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# APPROXIMATION TO MINIMUM *K*-EXTENDED BAYES RISK IN SEQUENCES OF FINITE STATE DECISION PROBLEMS AND GAMES

BY

STEPHEN B. VARDEMAN

0. **Summary.** This paper treats a sequence version of the finite state compound decision problem. $k$-extended standards for the risk of sequence compound procedures are described. Bounds are developed for the risks of a family of procedures employing artificial randomization. In addition it is noted that the given formulation of the problem includes a game theoretic situation and three additional solutions are offered for this specialization.

1. **Introduction.** Simultaneous consideration of a number of independent structuarlly identical decision problems with the goal of controlling the average or total risk incurred across the problems was first suggested by Robbins (1951). Robbins termed his original example involving $N$ independent discriminations between normal (1, 1) and normal (−1, 1) distributions a compound decision problem. The procedure he proposed has total risk approximating $N$ times the minimum Bayes risk versus the normalized empirical distribution of states in the component testing problem, and finding procedures with similar total risk performance became the usual objective in compound decision theory.

Hannan (1956), (1957) in rather general finite state settings showed that the usual compound decision theoretic goal is achievable not only in situations in which all $N$ of the problems are considered

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simultaneously, but also in situations where the independent structurally identical problems are faced serially. Such a modification of the compound setting has become known as a sequence compond decision problem. Hannan's procedures involve artificial randomization and are applicable in situations in which before making the $\alpha$th decision one has available either the exact empiric distribution of states through the $(\alpha-1)$ *st* problem or an estimate of the same.

Johns (1967) and Gilliland and Hannan (1969) suggested standards of performance for compound procedures which are appropriate to sequence versions of compound problems and asymptotically more strigent than the usual standards. These standards, which take into account $k$th order empirical dependencies in the sequence of states have become known as $k$-extended standards. Work of Ballard (1974) and Ballard, Gilliland and Hannan (1974) shows that in Van Ryzin's (1966) finite state finite act statistical setting, generalizations of his non-randomized procedures have risk approximating these $k$-extended objectives.

In this paper we treat a particularly tractable, yet quite general finite state sequence compound decision problem. The generality of the problem derives primarily from the fact that a risk structure rather than action space and loss structure is assumed for the component problem. In §2 this finite state restricted risk component sequence compound decision problem is described along with the unextended and $k$-extended standards for the problem. The estimation of $k$th order empirical distributions of states is also very briefly considered. The notation and usefulness of the $k$-extended ideas presented in §2 are illustrated by a simple example in §3. Section 4 contains the description of procedures which are generalizations of the procedures involving randomization proposed originally by Hannan. We bound the total risk of the procedures and note that appropriate choice of arbitrary constants yields bounds approximating the $k$-extended standards at a $O(N^{1/2})$ rate. In §5 it is noted that the problem includes a game theoretic situation in which the component risk set is composed of the risk points available to player II, and before each repetition of the game II is furnished with the empirical distribution of I's moves through the previous play. After noting a simple game theoretic decomposition of the $k$-extended standard, three procedures are provided in addition to the game theoretic specialization of the procedure from §3, which have risk approximating the $k$-extended standard at a $O(N^{1/2})$ rate. The basic technique employed in §5.3 and §5.4 has been used independently by Cover and Shenhar (1974) in the situation of sequential prediction of binary sequences.

## 2. The $k$-extended finite state restricted risk component sequence compound decision problem.

**2.1. The component problem.** Consider a (component) decision problem with states $\theta \in \Theta = \{1, 2, \cdots, m\}$ and risk set $S \subset [0, \infty)^m$. For each $\theta \in \Theta$ let $P_\theta$ be a probability on a measurable space $(\mathscr{X}, \mathscr{G})$. In all that follows we shall assume that $S$ is bounded, $\|s\|_\infty \leq B < \infty$ for each $s \in S$, where $\|\cdot\|_\infty$ denotes the supremum norm for $m$-vectors. $S$ may be a proper subset of the largest possible risk set for a given action space and loss function and Gilliland and Hannan (1974) have used the term "restricted risk component" to indicate this possibility. For $w$ a vector in $R^m$ and $s \in S$ let $ws$ denote the vector inner product of $w$ and $s$. In the case where $w$ is a probability vector, $ws$ is the Bayes risk of $s$ against the prior $w$. Agreeing to let $\theta \in \Theta$ correspond to the $\theta$th standard basis vector in $R^m$ $\theta s$ is then $\theta$ coordinate of the risk $s$.

**2.2. The sequence compound problem.** We study a sequence compound problem composed of $N$ independent repetitions of the component problem, where the choice of risk function in component $\alpha$ is allowed to depend upon independent, $P_{\theta_\beta}$ distributed observations for $\beta \leq \alpha - 1$, and the compound risk is taken as the sum of risks in components 1 through $N$. More precisely, let $k$ be a positive integer and $\boldsymbol{s} = (s_1, s_2, \cdots, s_N)$ be such that $s_\alpha$ is a $\mathscr{G}^{\alpha+k-2}$ measurable mapping into $S$. For $\boldsymbol{\theta}_N = (\theta_{2-k}, \cdots, \theta_N)$ we suppose that $\boldsymbol{X}_N = (X_{2-k}, \cdots, X_N)$ is distributed as $\boldsymbol{P}_N = P_{\theta_{2-k}} \times \cdots \times P_{\theta_N}$. (The purpose of allowing indices $\alpha < 1$ in the case $k > 1$ here is to simplify later notation.) The compound risk of the sequence rule $\boldsymbol{s}$ is

$$\sum_{\alpha=1}^{N} E\theta_\alpha s_\alpha(\boldsymbol{X}_{\alpha-1}) = \sum_{1}^{N} \int \theta_\alpha s_\alpha \, d\boldsymbol{P}_{\alpha-1}\,. \tag{1}$$

