BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 10, Number 1, March 1982

HOMOGENEOUS SPACE WITH POSITIVE RICCI CURVATURES

BY

HENG-LUNG LAI* (賴恒隆)

Abstract. We prove that a homegeneous space $M=G/H$ carries a G-invariant metric with strictly positive Ricci curvatures if and only if M allows a compact semisimple group S acts transitively on it.

1. **Introduction.** Let M be a Riemannian manifold on which a connected Lie group G acts transitively, so that M is a homogeneous space G/H, where H is the isotropy subgroup at some fixed point. We like to study the problem when does M carry a G-invariant metric with strictly positive Ricci curvature?

According to the classical theorem of Myers, such a homogeneous space must be compact with finite fundamental group. In case $M = G$ is a connected Lie group, this implies that G must be compact semisimple, it is well known that the converse is also true. The purpose of this note is to prove that a similar result holds for G/H to admit G-invariant metric of positive Ricci curvature:

THEOREM. *Let G be a connected Lie group, H a closed subgroup. Then $M = G/H$ admits an invariant metric with strictly positive Ricci curvatures if and only if M allows a compact semisimple group S acts transitively on it.*

We consider a similar problem for $M = G/H$ to carry an invariant metric with nonnegative Ricci curvature. Goto-Uesu [3] gives a complete answer for the case $M = G$ is a Lie group, using the decomposition theorem of Cheeger-Gromoll. Their proof can be easily carried over to the general case, and their result can be

Received by the editors August 5, 1981.
* Partially supported by the National Science Council, R. O. C.

stated as follows: A homogeneous space M on which G acts effectively and transitively admits a G-invariant metric of nonnegative Ricci curvature if and only if 1) $G=AB$ (semidirect), A carries biinvariant metric acting orthogonally on $\mathfrak{B}$ (the Lie algebra of B), B an abelian normal subgroup; and 2) the isotropy subgroup H is contained in A except possibly central elements of G. If the first condition is satisfied, G carries a left invariant metric of nonnegative sectional curvature. When the second condition also holds, so does $M=G/H$.

I like to thank Prof. H. S. Lue for valuable discussions in the preparation of this paper.

2. **Let us prove the easy direction first.** Let G be a (connected) compact semisimple Lie group with Lie algebra $\mathfrak{G}$, H a closed subgroup of G with Lie subalgebra $\mathfrak{h}$. The biinvariant metric on G induces a normal metric on G/H which is G-invariant. We are going to show that with respect to this normal metric, the Ricci curvature is positive in any direction.

Identify the tangent space to G/H at eH with the orthogonal complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{G}$. Note that $[\mathfrak{h}, \mathfrak{p}]\subset\mathfrak{p}$.

According to the submersion formula of O'neill [6], for orthonormal vectors x, y in $\mathfrak{p}$, the sectional curvature of the plane determined by x, y is given by

$$k(x, y)=\frac{1}{4}\|[x, y]_{\mathfrak{p}}\|^2+\|[x, y]_{\mathfrak{h}}\|^2,$$

where $[x, y]_{\mathfrak{h}}$, $[x, y]_{\mathfrak{p}}$ are the $\mathfrak{h}$- and $\mathfrak{p}$-components of $[x, y]$ respectively. In particular, $k(x, y)\geq 0$.

For any fixed unit vector $x\in\mathfrak{p}$, let $\{e_1,\cdots, e_n\}$ be an orthonormal basis of $\mathfrak{p}$ with $e_1=x$. Then the Ricci curvature in the direction x is

$$r(x)=\sum_{i=1}^{n} k(x, e_i)\,.$$

Since $k(x, e_i)\geq 0$, so that $r(x)\geq 0$, and $r(x)=0$ if and only if $k(x, e_i)=0$ for any i. This happens if and only if $[x, e_i]_{\mathfrak{h}}=[x, e_i]_{\mathfrak{p}}=0$ for $i=1,\cdots, n$, i.e. $[x, e_i]=0$ for $i=1,\cdots, n$. But $\{e_1,\cdots, e_n\}$ is an orthonormal basis for $\mathfrak{p}$, this implies that

$$[x, \mathfrak{p}] = 0 .$$

Consider the Lie subalgebra u of $\mathfrak{G}$ generated by the subspace $\mathfrak{p}$

$$u = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] + [\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] + \cdots .$$

