assume that $a \notin Z$ and proceed to show that $U \subseteq Z$.

Assume first that $d(Z)\neq 0$; let $\alpha\in Z$ with $d(\alpha)\neq 0$. For $u\in U$, $x\in R$ we have that $ad(\alpha[u, x])=ad([u, \alpha x])\in Z$. That is, $d(\alpha)\,a[u, x]+\alpha\, ad([u, x])\in Z$ so $d(\alpha)\,a[u, x]\in Z$ and hence $a[u, x]\in Z$ for all $u\in U$, $x\in R$. Commuting this with a we get $a[a, [U, R]]=0$. By [1; Lemma 7] $[U, R]\subseteq Z$ and so $U\subseteq Z$. Therefore we may assume that $d(Z)=0$ in all that follows.

For $u\in U$ we have $ad([a, u])\in Z$. That is, $a[d(a), u]+a[a, d(u)]\in Z$. Since $a[a, d(u)]=[a, ad(u)]=0$, it follows that $a[d(a), U]\subseteq Z$. Let $v\in[U, U]$; then $ad(v)\in Z$ so $d(ad(v))=0$, that is, $d(a)\ d(v)+ad^2(v)=0$. But $d(v)\in U$, so $ad^2(v)\in Z$ and hence $d(a)\,d(v)\in Z$ for all $v\in[U, U]$. Now $a[d(a), d(v)]d(v)=a[d(a)\,d(v), d(v)]=0$. Since $a[d(a), d(v)]\in a[d(a), U]\subseteq Z$, we obtain that $a[d(a), d(v)]=0$ for all $v\in[U, U]$.

Suppose that $d(a)\neq 0$. Then $d(a)d([U, U])\neq 0$, say, $d(a)\,d(v_0)\neq 0$ for some $v_0\in[U, U]$. Set $\alpha=d(a)\,d(v_0)\in Z$ and $\beta=ad(v_0)\in Z$. From $a[d(a), d(v_0)]=0$ we have $\alpha a=\beta d(a)$. Thus $\alpha a[a, u]=a[\alpha a, u]=a[\beta d(a), u]=\beta a[d(a), u]\in Z$ for all $u\in U$. Since $\alpha\neq 0$ we end up with $a[a, u]\in Z$. Then $d(a[a, u])=0$, that is, $d(a)[a, u]+a[d(a), u]=0$. Multiplying β and using $\beta d(a)=\alpha a$ we get $2\alpha a[a, u]=0$ so $a[a, u]=0$ for all $u\in U$. Hence $U\subseteq Z$ since $a\notin Z$.

Now to the case when $d(a)=0$. For $u\in U$ we have $ad(u)\in Z$ so $ad^2(u)=d(ad(u))=0$. Let $x\in R$, $w\in W=[[U, U], [U, U]]$; then from $ad([x, d^2(w)])\in Z$ we have $ad(x)\,d^2(w)+axd^3(w)\in Z$ because $ad^2(w)=ad^3(w)=0$. For $x\in d(U)$ we get $ad(U)\,d^3(W)\subseteq Z$. Note that $ad(U)\subseteq Z$. If $ad(U)=0$ then $U\subseteq Z$. Assume that $ad(U)\neq 0$. Then $d^3(W)\subseteq Z$. But $ad^3(W)=0$ so it follows that $d^3(W)=0$. However, if $d^3(W)=0$ we get $ad(x)d^2(w)\in Z$ for all $x\in R$, $w\in W$. In particular, $ad(U)\,d^2(W)\subseteq Z$ and again $d^2(W)\subseteq Z$. By Theorem 1 we have $W=[[U, U],[U, U]]\subseteq Z$ so $[U, U]\subseteq Z$ and hence $U\subseteq Z$. This completes the proof.

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DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, TAIPEI, TAIWAN.