When $\boldsymbol{s}=(s, s,\cdots, s)$ for a fixed $s \in \boldsymbol{S}$ the risk (1) becomes

$$\sum_{1}^{N} \theta_\alpha s = \left(\sum_{1}^{N} \theta_\alpha\right) s = G_N s$$

where $G_N$ is the vector of frequencies of states in problems 1 through $N$. For $w \in R^m$ let $\Psi(w)$ denote $\bigwedge_{s\in S} ws$. Hannan (1957), (1956) first exhibited procedures in versions of the sequence compound problem with compound risks asymptotically approximaitng $\Psi(G_N) = \bigwedge_{s\in S} G_N s$.

Let $S^*$ be a class of $\mathcal{G}^{k-1}$ measurable mappings into $\boldsymbol{S}$ and consider sequence compound rules of the form $\boldsymbol{s}=(s^*,\cdots, s^*)$ for a fixed $s^* \in \boldsymbol{S}^*$. Denoting $(X_{\alpha-k+1},\cdots, X_{\alpha-1})$ as $\boldsymbol{X}^*_{\alpha-1}$ and $P_{\theta_{\alpha-k+1}} \times\cdots\times P_{\theta_{\alpha-1}}$ as $\boldsymbol{P}^*_{\alpha-1}$, the risk (1) for such an $\boldsymbol{s}$ reduces to the functional

$$\sum_{1}^{N} E\theta_\alpha s^*(\boldsymbol{X}^*_{\alpha-1}) = \sum_{1}^{N} \int \theta_\alpha s^* \, d\boldsymbol{P}^*_{\alpha-1} \tag{2}$$

of the empirical distribution on $\Theta^k$ of the vectors $\{(\theta_{2-k},\cdots, \theta_1),\cdots, (\theta_{N-k+1},\cdots, \theta_N)\}$. In terminology similar to that of Swain (1965), Johns (1967), and Gilliland and Hannan (1969), we will term

$$\bigwedge_{s^*\in S^*} \sum_{1}^{N} E\,\theta_\alpha s^*(\boldsymbol{X}^*_{\alpha-1}) \tag{3}$$

a $k$-extended simple envelope for the sequence compound problem. The purpose of this work is to exhibit sequence compound rules which achieve risk (3) asymptotically with rate.

2.3. **Bayes rules in the component problem.** We will make the assumption that $S$ is not only bounded, but also closed. For any $w \in R^m$, $\bigwedge_{s\in S} ws$ is then attained and we denote an infimizing $s$ by $\sigma(w)$. It is a simple consequence of Corollary 1 of Brown and Purves (1973) that there is a Borel measurable determination of $\sigma(\cdot)$. In addition we may assume that $\sigma(\cdot)$ has the property that $\sigma(pw) = \sigma(w)$ for $p > 0$. (If not, we replace $\sigma(w)$ by $\sigma(w/\|w\|)$ for $\|w\| \neq 0$, where $\|\cdot\|$ is the usual Euclidean vector norm.) Notice that with this notation we have

$$\Psi(G_N) = \bigwedge_{s\in S} G_N\, s = G_N\, \sigma(G_N)\,.$$

There is no essential loss of generality in the assumption that $S$ is closed. If $\bar{S}$ denotes the closure of $S$ in $R^m$, for any $\varepsilon > 0$ and any sequence compound rule $\mathbf{s}' = (s'_1, s'_2, \cdots, s'_N)$ where $s'_\alpha$ is a $\mathcal{G}^{\alpha+k-2}$ measurable mapping, into $\bar{S}$, there exists a rule $\mathbf{s} = (s_1, \cdots, s_N)$ such that $s_\alpha$ is a $\mathcal{G}^{\alpha+k-2}$ measurable mapping into $S$ with $|\theta_\beta s_\alpha(\cdot) - \theta_\beta s'_\alpha(\cdot)| \le 2^{-\alpha}\varepsilon$ for each $\theta_\beta \in \Theta$. Hence $|\sum_1^N E\,\theta_\alpha s_\alpha(\boldsymbol{X}_{\alpha-1}) - \sum_1^N E\,\theta_\alpha s'_\alpha(\boldsymbol{X}_{\alpha-1})| < \varepsilon$ for all $\boldsymbol{\theta}_N$, and theorems concerning the risks of $\bar{S}$ valued rules have $\varepsilon$ analogues for $S$ valued rules.