Note that $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$, and Jacobi's identity implies that

$$[\mathfrak{h}, [\mathfrak{p}, \mathfrak{p}]] \subset [[\mathfrak{h}, \mathfrak{p}], \mathfrak{p}] \subset [\mathfrak{p}, \mathfrak{p}],$$

and similarly

$$[\mathfrak{h}, [\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]]] \subset [\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]], \cdots$$

i. e.

$$[\mathfrak{h}, u] \subset u .$$

It is clear to see that $[\mathfrak{p}, u] \subset u$. Since $\mathfrak{G} = \mathfrak{h} \oplus \mathfrak{p}$ (as vector spaces), we conclude that u is an ideal of $\mathfrak{G}$. Therefore u must be a semisimple Lie algebra becuase $\mathfrak{G}$ is so. But $x \in u$ (because $x \in \mathfrak{p}$), and $[x, \mathfrak{p}] = 0$ implies that $[x, u] = 0$. This is absurd. Thus $r(x)$ must be positive for any unit vector x.

3. Assume that $M = G/H$ carries a G-invariant metric, i. e. an inner product $\langle , \rangle$ on $\mathfrak{G}/\mathfrak{h}$ (identified as the tangent space to M at eH) which are invariant under the action of Adh on $\mathfrak{G}/\mathfrak{h}$ ($h \in H$, Ad: $G \to GL(\mathfrak{G})$ the adjoint representation).

First, note that we may assume that G acts effectively on $M = G/H$ (G acts on M via left translation $L_g(xH) = gxH$), i. e. L_g is the identity map only when $g = e$. Suppose the action is not effective, let H_0 be the largest subgroup of H which is normal in G. Set $G^* = G/H_0$, $H^* = H/H_0$. Then G^* acts transitively and effectively on G^*/H^*. It is easy to see that G^*/H^* is diffeomorphic to G/H, and $(\mathfrak{G}/\mathfrak{h}_0)/(\mathfrak{h}/\mathfrak{h}_0)$ is isomorphic with $\mathfrak{G}/\mathfrak{h}$. The action Adh ($h \in H$) induces an action on $\mathfrak{G}/\mathfrak{h}_0$, which induces an action on $(\mathfrak{G}/\mathfrak{h}_0)/(\mathfrak{h}/\mathfrak{h}_0)$, the tangent space to G^*/H^* at eH^*. If we define an inner product on this space by the given one on $\mathfrak{G}/\mathfrak{h}$, it is clear that this inner product is invariant under the action of Adh ($h \in H$). Therefore, replacing G with G^*, H with H^*, we may assume that G acts transitively and effectively on $M = G/H$.

Since G/H carries a G-invariant metric, the Lie group G may be identified with a Lie subgroup of $I_0(M)$: the identity component of the group of all isometries of M. In particular, the Lie algebra $\mathfrak{G}$ of G may be considered as a Lie subalgebra of $\mathfrak{L}$, the Lie algebra of $I_0(M)$.

If the Ricci curvature corresponding to the given invariant metric is positive, Myers' theorem implies that M is compact with finite fundamental group. For a compact manifold M, the identity component of the isometry group $I_0(M)$ is a compact Lie group. Therefore, the Lie algebra $\mathfrak{L}$ of $I_0(M)$ is a compact Lie algebra, so that $\mathfrak{G}$ is also a compact Lie algebra. Being compact, $\mathfrak{G}$ has a direct sum decomposition of ideals $\mathfrak{G} = s \oplus z$, where $s = [\mathfrak{G}, \mathfrak{G}]$ is (compact) semisimple, z the center of $\mathfrak{G}$.

Let $\tilde{G}$ be the universal covering group of G. The simply connected group $\tilde{G}$ has a direct product decomposition into closed normal subgroups corresponding to the direct sum decomposition of its Lie algebra $\mathfrak{G}$: $\tilde{G} = S \times Z$, where S is compact semisimple, Z the radical of $\tilde{G}$, which is exactly the identity component of the center of $\tilde{G}$, and isomorphic to a vector group $\boldsymbol{R}^n$ for some (fixed) n.

