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EXPONENTIAL MAP ON A SIMPLE LIE GROUP OF CLASSICAL TYPE

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0. Let $\mathfrak{G}$ be a connected Lie group with Lie algebra G. In general, the exponential map $\exp: G \to \mathfrak{G}$ is not onto. Following Goto [3], for $g \in \mathfrak{G}$, we define the index (of the exponential map) ind (g) of g to be the smallest positive integer q such that $g^q \in \exp G$ if it exists. Otherwise, $\text{ind}(g) = \infty$. The indx ind$(\mathfrak{G})$ of $\mathfrak{G}$ is defined to be the least common multiple of all ind (g) $(g \in \mathfrak{G})$.

In Lai [7], the author proved that ind$(\mathfrak{G})$ is finite when $\mathfrak{G}$ is a connected semisimple Lie group with finite center. On the other hand, consider the universal covering group of $SL(2, \boldsymbol{R})$, which has an infinite cyclic center, we found an element with infinite index.

The results in [3], [7], [8] suggest that in a connected simple Lie group with infinite center, we may find an element with infinite index. It turns out that this conjecture is not true. In this paper, using the result in Djoković [2], we will show that ind$(g) < \infty$ for any $g \in \mathfrak{G}$, when the Lie algebra G of $\mathfrak{G}$ is of type $AIII(p, q) = su(p, q)$ $(p > q \geq 1)$ or of type $DIII(n) = so^*(2n)$ (n odd, $n \geq 3$), even when $\mathfrak{G}$ contains central elements of infinite order. (We follow Cartan's notation as used in Helgason [6].) Indeed, we will show that, for a connected simple Lie group of classical type, ind$(g) < \infty$ for any $g \in \mathfrak{G}$, if and only if either i) $\mathfrak{G}$ has finite center, or ii) the Lie algebra G of $\mathfrak{G}$ is of type $AIII(p, q)$ $(p > q)$ or $DIII(n)$ (n odd).

1. A real simple Lie algebra G of classical type is one of the

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complex Lie algebras $\mathrm{sl}(n, \boldsymbol{C})$, $\mathrm{so}(n, \boldsymbol{C})$, $\mathrm{sp}(n, \boldsymbol{C})$ or one of their real forms. When G is one of the complex simple Lie algebras, $\mathfrak{G}$ a connected Lie group with G as its Lie algebra, we have determined $\{\mathrm{ind}(g) \mid g \in \mathfrak{G}\}$ completely in [8]. In case G is compact, it is well-known that $\exp : G \to \mathfrak{G}$ is always onto. So we only have to consider the noncompact real forms of $\mathrm{sl}(n, \boldsymbol{C})$, $\mathrm{sp}(n, \boldsymbol{C})$, $\mathrm{so}(n, \boldsymbol{C})$.

From now on, by a simple Lie algebra G, we always mean one of the noncompact real simple Lie algebras of classical type, unless otherwise stated; $\mathfrak{G}$ ($\mathfrak{H}$, $\mathfrak{K}, \cdots$) a connected Lie group whose Lie algebra we denote by $G(H, K, \cdots)$.

Given a simple Lie algebra G, if the maximal compact subalgebra of G is semisimple, then $\mathfrak{G}$ always has finite center, so the result in [7] implies that $\mathrm{ind}(\mathfrak{G}) < \infty$ in this case. This happens when G is one of the following:

$$\mathrm{AI}(n) = \mathrm{sl}(n, \boldsymbol{R}) \quad (n \geq 3),$$

$$\mathrm{AII}(n) = \mathrm{su}^*(2n),$$

$$\mathrm{BDI}(p, q) = \mathrm{so}(p, q) \quad (p \geq q,\ p \neq 2,\ q \neq 2),$$

or

$$\mathrm{CII}(p, q) = \mathrm{sp}(p, q).$$

Notation. For an element x in a Lie algebra G, we define its centralizer in G, $\mathfrak{G}$ to be $z_G(x) = \{y \in G \mid [x, y] = 0\}$ and $Z_{\mathfrak{G}}(x) = \{g \in \mathfrak{G} \mid \mathrm{Ad}g \cdot x = x\}$ respectively. Clearly, $z_G(x)$ is the Lie algebra of $Z_{\mathfrak{G}}(x)$. We shall denote the connected component of $Z_{\mathfrak{G}}(x)$ containing the identity element by $Z^0_{\mathfrak{G}}(x)$. The center of $\mathfrak{G}$ will be denoted by $Z(\mathfrak{G})$.

2. When G is the normal real form of the corresponding complex simple Lie algebra G^c. There exists a subalgebra H of G whose complexification H^c is a Cartan subalgebra of G^c. Denote by Δ the root system of G^c with respect to H^c, we can choose the root vector e_α from G for any $\alpha \in \Delta$. Consider $x = \sum_{\alpha \in \Delta^+} e_\alpha$, which is a nilpotent element of G, regular in G^c. According to Steinberg [9] (pp. 110–112), $z_{G^c}(x)$ (and hence $z_G(x)$) consists only of nilpotent elements.