2.4. **The $\Gamma^k$ construct.** In order to describe compound rules achieving risk (3) asymptotically, we introduce a variant of (Gilliland and Hannan's $\Gamma^k$ decision problem. The $\Gamma^k$ problem has finite state space $\Theta^k$ and risk set $\tilde{S} \subset [0, +\infty)^{mk}$ of the form $\tilde{S} = \{\tilde{s} \in R^{m^k} \mid (\theta_1, \cdots, \theta_k)\tilde{s} = \int \theta_k\, s^*(\cdot)\, dP_{\theta_1} \times \cdots \times P_{\theta_{k-1}}$ for some $s^* \in S^*\}$, where we are indexing the components of $R^{m^k}$ vectors by $k$-vectors of $\theta$'s, letting $(\theta_1, \cdots, \theta_k)$ correspond to the standard basis vector in $R^{m^k}$ with a 1 in the $(\theta_1, \cdots, \theta_k)$ position and continuing to denote vector inner product by juxtaposition. The $\Gamma^k$ problem inherits the property of bounded risk from the component problem. $\tilde{\sigma}(v)$ will denote a Borel measurable, positive homogeneous minimizer of $v\tilde{s}$. (That no essential generality will be lost by the assumption that $\tilde{S}$ is closed follows from a comment similar to that made in the previous paragraph.)

Letting $G_N^k$ denote the $m^k$-vector of frequencies of $k$-vectors of $\theta$'s among $\{(\theta_{2-k}, \cdots, \theta_1), \cdots, (\theta_{N-k+1}, \cdots, \theta_N)\}$ and using the $\Gamma^k$ notation, we have from (2) and (3)

$$\begin{aligned}(3) &= \bigwedge_{s^*\in S^*} \sum_1^N \int \theta_\alpha\, s^*\, d\boldsymbol{P}^*_{\alpha-1} \\ &= \bigwedge_{\tilde{s}\in\tilde{S}} \sum_1^N (\theta_{\alpha-k+1}, \cdots, \theta_\alpha)\, \tilde{s} \\ &= \bigwedge_{\tilde{s}\in\tilde{S}} G_N^k\, \tilde{s} \\ &= G_N^k\, \tilde{\sigma}(G_N^k)\,.\end{aligned}$$

For $v \in R^{m^k}$ define

$$\Psi^k(v) = v\,\tilde{\sigma}(v) \tag{4}$$

and thus we may write $(3) = \Psi^k(G_N^k)$.

It will be important to recover elements of $S^*$ which give rise to values of the minimizer $\tilde{\sigma}(\cdot)$. Thus assume that $s^*(\cdot,\ \cdot)$ is a mapping from $R^{mk} \times \mathscr{X}^{k-1}$ into $S$ with the property that for $v \in R^{mk}$, $s^*(v,\ \cdot) \in S^*$ such that

$$\int \theta_k\, s^*(v,\ \cdot)\, dP_{\theta_1} \times \cdots \times P_{\theta_{k-1}} = (\theta_1, \cdots, \theta_k)\, \tilde{\sigma}(v) \tag{5}$$

for each $\boldsymbol{\theta} \in \Theta^k$. Notice that in the case $k = 1$, the $\Gamma^k$ construct is identical with the original component problem and we take $s^*(v) = \tilde{\sigma}(v) = \sigma(v)$.

2.5. **Assumption on the $P_\theta$ and estimation of empirics.** We will assume that $\mathscr{P} = \{P_\theta\}_{\theta\in\Theta}$ is a linearly independent family of measures. That is, for real numbers $a_1, \cdots, a_m$, $\sum_{\theta\in\Theta} a_\theta P_\theta$ is the zero signed measure only if each $a_\theta = 0$. Robbins (1964), Van Ryzin (1966), and Ballard (1974) discuss the estimation of mixtures of a finite number of linearly independent distributions.

In the linearly independent situation there are $R^m$ valued, bounded, $\mathscr{G}$ measurable mappings $t$ with the property that $\int t(\cdot)\, dP_\theta$ is the $m$-vector with all zero entries except a 1 in the $\theta$ position. $t(X_\alpha)$ is then an unbiased estimate of the $m$-vector corresponding to $\theta_\alpha$. Ballard (1974) uses vectors of all possible products of coordinates of $k$ such mappings $t$ to construct $R^{mk}$ valued, bounded $\mathscr{G}^k$ measurable mappings $\tilde{t}$ with the property that $\int \tilde{t}(\cdot)\, dP_{\theta_1} \times \cdots \times P_{\theta_k}$ is the $m^k$-vector with all zero entries except a 1 in the $(\theta_1, \theta_2, \cdots, \theta_k)$ position. $\tilde{t}(X_{\alpha-k+1}, \cdots, X_\alpha)$ is then an unbiased estimate of the $m^k$-vector corresponding to $(\theta_{\alpha-k+1}, \cdots, \theta_\alpha)$, and $\sum_{j=1}^{\alpha} \tilde{t}(X_{j-k+1}, \cdots, X_j)$ is an unbiased estimate of $G_\alpha^k$.

We will not assume a special product structure for our estimates but will assume only that $\tilde{t}$ is an $\mathscr{G}^k$ measurable mapping into $R^{mk}$ with the properties

$$\int \tilde{t}\, dP_{\theta_1} \times \cdots \times P_{\theta_k} = (\theta_1, \cdots, \theta_k) \in R^{mk}, \tag{5}$$

and

$$\bigvee_{\theta \in \Theta^k} \int (\|\tilde{t}\|_1)^2 \, dP_{\theta_1} \times \cdots \times P_{\theta_k} \equiv \tau^2 < \infty, \tag{7}$$

where $\|\cdot\|_1$ is the usual $l_1$ vector norm. (Ballard's product kernels provide examples of functions satisfying (6) and (7).) Let $\tilde{t}_\alpha$ denote $\tilde{t}(X_{\alpha-k+1}, \cdots, X_\alpha)$ and $\tilde{T}_\alpha$ denote $\sum_{j=1}^{\alpha} \tilde{t}_j$ for $\alpha \geq 1$ and 0 otherwise.

