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REMARKS ON SOME SUBMANIFOLDS OF SASAKIAN MANIFOLDS

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Let M be a $(2m+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g). An n-dimensional submanifold N of M is said to be ***invariant*** (or ***ϕ-invariant***) in M if $\phi T_x(N) \subset T_x(N)$ holds for every $x \in N$, where $T_x(N)$ is the tangent space of N at x. It is said to be ***anti-invariant*** (or ***anti-ϕ-invariant***) in M if $\phi T_x(N) \subset T_x^{\perp}(N)$ holds for each $x \in N$, where $T_x^{\perp}(N)$ denotes the orthogonal complement of $T_x(N)$ in $T_x(M)$ (Yano-Kon [1]). A plane section p in $T_x(N)$, $x \in N$, is said to be ***ϕ-invariant*** if $\phi p = p$. It is said to be anti-ϕ-invariant if ϕp is perpendicular to p.

A submanifold N of a Sasakian manifold M is called a ***Sasakian submanifold*** of M if the structure vector field ξ is tangent to N everywhere on N and ϕX is tangent to N for every vector X tangent to N, that is N is invariant [1].

It is known that if N is an n-dimensional submanifold of a $(2m+1)$-dimensional Sasakian manifold M and the structure vector field ξ is normal to N, then N is an anti-invariant submanifold in M and $n \leq m$. It is also known that if N is an anti-invariant submanifold of dimension $n = m+1$, then the structure vector field ξ is tangent to N [1].

1. Submanifolds of Sasakian manifolds in which the structure vector field ξ is either always tangent or always normal.

PROPOSITION 1. *Let N be a submanifold of a Sasakian manifold. Then the structure vector field ξ is either tangent to N at every point*

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of N or normal to N at every point of N if and only if $\phi^2 X$ is tangent to N for every tangent vector X on N.

Proof. Suppose that $\phi^2 X$ is tangent to N for every tangent vector X at $x \in N$. Then, since $\phi^2 X = -X + \eta(X)\xi$ we have $\eta(X)\xi \in T_x(N)$. Then either (i) $\eta(X) \neq 0$ for some tangent vector X of N at x, or (ii) $\eta(X) = 0$ for every tangent vector X of N at x. In case (i) we have $\xi \in T_x(N)$. In case (ii) we have $\eta(X) = g(X, \xi) = 0$ for every tangent vector X of N at x. Thus in this case ξ is normal to $T_x(N)$. So at each point $x \in N$, ξ is either tangent to N or normal to N. It follows from continuity of the vector field ξ and the connectedness of N that ξ is either always tangent to N or always perpendicular to N.

Conversely, if ξ is tangent to N at every point of N then $\phi^2 X = -X + \eta(x)\xi$ is also tangent to N for any vector X tangent to N. If ξ is normal to N at every point of N, then $\eta(X) = g(X, \xi) = 0$ for any tangent vector X of N. Thus $\phi^2 X = -X$ is tangent to N.

PROPOSITION 2. *A submanifold N of a Sasakian manifold M is a Sasakian submanifold if and only if every plane section of $T_x(M)$, $x \in N$, determined by a tangent vector X of N and ϕX is a tangent plane section of N at every point $x \in N$.*

Proof. Assume that the plane section determined by X and ϕX is tangent to N for every vector X tangent to N. Then since ϕX is tangent to N we have $\phi^2 X$ tangent to N. By proposition 1 ξ is tangent to N at every point of N, or ξ is normal to N at every point of N. If ξ is normal to N at every point of N then N is anti-invariant [Yano-Kon [1]] contradicting our assumption that ϕX is tangent to N. Thus ξ is tangent to N at every point of N. Since N is invariant it is a Sasakian submanifold.

The converse is clear.

PROPOSITION 3. *Let N be a submanifold of a Sasakian manifold M. Furthermore, suppose that the structure vector field ξ is either tangent to N at every point or normal to N at every point of N.*

Then N is an anti-invariant submanifold of M if and only if every plane section $p \subset T_x(N)$ perpendicular to ξ is anti-ϕ-invariant (*see Chen-Ogiue* [2]).