Let $c \in Z(\mathfrak{G})$, $g = c \exp x \in \mathfrak{G}$ (x as above). If $g^n = \exp y$ for some $y \in G$, then $y = s + nx$ with $s \in G$ semisimple and $[s, x] = 0$ (because exp is one to one on the set of nilpotent elements). This can happen only when $s = 0$, i.e. $g^n \in \exp G$ if and only if $c^n = 1$.

THEOREM. *When $\mathfrak{G}$ is a connected real simple Lie group whose Lie algebra G is the normal real form of G^c, we can find $x \in G$, such that for any central element c of order r,* $\operatorname{ind}(c \exp x) = r$. *In particular, if $Z(\mathfrak{G})$ is infinite, we can choose a central element c of infinite order, and* $\operatorname{ind}(c \exp x) = \infty$.

This happens when G is either $\mathrm{AI}(2) = \mathrm{sl}(2, \boldsymbol{R})$ or $\mathrm{CI}(n) = \mathrm{sp}(n, \boldsymbol{R})$.

3. There remains to consider the case when G is one of the following:

$$\mathrm{AIII}(p, q) = \mathrm{su}(p, q) \quad (p \geq q),$$
$$\mathrm{BDI}(p, 2) = \mathrm{so}(p, 2) \quad (p \neq 2),$$
$$\mathrm{DIII}(n) = \mathrm{so}^*(2n) \quad (n \geq 3).$$

In the next three sections, we shall prove a result similar to the above theorem when G is of type $\mathrm{AIII}(p, p)$, $\mathrm{BDI}(p, 2)$, or $\mathrm{DIII}(n)$ (n even). Then in §7, we shall show that $\operatorname{ind}(g)$ ($g \in \mathfrak{G}$) is always finite when G is of type $\mathrm{DIII}(n)$ (n odd, $n \geq 3$), and do the same thing when G is of type $\mathrm{AIII}(p, q)$ ($p > q \geq 1$) in §8.

Before we go on, note the following: Given any element x in a semisimple Lie algebra G, x has a unique decomposition $x = x_c + x_r + x_n$ satisfying $[x_c, x_r] = [x_c, x_n] = [x_r, x_n] = 0$ where the linear transformations $\operatorname{ad} x_n$ is nilpotent, $\operatorname{ad} x_r$ $\boldsymbol{R}$-diagonalizable on G, and $\operatorname{ad} x_c$ semisimple on G^c with purely imaginary eigenvalues. For simplicity, we call x_c, x_r, x_n the compact, $\boldsymbol{R}$-diagonalizable, and nilpotent parts of x respectively. It is not difficult to see that for any connected Lie group $\mathfrak{G}$ with G as its Lie algebra, exp is one to one on the set of nilpotent elements as well as on the set of $\boldsymbol{R}$-diagonalizable elements.

In the following, we denote by E_{st} the square matrix whose (ij) entry is $\delta_{is}\ \delta_{jt}$.

4. We consider $G = \mathrm{AIII}(p, p)$ in this section, we shall find an $\boldsymbol{R}$-diagonalizable element which play the same role as the regular nilpotent element x we chose in §2.

$$G = \mathrm{su}(p,\ p) = \left\{x = \begin{pmatrix} z_1 & z_2 \\ {}^t\bar{z}_2 & z_3 \end{pmatrix} \middle| z_1,\ z_3 \in u(p),\right.$$

$$\left. z_2 \in \mathrm{gl}(p, \boldsymbol{C}),\ \mathrm{tr}(z_1 + z_3) = 0\right\}.$$

When $p = 1$, $\mathrm{su}(1, 1) \cong \mathrm{sl}(2, \boldsymbol{R})$, which has been discussed in §2.

Assume that $p \geq 2$. The Cartan decomposition $G = K \oplus P$ may be chosen in such a way that $K = \{x \in G | z_2 = 0\} \cong s(u(p) \oplus u(p))$ and $P = \{x \in G | z_1 = z_3 = 0\}$. A maximal abelian subalgebra of P is given by

$$A = \sum_{j=1}^{p} \boldsymbol{R}(E_{j\ p+j} + E_{p+j\ j})\,.$$

Let $a = \sum_{j=1}^{p} \lambda_j(E_{j\ p+j} + E_{p+j\ j}) \in A$ be chosen in such a way that λ_j^2 are all distinct positive reals. First we porve that a is regular in A.

If

$$Z = \sum_{i,j=1}^{p} z_{ij}\, E_{i\ p+j} + \bar{z}_{ji}\, E_{p+i\,j} \in P \qquad (z_{ij} \in \boldsymbol{C})$$

commutes with a, since

$$aZ = \sum_{i,j} \lambda_j(\bar{z}_{ij}\, E_{ji} + z_{ji}\, E_{p+j\ p+i})$$

and

$$Za = \sum_{i,j} \lambda_j(z_{ij}\, E_{ij} + \bar{z}_{ji}\, E_{p+i\ p+j}),$$

so we have $\lambda_i\, \bar{z}_{ji} = \lambda_j\, z_{ij}$ and $\lambda_i\, z_{ij} = \lambda_j\, \bar{z}_{ji}$. When $i \neq j$, the assumption $\lambda_i^2 \neq \lambda_j^2$ implies that $z_{ij} = \bar{z}_{ji} = 0$. When $i = j$, $\bar{z}_{ii} = z_{ii}$, i. e. $z_{ii} \in \boldsymbol{R}$. Therefore $Z \in A$, i. e. a is regular in A.