3. **An example.** To illustrate the usefulness of the $k$-extended ideas, consider one of the simplest of all possible sequence compound decision problems, $N$ independent discriminations between fair and two headed coins. That is, let $\Theta = \{1, 2\}$, $\mathcal{X} = \{0, 1\}$, $P_1$ be a probability placing mass $\frac{1}{2}$ on each element of $\mathcal{X}$ and $P_2$ degenerate on $\{1\}$. For this example will take $k = 2$ and consider making decisions about the values of $\theta_1, \cdots, \theta_N$ based on $\boldsymbol{X}_N = (X_0, X_1, \cdots, X_N)$ distributed as $\boldsymbol{P}_N = P_{\theta_0} \times P_{\theta_1} \times \cdots \times P_{\theta_N}$.

Supposing the component problem action space to be $A = \Theta$ and loss function to be

$$L(\theta, a) = \begin{cases} 1 \text{ if } a \neq \theta \\ 0 \text{ if } a = \theta \end{cases}$$

the risk set generated by all possible component decision rules is the subset of $[0, 1]^2$ composing the convex hull of the points $(0, 1)$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, 1)$ and $(1, 0)$.

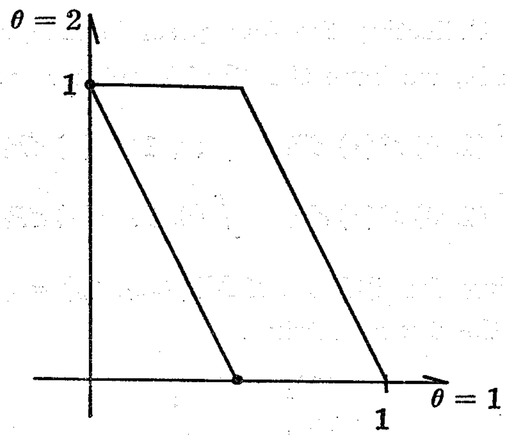


FIGURE 1.

We will suppose that only risk points corresponding to admissible nonrandomized component decision rules are of interest and hence take $S = \{(0, 1), (\frac{1}{2}, 0)\}$. It is easily verified that the function $\sigma$ defined by

$$\sigma((w_1, w_2)) = \begin{cases} \left(\frac{1}{2}, 0\right) & \text{if} \quad w_2 \geq \frac{1}{2} w_1 \\ (0, 1) & \text{if} \quad w_2 < \frac{1}{2} w_1 \end{cases}$$

has the property that for $w \in R^2$, $\bigwedge_{s \in S} ws = w\sigma(w)$.

Should one employ either of the two component risk functions (equivalently, the corresponding component decision rule) in each of the $N$ decisions, the total risk suffered is then $(N_1, N_2)s$ where $N_1$ is the frequency of states 1 amongst $\theta_1, \cdots, \theta_N$, $N_2$ is the frequency of states 2, and as always juxtaposition indicates inner product. $\Psi(G_N) = (N_1, N_2)\sigma(N_1, N_2)) = (\frac{1}{2} N_1) \wedge N_2$ is then the usual total risk objective for compound procedures specialized to this example. ($\Psi(G_N)$ is essentially the best one could hope to do in terms of total risk if he determined to i) use only $X_\alpha$ in the $\alpha$th decision and ii) use the same function of the observation to make each decision.)

Let $S^*$ be the set of 4 possible functions from $\mathscr{X}$ to $S$. Notice that for $s^* \in S^*$, if risk function $s^*(X_{\alpha-1})$ (or equivalently, the randomly determined corresponding component decision rule) were used in problem $\alpha$, the single component risk suffered would be $\int \theta_\alpha s^*(\cdot) dP_{\theta_{\alpha-1}}$. Collecting the four possible integrals of this form into a $2 \times 2$ matrix, we have the $\Gamma^2$ risk point corresponding to $s^*$

$$\tilde{s} = \begin{pmatrix} \int (1, 0)\, s^*(\cdot)\, dP_1 & \int (0, 1)\, s^*(\cdot)\, dP_1 \\ \int (1, 0)\, s^*(\cdot)\, dP_2 & \int (0, 1)\, s^*(\cdot)\, dP_2 \end{pmatrix}.$$

With $N_{\theta,\theta'}$ standing for $\#\{1 \leq \alpha \leq N \mid (\theta_{\alpha-1}, \theta_\alpha) = (\theta, \theta')\}$ we may represent $G_N^2$ as the $2 \times 2$ matrix

$$\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

and notice that with the obvious inner product between $2 \times 2$ matrices, the *total* risk suffered using risk $s^*(X_{\alpha-1})$ in each problem

$\alpha$ is then $G_N^2\tilde{s}$. As $S^*$ contains the two constant maps from $\mathscr{X}$ to $S$ it is the case that $\Psi^2(G_N^2) = \bigwedge_{s^*\in S^*} G_N^2\tilde{s} \le \Psi(G_N)$, and for this example the 2-extended risk objective is at least as stringent as the usual objective. ($\Psi^2(G_N^2)$ is essentially the best one could hope to do in terms of total risk if he determined to i) use only $X_{\alpha-1}$ and $X_\alpha$ in the $\alpha$th decision and ii) use the same function of the observations to make each decision.)