Let $\tilde{H} = p^{-1}(H)$, where p: $\tilde{G} \rightarrow G$ is the covering map, Then $\tilde{G}/\tilde{H}$ is diffeomorphic to G/H. Note that $\tilde{G}$ cannot be considered as a Lie subgroup of $I_0(M)$, which is the only property destroyed by considering the universal covering group. We still have a homogeneous manifold M of positive Ricci curvature, and $\tilde{G}$ acts transitively on M.

Let $\tilde{H}_e$ be the identity component of $\tilde{H}$. Then $\tilde{G}/\tilde{H}_e \rightarrow \tilde{G}/\tilde{H}$ is a covering, and the induced metric on $\tilde{G}/\tilde{H}_e$ has the same curvature properties as that of $\tilde{G}/\tilde{H}$. Changing notation, we are given: a homogeneous manifold G/H of positive Ricci curvature, where G is simply connected with compact Lie algebra $\mathfrak{G}$, H a connected closed subgroup of G. $G = S \times Z$, S compact semisimple, Z the radical of G ($\simeq \boldsymbol{R}^n$ vector group).

The connected Lie group H has a similar decomposition corresponding to the decomposition of its Lie algebra $\mathfrak{h} = s_1 \oplus z_1$ (which is also compact),

$$H = S_1 C_1 \qquad (S_1 \cap C_1 \text{ is finite}),$$

where S_1 is compact semisimple (so is closed), C_1 the radical of H (so is closed, connected, abelian, normal in H). The connected abelian group C_1 can be further decomposed into the direct product of a torus T and a vector group C_2. So we have

$$H = KC_2,$$

where K is maximal compact in H, C_2 is a closed vector subgroup $\boldsymbol{R}^l$, and $K \cap C_2 = \{e\}$. Since H is closed in G, so C_2 is a closed vector subgroup of G. The compact subgroup K must be contained in the maximal compact subgroup S of G.

Consider the adjoint representation $Ad\colon G \to GL(\mathfrak{G})$. It is easy to see that C_2 is the identity component of the inverse image of AdC_2, and Z is the identity component of the inverse image of AdZ. Thus the vector group C_2Z is the identity component of the inverse image of $AdC_2 \cdot AdZ = Ad(C_2 Z)$, the later subgroup is a torus, which is compact, and therefore is closed. In particular, the vector group $C = C_2Z$ is closed in G. Since K is compact, $HZ = K C_2Z = KC$ is also closed in G. This implies that $H/H \cap Z$ is closed in G/Z because $\pi^{-1}(H/H \cap Z) = HZ$, where $\pi\colon G \to G/Z$ denotes the canonical map. But $G/Z \simeq S$ is compact, so $H/H \cap Z$ is compact.

On the other hand, $H \cap Z$ is closed normal in H, K is maximal compact in H, a theorem of Iwasawa implies that $K/H \cap Z$ is maximal compact in $H/H \cap Z$. The above discussion implies that $K/H \cap Z = H/H \cap Z$, and $H = K(H \cap Z) = KZ_1$ which is indeed a direct product, as is easy to see.

We have proved that

$$G = S \times Z, \qquad S \text{ compact semisimple}, \qquad Z \simeq \boldsymbol{R}^n;$$

$$H = K \times Z_1, \qquad K = S \cap H, \qquad Z_1 = Z \cap H \simeq \boldsymbol{R}^l.$$

Consider $M' = S/K \times Z/Z_1 \simeq S/K \times \boldsymbol{R}^{n-l}$. For $g = sx \in G$, $m' = (tK, yZ_1) \in M'$, we define $g\, m'$ to be $(stK, xy Z_1)$. This gives a transitive action of G on M'. The isotropy subgroup of this action at $m_0 = (eK, eZ_1)$ is exactly H. (When $gm_0 = m_0$ with $g = sx$, then $sK = K$, $xZ_1 = Z_1$, so $s \in K$, $x \in Z_1$ and $g \in H$. Conversely, if $g = sx \in H$, then $s \in K$, $x \in Z_1$ and $gm_0 = m_0$.) Therefore

$M' = G/H$, ie, $M = G/H \simeq S/K \times \boldsymbol{R}^{n-l}$. The compactness of M implies that $n = l$, or, $Z = Z_1 \subset H$, and $G/H \simeq S/K$. Hence M allows a compact semisimple group S acting transitively on it This finishes our proof.

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MATHEMATICS DEPARTMENT, TSING-HUA UNIVERSITY, HSIN-CHU, TAIWAN, R. O. C.