Next, we like to compute $z_G(a)$, which is clearly equal to $z_K(a) \oplus A$, because a is regular in A and $[K, P] \subset P$, $[P, P] \subset K$. To find $z_K(a)$, consider

$$X = \sum_{i,j=1}^{p} (x_{ij} E_{ij} + y_{ij} E_{p+i\,p+j}) \in K,$$

computation shows that

$$aX = \sum_{i,j} \lambda_i (y_{ij} E_{i\,p+j} + x_{ij} E_{p+i\,j})$$

and

$$Xa = \sum_{i,j} \lambda_j (x_{ij} E_{i\,p+j} + y_{ij} E_{p+i\,j}).$$

So that $[X, a] = 0$ if and only if for $i, j = 1, \cdots, p$,

$$\lambda_i y_{ij} = \lambda_j x_{ij} \quad \text{and} \quad \lambda_i x_{ij} = \lambda_j y_{ij} \quad \text{hold.}$$

When $i \neq j$, we have $x_{ij} = y_{ij} = 0$ because $\lambda_i^2 \neq \lambda_j^2$. When $i = j$, we get $x_{ii} = y_{ii}$ for $i = 1, \cdots, p$. Therefore

$$X = \sum_{j=1}^{p} y_j (E_{jj} + E_{p+j\,p+j}), \quad \text{with} \quad y_j \in i\boldsymbol{R} \quad \text{and} \quad \sum_{j=1}^{p} y_j = 0.$$

Hence

$$z_K(a) = \left\{ x = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \middle| \; y = \operatorname{diag}(iy_1, \cdots, iy_p) \in \mathrm{su}(p) \right\} \subset \mathrm{su}(p) \oplus \mathrm{su}(p)$$

and

$$z_G(a) = z_K(a) \oplus A$$

(direct sum of abelian ideals). Notice that $\mathrm{su}(p) \oplus \mathrm{su}(p)$ is (compact) semisimple when $p > 1$, so $z_K(a)$ is contained in a compact semisimple subalgebra of G.

REMARK. For $\mathrm{su}(p, q)$ $(p > q)$, we may choose a regular element a in similar way, computation shows that $z_K(a)$ consists of

$$X = \sum_{j=1}^{q} y_j (E_{jj} + E_{p+j\,p+j}) + Z \quad \text{with} \quad y_j \in i\boldsymbol{R}, \quad Z \in u(p-q)$$

and

$$2 \sum_{j=1}^{q} y_j + \operatorname{tr} Z = 0,$$

it is no longer ture that $z_K(a)$ is contained in a semisimple subalgebra of G.

Notice that a is a real symmetric matrix, so a and hence $\operatorname{ad} a$ (considered as a linear operator on $\mathrm{gl}(2p, \boldsymbol{C})$) is semisimple with all eignvalues real. Since G is a subalgebra of $\mathrm{gl}(2p, \boldsymbol{C})$ and $a \in G$ (so that G is invariant under ad a), we conclude that ad a is $\boldsymbol{R}$-diagonalizable as a linear operator on G.

THEOREM, *Let $\mathfrak{G}$ be a connected Lie group with Lie algebra $G = \mathrm{su}(p, p)$ $(p \geq 2)$. Assume that $Z(\mathfrak{G})$ is infinite, c be a central element of infinite order, then* ind $(c \cdot \exp a) = \infty$ *(a as above).*

Proof. Let $g = c \exp a$. If $g^n = \exp x$ for some $x \in G$ (n a positive integer), then

$$c^n \exp na = \exp(x_c + x_r + x_n) = \exp x_c \cdot \exp x_r \cdot \exp x_n,$$

where x_c, x_r, x_n have the usual meanings. Then $x_n = 0$, $x_r = na$, and $x_c + x_r \in z_G(a) = z_K(a) \oplus A$. Since x_c is a compact element, we have $x_c \in z_K(a) \subset \mathrm{su}(p) \oplus \mathrm{su}(p)$. Therefore, $\exp \boldsymbol{R} x_c \subset \exp(z_K(a)) \subset \exp(\mathrm{su}(p) \oplus \mathrm{su}(p))$, the later is a compact subgroup of $\mathfrak{G}$ because $\mathrm{su}(p) \oplus \mathrm{su}(p)$ is compact semisimple. But $c^n \exp na = \exp x_c \cdot \exp na$ implies that $c^n = \exp x_c$. Therefore the discrete central subgroup $\{c^{mn} | m \in \boldsymbol{Z}\}$ $(\subset \exp(\mathrm{su}(p) \oplus \mathrm{su}(p)))$ must have compact closure, which is absurd because c is of infinite order and $n \neq 0$.

We conclude that $(c \exp a)^n \notin \exp G$ for any positive integer n, i.e. $\mathrm{ind}(c \exp a) = \infty$.