That $\Psi^2(G_N^2)$ can be substantially less than $\Psi(G_N)$ for $\boldsymbol{\theta}_N$ with marked dependencies in $G_N^2$ is easily verified by consideration of $\boldsymbol{\theta}_N$ with $\theta_0$ through $\theta_{N/2}$ equal to 1 and $\theta_{N/2+1}$ through $\theta_N$ equal to 2. For such a $\boldsymbol{\theta}_N$, $\Psi(G_N) = N/4$ and it is easy to verify that $\Psi^2(G_N^2)$ is essentially $N/8$. The usefulness of the sequence compound procedures of §4 is that their $N$ problem risk approximates $\Psi^k(G_N^k)$ whatever be $\boldsymbol{\theta}_N$.

4. **A bound on the risk of a *k*-extended sequence compound procedure employing artificial randomization.** We introduce a generalization of a strategy in the sequence compound problem proposed by Hannan (1957), (1956) and bound its risk. When constants appearing in both the description of the procedure and the bound are appropriately chosen, the strategy is seen to achieve risk $\Psi^k(G_N^k) + O(N^{1/2})$.

4.1. **Two lemmas.** Forms of the two lemmas which follow appeared first in Hannan (1957) and variations of one or the other have since appeared in Samuel (1963), (1965), Swain (1965), Van Ryzin (1966), Gilliland (1969) and Gilliland and Hannan (1969).

LEMMA 1. (Hannan) *Let $f_1, f_2, \cdots, f_N$ be real-valued functions on some set $\mathscr{D}$. Let $F_\alpha = \sum_1^\alpha f_j$ and suppose that for each $1 \le \alpha \le N$, $\exists\, \boldsymbol{d}_\alpha \in \mathscr{D}$ such that $F_\alpha(\boldsymbol{d}_\alpha) = \bigwedge_{d\in\mathscr{D}} F_\alpha(d)$. Let $\boldsymbol{d}_0$ be arbitrary. Then $\sum_{\alpha=1}^N f_\alpha(\boldsymbol{d}_\alpha) \le F_N(\boldsymbol{d}_N) \le \sum_{\alpha=1}^N f_\alpha(\boldsymbol{d}_{\alpha-1})$.*

**Proof.** $\sum_1^N f_\alpha(\boldsymbol{d}_\alpha) = F_N(\boldsymbol{d}_N) - \sum_1^{N-1}(F_\alpha(\boldsymbol{d}_{\alpha+1}) - F_\alpha(\boldsymbol{d}_\alpha))$. But for each $\alpha$, $F_\alpha(\boldsymbol{d}_{\alpha+1}) - F_\alpha(\boldsymbol{d}_\alpha) \ge 0$. Also $\sum_1^N f_\alpha(\boldsymbol{d}_{\alpha-1}) = F_N(\boldsymbol{d}_N) + \sum_1^N (F_\alpha(\boldsymbol{d}_{\alpha-1}) - F_\alpha(\boldsymbol{d}_\alpha))$. And for each $\alpha$, $F_\alpha(\boldsymbol{d}_{\alpha-1}) - F_\alpha(\boldsymbol{d}_\alpha) \ge 0$. $\square$

The application we will make of this lemma is to take $\mathcal{D} = \tilde{S}$, $f_\alpha(\tilde{s}) = v_\alpha \tilde{s}$ for $v_\alpha \in R^{mk}$, let $V_\alpha = \sum_{\beta=1}^{\alpha} v_\beta$ and conclude

$$\sum_{\alpha=1}^{N} v_\alpha \tilde{\sigma}(V_\alpha) \leq \Psi^k(V_N) \leq \sum_{\alpha=1}^{N} v_\alpha \tilde{\sigma}(V_{\alpha-1}) . \tag{8}$$

We will not prove the next lemma. Apart from slight notational differences, the proof of a similar lemma in section 3 of Gilliland (1969) applies in our case also. Gilliland's assumptions that $v$, $v'$, $z$ are $R^\infty$ vectors may be altered to $v$, $v'$, $z$ being $R^{mk}$ vectors without change in the form of the proof, his $B$ may be re-interpreted as our supremum norm on $S$, and his assumption that the co-ordinates of $v$, $v'$ are non-negative may be dropped.