Proof. Let X be an arbitrary vector in $T_x(N)$ which is perpendicular to ξ. If ξ is tangent to N at every point of N, then let $e_0 = \xi$, $e_1 = X$, $e_2, \cdots, e_n$ be a local orthonormal basis for N. If ξ is normal to N at every point of N, let $e_1 = X$, $e_2, \cdots, e_n$ be a local orthonormal basis for N. In this case all e_i's are perpendicular to ξ. Denote by p_{ij}, where $1 \leq i,\ j \leq n$, the plane section spanned by e_i and e_j, $i \neq j$. Suppose that every tangent plane section of N orthogonal to ξ is anti-ϕ-invariant. Then ϕp_{ij} is perpendicular to p_{ij}. Hence $\phi e_1 = \phi X$, $\phi e_2, \cdots, \phi e_n$ are perpendicular to e_j for $j = 1, \cdots, n$. Thus, if ξ is normal to N, ϕX belongs to $T_x^{\perp}(N)$ and N is anti-invariant. If ξ is tangent to N, since $g(\phi X, \xi) = \eta(\phi X) = 0$, ϕX belongs to $T_x^{\perp}(N)$ again. In this case, if X is any tangent vector, then $X = \alpha\xi + \sum_{i=1}^{m} \alpha_i e_i$. Thus $\phi X = \sum_{i=1}^{m} \alpha_i \phi e_i$ belongs to $T_x^{\perp}(N)$. Therefore, N is also an anti-invariant submanifold.

Conversely, suppose that N is anti-invariant. Let p be any plane section perpendicular to ξ. Then p is spanned by tangent vectors X and Y which are perpendicular to ξ. Then ϕX and ϕY are normal to N, and the plane section spanned by ϕX and ϕY is perpendicular to p.

As a consequence of proposition 3, we see that if the structure vector field ξ is normal to N everywhere on N, then every tangent plane section of N is anti-ϕ-invariant.

2. **Characterization of submanifolds which are invariant with respect to the curvature transformation.** A submanifold N of a Sasakian manifold M is said to be *invariant with respect to the curvature transformation* if $R(X, Y)\, T_x N \subset T_x N$ holds for all tangent vectors X and Y of N, where $R(X, Y)$ denotes the curvature tensor of the Sasakian manifold M [1].

It is easy to see that in a Sasakian space form $M(c)$, $c \neq 1$, Sasakian submanifolds, anti-invariant submanifolds tangent to the structure vector field ξ, and anti-invariant submanifolds normal to the structure vector field ξ are all invariant with respect to the curvature transformation.

In fact, for a submanifold N in a Sasakian space form $M(c)$ we have

$$
\begin{aligned}
&R(X, Y)Z \\
&\quad = \frac{1}{4}(c+3)(g(Y, Z)X - g(X, Z)Y) \\
(1)\qquad &\qquad - \frac{1}{4}(c-1)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\
&\qquad + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\
&\qquad - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z)
\end{aligned}
$$

for all tangent vectors X, Y and Z of N.

(i) If N is a Sasakian submanifold, X, Y, ξ, ϕX, ϕY and ϕZ are all tangent to N, hence $R(X, Y)Z \in T_x(N)$.

(ii) If N is an anti-invariant submanifold tangent to ξ, then $g(\phi Y, Z) = g(\phi X, Z) = g(\phi X, Y) = 0$. Hence

$$
\begin{aligned}
&R(X, Y)Z \\
&\quad = \frac{1}{4}(c+3)(g(Y, Z)X - g(X, Z)Y) \\
(2)\qquad &\qquad - \frac{1}{4}(c-1)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\
&\qquad + g(YZ)\eta(X)\xi - g(X, Z)\eta(Y)\xi).
\end{aligned}
$$

Thus, $R(X, Y)Z \in T_x(N)$, since $\xi \in T_x(N)$.

(iii) If N is an anti-invariant submanifold normal to ξ, then X, Y and Z are orthogonal to ξ. Hence $\eta(X) = \eta(Y) = \eta(Z) = 0$ also hold. Therefore

$$
(3)\qquad R(X, Y)Z = \frac{1}{4}(c+3)(g(Y, Z)X - g(X, Z)Y)
$$

and

$$
R(X, Y)Z \in T_x(N).
$$

It is also obvious that if $c = 1$, then any submanifold N of $M(1)$ is invariant with respect to the curvature transformation.

PROPOSITION 4. *Let N be a submanifold of a Sasakian space form $M(c)$, $c \neq 1$, such that the structure vector field ξ is tangent* *to* N *at every point of* N, *and* $\dim N \geq 3$. *Then* N *is invariant with respect to the curvature transformation if and only if* N *is either invariant or anti-invariant in* M *(see Chen-Ogiue* [2]).