5. When G is of type $\mathrm{BDI}(p, q) = \mathrm{so}(p, q)(p \geq q \geq 1)$. The maximal compact subalgebra $\mathrm{so}(p) \oplus \mathrm{so}(q)$ is not semisimple only when $p = 2$, $q = 1$ or $p \geq q = 2$. But $\mathrm{BDI}(2, 1) = \mathrm{AI}(2)$, $\mathrm{BDI}(3, 2) = \mathrm{CI}(2)$, $\mathrm{BDI}(4,2) = \mathrm{AIII}(2, 2)$, which we have discussed in §2, 4, and $\mathrm{BDI}(2, 2)$ is not simple $(\cong \mathrm{AI}(2) \times \mathrm{AI}(2)$, although our argument works for this case too). So we may and shall assume that $p \geq 5$, $q = 2$.

$$G = \mathrm{so}(p, 2) = \left\{ X = \begin{pmatrix} x & B \\ {}^tB & y \end{pmatrix} \middle| \, x \in \mathrm{so}(p), \; y \in \mathrm{so}(2),\ B\ p \text{ by } 2 \text{ real matrix} \right\}$$

has a Cartan decomposition $G = K \oplus P$, where $K = \{X \in G \mid B = 0\}$ and $P = \{X \in G \mid x = y = 0\}$. A maximal abelian subalgebra A of P is given by

$$A=\sum_{j=1}^{2} \boldsymbol{R}(E_{j\ p+j}+E_{p+j\ j}).$$

Let $a=\sum_{j=1}^{2}\lambda_j(E_{j\ p+j}+E_{p+j\ j})\in A$, where $\lambda_1^2\neq\lambda_2^2$ are positive reals. Computation shows that a is regular in A. Indeed, for

$$b=\sum_{i=1}^{p}\sum_{j=1}^{2} b_{ij}(E_{i\ p+j}+E_{p+j\ i})\in P,$$

we have

$$ab=\sum_{i=1}^{p}\sum_{j=1}^{2}\lambda_j b_{ij}E_{ji}+\sum_{i,j=1}^{2}\lambda_i b_{ij}E_{p+i\ p+j},$$

and

$$ba=\sum_{i=1}^{p}\sum_{j=1}^{2}\lambda_j b_{ij}E_{ij}+\sum_{i,j=1}^{2}\lambda_i b_{ij}E_{p+j\ p+i}.$$

So $ab=ba$ happens exactly when

$$\lambda_j b_{ij}=0\ (i\geq 3)\quad\text{and}\quad \lambda_1 b_{12}=\lambda_2 b_{21},\ \lambda_1 b_{21}=\lambda_2 b_{12},$$

i.e. $b_{ij}=0$ $(i\geq 3)$, $b_{12}=b_{21}=0$ (because $\lambda_1^2\neq\lambda_2^2$), and $b\in A$.

To find $z_G(a)$, which is clearly equal to $z_K(a)\oplus A$, consider

$$X=\sum_{i,j=1}^{p} x_{ij}E_{ij}+c(E_{p+1\ p+2}-E_{p+2\ p+1})\in K$$

$$(\text{i.e. } (x_{ij})\in \mathrm{so}(p)).$$

Then

$$aX=\lambda_1\sum_{j=1}^{p}x_{1j}E_{p+1\ j}+\lambda_2\sum_{j=1}^{p}x_{2j}E_{p+2\ j}$$
$$+\lambda_1 cE_{1\ p+2}-\lambda_2 cE_{2\ p+1},$$

and

$$Xa=\lambda_1\sum_{i=1}^{p}x_{i1}E_{i\ p+1}+\lambda_2\sum_{i=1}^{p}x_{i2}E_{i\ p+2}$$
$$-\lambda_1 cE_{p+2\ 1}+\lambda_2 cE_{p+1\ 2}.$$

So $[a, X]=0$ when and only when

$$\lambda_1 x_{1j}=\lambda_2 x_{2j}=\lambda_1 x_{i1}=\lambda_2 x_{i2}=0\qquad (i,\ j\geq 3),$$
$$\lambda_1 x_{11}=\lambda_2 x_{22}=0,$$
$$\lambda_1 x_{12}=\lambda_2 c,\qquad \lambda_2 x_{21}=-\lambda_1 c,$$
$$\lambda_1 x_{21}=-\lambda_2 c,\qquad \lambda_2 x_{12}=\lambda_1 c.$$

This is equivalent to $x_{11} = x_{22} = x_{1j} = x_{2j} = x_{i1} = x_{i2} = 0$ $(i, j \geq 3)$, and the assumption $\lambda_1^2 \neq \lambda_2^2$ implies that $x_{12} = x_{21} = c = 0$, i. e.

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{so}(p) \oplus \mathrm{so}(2), \qquad \text{so} \quad y \in \mathrm{so}(p-2).$$

We conclude that $z_G(a) = K_1 \oplus A$ (direct sum of ideals), where $K_1 = z_K(a)$ is a subalgebra of K, isomorphic with $\mathrm{so}(p-2)$, and K_1 is (compact) semisimple when $p \geq 5$ $(z_G(a) = A$ when $p = 2, 3)$.