LEMMA 2. *Let $Z$ be uniformly distributed on* $[0, 1]^{mk}$ *and let* $\mu$ *be the distribution of $Z$. For any $v$, $v'$ belonging to $R^{mk}$ and any* $\boldsymbol{\theta} \in \Theta^k$

$$|\mu\, \boldsymbol{\theta}(\tilde{\sigma}(v + z) - \tilde{\sigma}(v' + z))| \leq B\|v - v'\|_1,$$

*where operator notation is used to indicate integration.*

The lemma then gives

$$\|\mu(\tilde{\sigma}(v + z) - \tilde{\sigma}(v' + z))\|_\infty \leq B\|v - v'\|_1 . \tag{9}$$

4.2. **Definition of the procedures $\hat{s}$.** Take $\{H_\alpha\}_{\alpha=1}^{\infty}$ to be a non-decreasing sequence of positive constants. Define $H_\alpha = 0$ for $\alpha \leq 0$ and let $h_\alpha = H_\alpha - H_{\alpha-1}$. Let $Z$ be a uniform $[0, 1]^{mk}$ random vector indepedent of $X_N$. We will consider the procedure $\hat{\boldsymbol{s}} = (\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_N)$ where

$$\hat{s}_\alpha = s^*(\tilde{T}_{\alpha-k} + H_{\alpha-k}\, Z, X^*_{\alpha-1}) . \tag{10}$$

(In the $\alpha$ component, the proposed procedure uses $s^*(X^*_{\alpha-1})$ for a $s^* \in S^*$ corresponding to an element of $\tilde{S}$ which is $\Gamma^k$ Bayes against a randomly perturbed estimate of $G^k_{\alpha-k}$.)

4.3. **A bound for the risk of $\hat{s}$.**

THEOREM.

$$\sum_{\alpha=1}^{N} E\, \theta_\alpha \hat{s}_\alpha \leq \Psi^k(G^k_N) + \frac{1}{2} B\, H_N\, m^k + B\, k\, \tau \left(1 + 2\tau \sum_{\alpha=1}^{N} \frac{1}{H_\alpha}\right),$$

*where for each* $\alpha$, $E\theta_\alpha \hat{s}_\alpha$ *is interpreted as an interated integral, the* *first integration with respect to the distribution of $X^*_{\alpha-1}$ on $\mathscr{X}^{k-1}$, and the second with respect to the distribution of $(X_{2-k},\cdots, X_{\alpha-k}, Z)$ on $\mathscr{X}^{\alpha-1}\times[0, 1]^{mk}$.*

**Proof.** Use operator notation to indicate integration and denote the distribution of $Z$ as $\mu$. Then

$$\sum_1^N E\,\theta_\alpha\,\hat{s}_\alpha=\sum_1^N \mu\,\boldsymbol{P}_{\alpha-k}\boldsymbol{P}^*_{\alpha-1}\,\theta_\alpha\,s^*(\tilde{T}_{\alpha-k}+H_{\alpha-k}\,z,\ X^*_{\alpha-1})\,,$$

$$=\sum_1^N \mu\,\boldsymbol{P}_{\alpha-k}(\theta_{\alpha-k+1},\cdots,\ \theta_\alpha)\,\tilde{\sigma}(\tilde{T}_{\alpha-k}+H_{\alpha-k}\,z)\,,$$

from (5). So

$$\sum_1^N E\,\theta_\alpha\,\hat{s}_\alpha=\sum_1^N \mu\,\boldsymbol{P}_N(\theta_{\alpha-k+1},\cdots,\ \theta_\alpha)\,\tilde{\sigma}(\tilde{T}_{\alpha-k}+H_{\alpha-k}\,z)\,. \tag{11}$$

Recalling that $\tilde{t}_\alpha$ is unbiased for $(\theta_{\alpha-k+1},\cdots, \theta_\alpha)$ and independent of $\tilde{T}_{\alpha-k}+H_{\alpha-k}Z$, (11) gives

$$\sum_1^N E\,\theta_\alpha\,\hat{s}_\alpha=\sum_1^N \mu\,\boldsymbol{P}_N\,\tilde{t}_\alpha\,\tilde{\sigma}\,(\tilde{T}_{\alpha-k}+H_{\alpha-k}\,z)$$

$$=\sum_1^N \boldsymbol{P}_N\,\tilde{t}_\alpha\,\mu(\tilde{\sigma}(\tilde{T}_{\alpha-k}+H_{\alpha-k}\,z)-\tilde{\sigma}(\tilde{T}_\alpha+H_\alpha\,z)) \tag{12}$$

$$+\sum_1^N \boldsymbol{P}_N\,\mu\,\tilde{t}_\alpha\,\tilde{\sigma}(\tilde{T}_\alpha+H_\alpha\,z)\,.$$

Denote the first sum on the right of (12) by $A$ and the second by $C$. We will set $U_\alpha=\tilde{T}_\alpha+H_\alpha Z$, $\tilde{\sigma}(U_\alpha)=\tilde{\sigma}_\alpha$ and bound $A$ and $C$ separately.

First consider $A$. For $\alpha>k$

$$|\tilde{t}_\alpha\mu(\tilde{\sigma}_{\alpha-k}-\tilde{\sigma}_\alpha)|\leq\|\tilde{t}_\alpha\|_1\|\mu(\tilde{\sigma}_{\alpha-k}-\tilde{\sigma}_\alpha)\|_\infty\,. \tag{13}$$