Proof. By (1), for $R(X, Y)Z \in T_x(N)$ to hold for tangent vectors X, Y and Z of N it is necessary that

$$-g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z \in T_x(N) \tag{4}$$

holds. Let $\{X, Y\}$ be an orthonormal set of tangent vectors such that $X \neq \xi$ and $Y \neq \xi$. By setting $Z = X$ in (4) for such an orthonormal set $\{X, Y\}$ we have

$$-g(\phi Y, X)\phi X + 2g(\phi X, Y)\phi X \in T_x(N). \tag{5}$$

But

$$g(\phi^2 X, \phi Y) = g(\phi X, Y) - \eta(\phi X)\eta(Y) = g(\phi X, Y).$$

On the other hand,

$$\begin{aligned} g(\phi^2 X, \phi Y) &= g(-X + \eta(X)\xi, \phi Y) \\ &= -g(X, \phi Y) + \eta(X)g(\xi, \phi Y) \\ &= -g(X, \phi Y). \end{aligned}$$

Therefore,

$$g(\phi X, Y) = -g(X, \phi Y). \tag{6}$$

Thus (5) and (6) imply that

$$g(X, \phi Y)\phi X \in T_x(N). \tag{7}$$

From (7) it follows that (i) if there is a vector $Y_1 \neq 0$ tangent to N at x such that $g(X, \phi Y_1) \neq 0$, then $\phi X \in T_x(N)$. Otherwise (ii) $g(X, \phi Y) = 0$ for all Y tangent to N at x. Then by (6), $g(\phi X, Y) = 0$ for all Y tangent to N at x. This means that $\phi X \in T_x^\perp(N)$.

Furthermore, we can show that if there is a vector field $X \neq 0$, ξ tangent to N at x such that $\phi X \in T_x(N)$, then there is no tangent vector $Z \neq 0$, ξ at x such that $\phi Z \in T_x^\perp(N)$.

In fact, suppose that $X \neq 0$, ξ and $Z \neq 0$, ξ are tangent vectors of N at x such that $\phi X \in T_x(N)$ and $\phi Z \in T_x^\perp(N)$. Then there is a tangent vector $Y_1 \neq 0$ at x such that $g(X, \phi Y_1) \neq 0$. On the other hand, $\phi Z \in T_x^{\perp}(N)$ implies that $g(Z, \phi Y) = 0$ for all tangent vectors Y at x. Then

$$\begin{aligned} g(X+Z, \phi Y_1) &= g(X, \phi Y_1) + g(Z, \phi Y_1) \\ &= g(X, \phi Y_1) \neq 0 . \end{aligned}$$

Thus $\phi(X+Z) = \phi X + \phi Z \in T_x(N)$. Hence $\phi Z \in T_x(N)$. Then $\phi Z \in T_x(N) \cap T_x^{\perp}(N)$ and $\phi Z = 0$. This implies that $Z = 0$ or $Z = \xi$, which contradicts our assumption.

Therefore, at a point x of N, either $\phi T_x(N) \subset T_x(N)$ or $\phi T_x(N) \subset T_x^{\perp}(N)$. By continuity of ϕ and the connectedness of N either $\phi T_x(N) \subset T_x(N)$ for every $x \in N$ or $\phi T_x(N) \subset T_x^{\perp}(N)$ for every $x \in N$. Thus N is either a Sasakian submanifold, or it is an anti-invariant submanifold tangent to ξ.

The converse is obvious from the above examples (i) and (ii).

Since, as a special case of a theorem stated above, a submanifold normal to the structure vector field ξ of a Sasakian space form is anti-invariant, so it is invariant with respect to the curvature transformation as mentioned in the above example (iii).

PROPOSITION 5. *Let N be an anti-invariant submanifold of a Sasakian space form $M(c)$, $c \neq 1$, such that* $\dim N \geq 2$. *Then N is invariant with respect to the curvature transformation if and only if the structure vector field ξ is either tangent to N or normal to N at every point of N.*

Proof. Suppose that N is invariant with respect to the curvature transformation. Since N is anti-invariant we have

$$g(\phi Y, Z) = g(\phi X, Z) = g(\phi X, Y) = 0 \tag{8}$$

for any vectors X, Y and Z tangent to N. Then formula (1) implies that

$$(g(Y, Z)\,\eta(X) - g(X, Z)\,\eta(Y))\,\xi \in T_x(N) . \tag{9}$$

Let $\{X, Y\}$ be an orthonormal set of tangent vectors to N at x. The vectors X or Y can be ξ, if ξ is tangent to N. By substituting $Z = X$ in (9) we have

$$\eta(Y)\,\xi \in T_x(N)\,. \tag{10}$$

If $\eta(Y)=0$ for every vector Y tangent to N at x, then ξ is normal to N at x. If $\eta(Y)\neq 0$ for some vector Y tangent to N at x, then ξ is tangent to N. As in the proof of proposition 1 we see that ξ is tangent to N at every point of N, or ξ is normal to N at every point of N. Therefore N is either an anti-invariant submanifold tangent to ξ or it is an anti-invariant submanifold normal to ξ.

The converse is obvious from the above examples (ii) and (iii).

References

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