THEOREM. *Assume that* $\mathfrak{G}$ *is a connected Lie group with Lie algebra* $G = \mathrm{so}(p, 2)$, *with infinite center* $Z(\mathfrak{G})$. *Let* c *be a central element of infinite order,* a *as above. Then* $\mathrm{ind}(c \exp a) = \infty$.

Proof. First note that ad a is $\boldsymbol{R}$-diagonalizable on G, because a is a real symmetric matrix and $(a \in) G$ is a subalgebra of $\mathrm{gl}(p+2, \boldsymbol{C})$.

If $(c \exp a)^n = \exp x$ for some $x \in G$, (n a positive integer), then x can be decomposed into $x = x_c + na$, $[x_c, na] = 0$. Therefore x_c is a compact element in $z_G(na) = z_G(a) = K_1 \oplus A$. But K_1 is the (unique) maximal compact subalgebra of $z_G(a)$, so that $x_c \in K_1 \cong \mathrm{so}(p-2)$. Therefore $\exp \boldsymbol{R} x_c \subset \exp K_1$, which is compact because $\mathrm{so}(p-2)$ is compact semisimple $(p \geq 5)$. On the other hand, the equation $c^n \exp na = \exp x_c \cdot \exp na$ implies that $c^n = \exp x_c$. Thus the discrete central subgroup $\{c^{mn} \mid m \in \boldsymbol{Z}\} \subset \exp K_1$ must have compact closure, which is absurd in case $n \neq 0$.

We have prove that $(c \exp a)^n \notin \exp G$ for any positive integer n, i. e. $\mathrm{ind}(c \exp a) = \infty$.

6. Consider the Lie algebra G of type $\mathrm{DIII}(n) = \mathrm{so}^*(2n)$ $(n \geq 3)$. We prove in this section a result similar to that in §4, 5 when n is even, $n = 2m$ $(m \geq 2)$.

$$G = \mathrm{so}^*(2n) = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \middle| \, z_1, z_2 \in \mathrm{gl}(n, \boldsymbol{C}), \; z_1 \text{ skew}, \; z_2 \text{ Hermitian} \right\}$$

has a Cartan decomposition $G = K \oplus P$, where we can choose K to be

$$K=\left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\,\middle|\, x\in \mathrm{so}(n),\ y \text{ symmetric}\right\}\cong u(n) \text{ via}$$

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\longrightarrow x+iy,$$

and

$$P=\left\{i\begin{pmatrix} x & y \\ y & -x \end{pmatrix}\,\middle|\, x,\ y\in \mathrm{so}(n)\right\}$$

(Helgason [6] p. 453). A maximal abelian subalgebra of P is given by (recall that $n=2m$).

$$A=\left\{\sum_{j=1}^{m} i\lambda_j((E_{2j-1\,2j}-E_{2j\,2j-1})-(E_{n+2j-1\;n+2j} - E_{n+2j\;n+2j-1}))\,|\,\lambda_j\in \boldsymbol{R}\right\}.$$

Let $a\in A$ be chosen such that λ_j^2 are all distinct, direct computation shows that a is regular in A, hence $z_G(a)=z_K(a)\oplus A$.

To find $z_K(a)$. Let $z=\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\in K$. Then $az=za$ if and only if $bx=xb$ and $by=-yb$, where $b=\sum_{j=1}^m \lambda_j(E_{2j-1\,2j}-E_{2j\,2j-1})$. Since $x\in \mathrm{so}(n)$ and b is regular in $\mathrm{so}(n)$, we see immediately that $x=\sum_{j=1}^m x_j(E_{2j-1\,2j}-E_{2j\,2j-1})$. On the other hand, for symmetric y, computation shows that $by=-yb$ happens if and only if $y_{11}=-y_{22}$, $y_{33}=-y_{44},\cdots, y_{n-1\,n-1}=-y_{nn}$, and $y_{ij}=0$ if $j>i+1$. In particular, $\operatorname{tr} x=\operatorname{tr} y=0$. If we identify z with $x+iy\in u(n)$, this implies that $x+iy\in z_K(a)$ can happen only when $\operatorname{tr}(x+iy)=0$, i.e. lies in $\mathrm{su}(n)$. Therefore $z_G(a)=z_K(a)\oplus A=K_1\oplus A$, where K_1 is a subalgebra of $\mathrm{su}(n)$.

THEOREM. *Let* $\mathfrak{G}$ *be a connected Lie group whose Lie algebra* G *is of type* DIII(n), *n even. If* $Z(\mathfrak{G})$ *is infinite, and c a central element of infinite order, then for the* $\boldsymbol{R}$*-diagonabizable element a we have chosen above*, $\operatorname{ind}(c\exp a)=\infty$.

The proof is the same as what we did for AIII(p, p) in §4, so we omit the details, note that $\mathrm{su}(n)$ ($n\geq 4$) is semisimple, and hence $\exp(\mathrm{su}(n))$ is a compact subgroup of $\mathfrak{G}$.

REMARK. When $n = 2m + 1$. For $x + iy \in z_K(a)$, computation shows that y_{nn} can be arbitrary, so that $\mathrm{tr}(y) \neq 0$, and $z_K(a)$ does not lie in a semisimple subalgebra of K (please see the following section).