We write $\tilde{\sigma}_{\alpha-k}-\tilde{\sigma}_\alpha=\tilde{\sigma}((T_{\alpha-k}/H_{\alpha-k})+z)-\tilde{\sigma}((\tilde{T}_\alpha/H_\alpha)+z)$ and (13) and (9) give

$$|\tilde{t}_\alpha\,\mu(\tilde{\sigma}_{\alpha-k}-\tilde{\sigma}_\alpha)|\leq\|\tilde{t}_\alpha\|_1\,B\left\|\frac{\tilde{T}_\alpha}{H_\alpha}-\frac{\tilde{T}_{\alpha-k}}{H_{\alpha-k}}\right\|_1$$

$$=B\|\tilde{t}_\alpha\|_1\left\|\frac{\sum_{j=1}^k\tilde{t}_{\alpha-k+j}}{H_\alpha}-\tilde{T}_{\alpha-k}\Big(\frac{1}{H_{\alpha-k}}-\frac{1}{H_\alpha}\Big)\right\|_1$$

$$\leq B\|\tilde{t}_\alpha\|_1\Big(\frac{1}{H_\alpha}\sum_{j=1}^k\|\tilde{t}_{\alpha-k+j}\|_1$$

$$+\Big(\frac{1}{H_{\alpha-k}}-\frac{1}{H_\alpha}\Big)\sum_{j=1}^{\alpha-k}\|\tilde{t}_j\|_1\Big)\,.$$

Hence

$$\sum_{\alpha=k+1}^{N} \boldsymbol{P}_N |\tilde{t}_\alpha \, \mu(\tilde{\sigma}_{\alpha-k} - \tilde{\sigma}_\alpha)|$$

(14) $$\leq B \sum_{\alpha=k+1}^{N} \boldsymbol{P}_N \left( \frac{1}{H_\alpha} \sum_{j=1}^{k} \|\tilde{t}_\alpha\|_1 \|\tilde{t}_{\alpha-k+j}\|_1 + \left( \frac{1}{H_{\alpha-k}} - \frac{1}{H_\alpha} \right) \sum_{j=1}^{\alpha-k} \|\tilde{t}_\alpha\|_1 \|\tilde{t}_j\|_1 \right).$$

The Schwarz inequality and (7) applied to (14) give

$$\sum_{\alpha=k+1}^{N} \boldsymbol{P}_N |\tilde{t}_\alpha \, \mu(\tilde{\sigma}_{\alpha-k} - \tilde{\sigma}_\alpha)|$$

$$\leq B \sum_{\alpha=k+1}^{N} \left( \frac{k}{H_\alpha} \tau^2 + \left( \frac{1}{H_{\alpha-k}} - \frac{1}{H_\alpha} \right) (\alpha-k)\tau^2 \right)$$

$$= B\tau^2 \left( 2k \sum_{k+1}^{N} \frac{1}{H_\alpha} + \sum_{\alpha=1}^{k} \frac{\alpha}{H_\alpha} - \sum_{N-k}^{N} \frac{\alpha}{H_\alpha} \right).$$

But

$$\sum_{\alpha=1}^{k} \boldsymbol{P}_N \, \mu |\tilde{t}_\alpha (\tilde{\sigma}_{\alpha-k} - \tilde{\sigma}_\alpha)|$$

$$\leq \sum_{\alpha=1}^{k} \boldsymbol{P}_N \, \mu \|\tilde{t}_\alpha\|_1 \|\tilde{\sigma}_{\alpha-k} - \tilde{\sigma}_\alpha\|_\infty \leq B \, k \, \tau$$

by the moment inequality. So

$$A \leq Bk\tau + B\tau^2 \left( 2k \sum_{k+1}^{N} \frac{1}{H_\alpha} + \sum_{\alpha=1}^{k} \frac{\alpha}{H_\alpha} - \sum_{N-k}^{N} \frac{\alpha}{H_\alpha} \right)$$

$$\leq Bk\tau + 2Bk\tau^2 \sum_{1}^{N} \frac{1}{H_\alpha}.$$

Now bound $C$.

$$C = \boldsymbol{P}_N \, \mu \sum_{1}^{N} \tilde{t}_\alpha \, \tilde{\sigma}_\alpha$$

$$\leq \boldsymbol{P}_N \, \mu \sum_{1}^{N} (\tilde{t}_\alpha + h_\alpha z) \, \tilde{\sigma}_\alpha \leq \boldsymbol{P}_N \, \mu \, \Psi^k(\tilde{T}_N + H_N z)$$

by (8). But by (4)

$$\begin{aligned} \boldsymbol{P}_N\, \mu\, \Psi^k(\tilde{\boldsymbol{T}}_N + H_N \boldsymbol{z}) &= \boldsymbol{P}_N\, \mu(\tilde{\boldsymbol{T}}_N + H_N \boldsymbol{z})\, \tilde{\sigma}(\boldsymbol{T}_N + H_N \boldsymbol{z}) \\ &\le \boldsymbol{P}_N\, \mu(\tilde{\boldsymbol{T}}_N + H_N \boldsymbol{z})\, \tilde{\sigma}(G_N^k) \\ &= \boldsymbol{P}_N \left(\tilde{\boldsymbol{T}}_N + \frac{1}{2} H_N \boldsymbol{1}\right) \tilde{\sigma}(G_N^k) \\ &= G_N^k\, \tilde{\sigma}(G_N^k) + \frac{1}{2} H_N \| \tilde{\sigma}(G_N^k) \|_1 \\ &\le \Psi^k(G_N^k) + \frac{1}{2} H_N\, m^k B\,. \end{aligned}$$

That is, $C \le \Psi^k(G_N^k) + \frac{1}{2} H_N m^k B$ and combining the bounds, the theorem is proved. □

It is clear from the proof that the result is basically a $\Gamma^k$ phenomenon, hence the iterated integral condition appears. Under conditions sufficient to allow a $\mathcal{B}^{mk} \times \mathcal{G}^{k-1}$ measurable choice of $s^*(\cdot, \cdot)$ the special interpretation of expectation becomes unnecessary.