7. Now consider the Lie algebra G of type $\mathrm{DIII}(n) = \mathrm{so}^*(2n)$, n odd (≥ 3). We shall prove that $\mathrm{ind}(g)$ ($g \in \mathfrak{G}$) is always finite. For this purpose, we may and shall assume that $\mathfrak{G}$ is simply connected. In this case, the center $Z(\mathfrak{G})$ is free cyclic with one generator (say c), see Goto-Kobayashi [5].

Denote by exp: $G \to \mathfrak{G}$ and Exp: $G \to \mathrm{SO}^*(2n)$ the exponential maps on $\mathfrak{G}$ and the corresponding classical group $\mathrm{SO}^*(2n)$ respectively, $\pi : \mathfrak{G} \to \mathrm{SO}^*(2n)$ the covering map. Note that $\mathrm{SO}^*(2n)$ has nontrival center $\{\pm 1\}$, so that $\pi^{-1}(-1) = \{c^{2p+1} \mid p \in Z\}$.

According to Djoković [2] (7.1), Exp is surjective. To prove that $\mathrm{ind}(g)$ is finite for any $g \in \mathfrak{G}$, it suffices to consider elements of the form $g = c^n \exp x$, where $n \in Z$ and $x \in G$.

Given $x \in G$, decompose it as usual $x = x_c + x_r + x_n$, write $x' = x_r + x_n$. Consider the centralizer $z_G(x')$ of x' in G, and the centralizer of x' in $\mathrm{SO}^*(2n)$: $\{g \in \mathrm{SO}^*(2n) \mid \mathrm{Ad}g \cdot x' = x'\}$, whose connected component containing the identity we denoted by $\mathfrak{H}$.

Since Exp: $G \to \mathrm{SO}^*(2n)$ is onto, we can find $y \in G$ such that $\mathrm{Exp}\, y = -\mathrm{Exp}\, x'$. Decompose y as usual, we see that $y = y_c + x'$ with $y_c \in z_G(x')$ and $\mathrm{Exp}\, y_c = -1$. Therefore $-1 \in \mathfrak{H}$.

For the given element $x = x_c + x'$ in G ($x_c \in z_G(x')$ the compact part of x), we may choose a maximal compact subgroup $\mathfrak{K}_1$ of $\mathfrak{H}$ containing $\mathrm{Exp}\ x_c$, then $-1 \in \mathfrak{K}_1$ because $-1 \in \mathfrak{H}$. and -1 is central in $\mathfrak{H}$. Let K_1 be the (compact) subalgebra of $z_G(x')$ such that $\mathrm{Exp}\ K_1 = \mathfrak{K}_1$, K a maximal compact subalgebra of G containing K_1. Denote by z_0 the central element in K satisfying $\exp\, z_0 = c$. Note that $(\mathrm{Exp}|_K)^{-1}(-1) = \{pz_0 + z \mid p$ odd integer, $z \in [K, K]$ and $\exp z = 1\}$ because $Z(\mathfrak{G})$ is infinite cyclic generated by c.

Now, both -1 and $\mathrm{Exp}\, x_c$ lie in $\mathfrak{K}_1$, we can find $y_c, x_1 \in K_1 = z_K(x') = z_G(x') \cap K$ such that $\mathrm{Exp}\, y_c = -1$ and $\mathrm{Exp}\ x_1 = -\mathrm{Exp}\, x_c$.

Hence $y_c = rz_0 + z$ with $\exp z = 1$, and $x_1 = qz_0 + y_1$, where y_1 may be chosen such that $\exp y_1 = \exp x_c$.

Since y_c and x_1 lie in the same compact subalgebra K_1, we may find $g = \operatorname{Exp} a \in \mathfrak{K}_1$ $(a \in K_1)$ such that $[\operatorname{Ad} g \cdot y_c, x_1] = 0$, denote $\operatorname{Ad} g \cdot y_c$ by y'. Since both x' and z_0 lie in the center of K_1, we have $\operatorname{Ad} g \cdot x' = x'$ and $\operatorname{Ad} g \cdot z_0 = z_0$. Therefore

$$[y', x'] = [\operatorname{Ad} g \cdot y_c, \operatorname{Ad} g \cdot x'] = \operatorname{Ad} g \cdot [y_c, x'] = \operatorname{Ad} g \cdot 0 = 0,$$

i. e.

$$y' = \operatorname{Ad} g \cdot y_c = rz_0 + z' \in z_G(x') \cap K = K_1,$$

where $\exp z' = \exp \operatorname{Ad} g \cdot z = (\exp a)^{-1} \exp z (\exp a) = 1$. We have $qz_0 + y_1,\ rz_0 + z' \in K_1$, so that $ry_1 - qz' \in K_1$.

In the above discussion, replace x' by $-x'$ if necessary $(z_G(x') = z_G(-x'))$, we may assume that r is a positive integer. Now we have

$$\begin{aligned}
&(c^n \exp x)^r \\
&\quad = c^{nr} \cdot \exp rx_c \cdot \exp rx' \\
&\quad = c^{nr} \cdot \exp ry_1 \cdot \exp rx' \qquad (\text{because } \exp x_c = \exp y_1) \\
&\quad = c^{nr} \exp(ry_1 - qz') \exp rx' \\
&\qquad\qquad (0 = [x_1, y'] = [qz_0 + y_1, rz_0 + z'] = [y_1, z'], \\
&\qquad\qquad\quad \text{and } \exp z' = 1) \\
&\quad = \exp n(rz_0 + z') \cdot \exp (ry_1 - qz' + rx') \\
&\qquad\qquad (\text{because } ry_1 - qz' \in K_1 \subset z_G(x')) \\
&\quad = \exp (n(rz_0 + z') + ry_1 - qz' + rx') \\
&\qquad\qquad (\text{because } rz_0 + z' \in K_1 \\
&\qquad\qquad\quad \text{and } 0 = [rz_0 + z', ry_1 - qz'] = r[z', y_1]).
\end{aligned}$$

Hence $\operatorname{ind}(c^n \exp x) \leq r$ for any $n \in \boldsymbol{Z}$.

This proves the following:

THEOREM. *Let $\mathfrak{G}$ be a connected Lie group with Lie algebra $G = \mathrm{so}^*(2n)$, n odd* (≥ 3). *Then* $\operatorname{ind}(g) < \infty$ *for any* $g \in \mathfrak{G}$.

REMARKS 1. In the above discussion, $z \neq 0$ in general. Because if G is a simple Lie algebra whose maximal compact subalgebra K has nontrivial center $\boldsymbol{R}z_0$, then we can show that $z_G(z_0) = K$.

2. We have shown that $\operatorname{ind}(g) < \infty$ for any $g \in \mathfrak{G}$, but we still cannot determine whether $\operatorname{ind}(\mathfrak{G}) = \text{l. c. m.}\{\operatorname{ind}(g) \mid g \in \mathfrak{G}\}$ is also finite, i. e. whether $\{\operatorname{ind}(g) \mid g \in \mathfrak{G}\}$ is bounded form above. For this problem, we have to study whether the set of positive integers r for which $c^r \in Z^0_{\mathfrak{G}}(x)$ $(x \in G)$ is bounded. (Of course, we only have to consider those $x \in G$ with compact part $x_c = 0$.) The auther has no idea about it.

3. In case n is even, $Z(\mathfrak{G}) \cong \boldsymbol{Z} \times \boldsymbol{Z}_2$ for simply connected $\mathfrak{G}$ (Goto-Kobayashi [5]). Given x' as above, it may happen that $Z^0_{\mathfrak{G}}(x') \cap Z(\mathfrak{G})$ contains only torsion elements (which also maps to -1 by π). Indeed, this must happen since we have found an $\boldsymbol{R}$-diagonalizable element $a \in G$ in §6 satisfying $\operatorname{ind}(c \exp a) = \infty$.

8. Finally, we consider the case when G is of type $\text{AIII}(p, q) = \mathfrak{su}(p, q)$ $(p > q \geq 1)$. Following a discussion similar to the one we did in the last section, we shall prove that $\operatorname{ind}(g)$ is also finite for any $g \in \mathfrak{G}$. For our purpose, we only have to consider the case when $\mathfrak{G}$ is simply connected. In this case, $Z(\mathfrak{G}) \cong \boldsymbol{Z} \times \boldsymbol{Z}_d$ with $d = \text{G. C. D.}(p, q)$, we fix a generator α for the free part and β for the torsion part of $Z(\mathfrak{G})$ respectively. To prove that $\operatorname{ind}(g) < \infty$, by taking a (finite) power if necessary, we only have to consider the case $g = \alpha^n \exp x$ $(n \in \boldsymbol{Z},\ x \in G)$.

Again denote by exp, Exp the exponential maps on $\mathfrak{G}$ and the corresponding classical group $SU(p, q)$ respectively, $\pi : \mathfrak{G} \to SU(p, q)$ the covering map. The center of $SU(p, q)$ is isomorphic to $\boldsymbol{Z}_{p+q}$, generated by $\nu = e^{2\pi i/p+q} I$, note that $\pi(\beta) = \nu^{p'+q'}$ $(p = p'd,\ q = q'd)$ generates $Z(SU(p)) \cap Z(SU(q))$.

According to Djoković [2] (3.7), ind: $SU(p, q) \to \boldsymbol{Z}$ consists of positive integers $\text{G. C. D.}(p_1, \cdots, p_s)$, where $p_1 \geq \cdots \geq p_s$ are positive integers satisfying

$$p_1 + \cdots + p_s = p + q \quad \text{and} \quad \left[\frac{p_1}{2}\right] + \cdots + \left[\frac{p_s}{2}\right] \leq q .$$

In case $p > q$, it is clear that $\operatorname{ind}(g) < p + q$ for any $g \in SU(p, q)$. In particular, $\operatorname{ind}(\nu \operatorname{Exp} x) < p + q$ for any $x \in G$. Therefore, for any $x \in G$, $Z^0_{SU(p,q)}(x)$ contains some nontrivial central element.

Given any $x \in G$, decompose it in the usual fashion: $x = x_c + x_r + x_n$, write $x' = x_r + x_n$. If we can prove that $\alpha^r \in Z^0_{\mathfrak{G}}(x')$ for some positive integer r, then an argument similar to the one in the preceeding section will show that $\text{ind}(\alpha^n \exp x) \le r$. So our problem reduces to the following: Given $x' = x_r + x_n$ as above, can we find nonzero integer r such that $\alpha^r \in Z^0_{\mathfrak{G}}(x')$?

As we noted earlier, $Z^0_{SU(p,q)}(x')$ contains some nontrivial central element, i.e. we can find $y_c \in z_G(x')$ such that $\exp y_c$ is a nontrivial central element in $\mathfrak{G}$. If we can show that $\exp y_c$ is not torsion, then we are done (by taking a power if necessary). Note that the central element $\exp y_c$ is torsion only when y_c lies in $[K, K]$ ($\cong \text{su}(p) \oplus \text{su}(q)$), where K ($\cong s(u(p) \oplus u(q))$) is any maximal compact subalgebra of G containing y_c.

Assume that $m = \text{ind}(\nu \, \text{Exp}\, x')$, the only case we have to worry is when $\nu^m \in \langle \pi(\beta) \rangle$, so that md is a multiple of $p+q$.

If $\nu^m = \text{Exp}\, y$ ($y \in z_G(x')$) is given in such a way that $y = y_1 + y_2$ with $y_1 \in K_1 \cong \text{su}(p)$, $y_2 \in K_2 \cong \text{su}(q)$. Check the argument in Djoković [2] (p. 80), we see that such thing can happen only when $p = p_1 + \cdots + p_r$, $q = q_1 + \cdots + q_s$, $m = \text{G.C.D.}(p_1, \cdots, p_r, q_1, \cdots, q_s)$ satisfying $m | p_i$, $m | q_j$. In particular, $m | d = \text{G.C.D.}(p, q)$. Write $p = md'p'$, $q = md'q'$, where $d = md'$ and $1 = \text{G.C.D.}(p', q')$. The fact that $(p+q) | md$ implies that $(p' + q') | m$, and our assumption $p > q$ implies that $p' - q' \ge 1$. Consider the condition

$$\left[\frac{p_1}{2}\right] + \cdots + \left[\frac{p_r}{2}\right] + \left[\frac{q_1}{2}\right] + \cdots + \left[\frac{q_s}{2}\right] \le q.$$

For any positive integer k, we have $[k/2] \ge (k-1)/2$. So the above inequality implies that

$$md'q' = q \ge \frac{1}{2}(p_1 - 1 + \cdots + p_r - 1 + q_1 - 1 + \cdots + q_s - 1)$$

$$= \frac{1}{2}(md'(p' + q') - (r + s)),$$

hence

$$r + s \ge md'(p' - q') \ge (p' + q')d' = (p+q)/m.$$

But $m|p_i$ and $m|q_j$ imply that $r+s\leq(p+q)/m$, and we conclude that $p_1=\cdots=p_r=q_1=\cdots=q_s=m$, it is also easy to see that m must be odd. (For example, $(p, q)=(6, 3), (12, 6), (15, 10)\cdots$.)

Now, the integers $t_1,\cdots, t_s$ as chosen in [2] (p. 81) to satisfy $\operatorname{tr}(\Delta)=0$ can have many choices, the only condition they need to satisfy is $t_1+\cdots+t_s=-q(\Gamma)$. It is clear that we may change these integers so that $y\in z_G(x')\cap(\operatorname{Exp}|_K)^{-1}(\nu^m)$ does not lie in $[K, K]$, i.e. $\exp y\in Z^0_{\mathfrak{G}}(x')$ is not a torsion element.

THEOREM. *Let* $\mathfrak{G}$ *be a connected Lie group with Lie algebra* $G=\mathrm{su}(p, q)(p>q\geq 1)$ *then* $\operatorname{ind}(g)<\infty$ *for any* $g\in\mathfrak{G}$.

We have proved that $\alpha^r\in Z^0_{\mathfrak{G}}(x')$ for some positive integer r, the rest of the proof is similar to the one in §7.

BIBLIOGRAPHY

1. N. Bourgoyne and R. Cushman, *Conjugacy classes in linear groups*, J. Algebra **44** (1977), 339-362.
2. D.Ž. Djoković, *On the exponential map in classical Lie groups*, J. Algebra **64** (1980), 76-88.
3. M. Goto, *Index of the exponential map of a semi-algebraic group*, Amer. J. Math. **100** (1978), 837-843.
4. M. Goto and F. Grosshans, *Semisimple Lie Algebras*, Marcel Dekker, New York 1978.
5. M. Goto and E.T. Kobayashi, *On the subgroups of the centers of simply connected simple Lie groups*, Osaka J. Math. **6** (1969), 251-281.
6. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, N.Y. 1978.
7. H.L. Lai, *Surjectivity of exponential map on semisimple Lie groups*, J. Math. Soc. Japan **29** (1977), 303-325.
8. ———, *Index of the exponential map on a complex simple Lie group*, Osaka J. Math. **15** (1978), 561-567.
— *Corrections and supplements*, Osaka J. Math. **17** (1980), 525-530.
9. R. Steinberg, *Conjugacy classes in algebraic groups*, Lecture Notes in Math. #366, Springer-Verlag, 1974.

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