COROLLARY. *With the choice* $H_\alpha = \alpha^{1/2}$ *each* $\alpha$, *there exists a constant* $\mathcal{K}$ *depending only on* $B$, $m$, $k$ *and* $\tau$ *such that*

$$\sum_{\alpha=1}^{N} E\, \theta_\alpha\, \hat{s}_\alpha \le \Psi^k(G_N^k) + \mathcal{K} N^{1/2}\,. \tag{15}$$

Notice that the corollary shows that on an average, rather than total risk scale, with $H_\alpha = \alpha^{1/2}$, the risk incurred by the strategy $\hat{\boldsymbol{s}}$ is bounded by $\Psi^k((1/N)\, G_N^k) + \mathcal{K} N^{-1/2}$.

5. ***k*-extended game theoretic results.** The framework introduce in section 2 is quite flexible. Both decision theoretic and game theoretic problems are covered. In this section we consider a game theoretic setting, that is a situation where the information about past states is assumed to be perfect.

5.1. **Specializations to a game theoretic setting.** We take $\mathcal{X} = \Theta$, let $\mathcal{G}$ be the set of all subsets of $\Theta$ and suppose each $\boldsymbol{P}_\theta$ to be degenerate at $\theta$. $\boldsymbol{S}^*$ becomes the set of all functions from $\Theta^{k-1}$ into $\boldsymbol{S}$ and we may take $\tilde{t}(\boldsymbol{\theta}) = \boldsymbol{\theta}$ for $\boldsymbol{\theta} \in \Theta^k$. The results of § 4.3 are in force in this situation so that specializations of the strategies $\hat{\boldsymbol{s}}$ provide asymptotic solutions of the $k$-extended sequence compound problem.

In addition, a simple decomposition of the $k$-extended envelope is available in this setting that allows us to modify solutions of the unextended problem to produce solutions of the $k$-extended problem.

5.2. **A decomposition of $\Psi^k(G_N^k)$ in the case $k>1$.** For each $\boldsymbol{\theta} \in \Theta^{k-1}$ define $G_\alpha^k|\boldsymbol{\theta}$ to be the $m$ vector with $\theta$th entry $(\boldsymbol{\theta}, \theta)G_\alpha^k$ and denote $(\theta_{\alpha-k+1},\cdots, \theta_{\alpha-1})$ as $\boldsymbol{\theta}_{\alpha-1}^*$.

LEMMA 3. *In the game theoretic context*

$$\Psi^k(G_N^k) = \sum_{\boldsymbol{\theta}\in\Theta^{k-1}} \Psi(G_N^k|\boldsymbol{\theta}) .$$

**Proof.** For $s^* \in \boldsymbol{S}^*$

$$
\begin{aligned}
E\sum_1^N \theta_\alpha\, s^*(\boldsymbol{X}_{\alpha-1}^*) &= \sum_1^N \theta_\alpha\, s^*(\boldsymbol{\theta}_{\alpha-1}^*) \\
&= \sum_{\boldsymbol{\theta}\in\Theta^{k-1}} \sum_{\alpha\ni\boldsymbol{\theta}_{\alpha-1}^*=\boldsymbol{\theta}} \theta_\alpha\, s^*(\boldsymbol{\theta}) \\
&= \sum_{\boldsymbol{\theta}\in\Theta^{k-1}} \Big( \sum_{\alpha\ni\boldsymbol{\theta}_{\alpha-1}^*=\boldsymbol{\theta}} \theta_\alpha \Big) s^*(\boldsymbol{\theta}) \qquad (16)\\
&= \sum_{\boldsymbol{\theta}\in\Theta^{k-1}} (G_N^k|\boldsymbol{\theta})\, s^*(\boldsymbol{\theta}) .
\end{aligned}
$$

But (16) is minimal if $s^*(\boldsymbol{\theta}) = \sigma(G_N^k|\boldsymbol{\theta})$ for each $\boldsymbol{\theta} \in \Theta^{k-1}$. Hence

$$\Psi^k(G_N^k) = \sum_{\boldsymbol{\theta}\in\Theta^{k-1}} (G_N^k|\boldsymbol{\theta})\, \sigma(G_N^k|\boldsymbol{\theta}) = \sum_{\boldsymbol{\theta}\in\Theta^{k-1}} \Psi(G_N^k|\boldsymbol{\theta}). \quad \square$$

The lemma suggests that given a strategy with risk approximating the unextended ($k=1$) envelope at some rate, it may be possible to achieve the $k$-extended envelope at the same rate by at stage $\alpha$ choosing a risk according to the risks used and states holding in those component problems with indices $\beta<\alpha$ for which $\boldsymbol{\theta}_{\beta-1}^* = \boldsymbol{\theta}_{\alpha-1}^*$. Two examples of the use of this kind of technique follow.

5.3. **A modification of Hannan's game theoretic strategy.** Hannan (1957) shows that for the case $k=1$, the risk incurred in a game theoretic setting by the specialization of $\hat{\boldsymbol{s}}$ defined in (10) with $H_\alpha = (6\alpha/m)^{1/2}$ achieves

$$(17)\quad -N^{1/2}\left(\frac{3}{2}m\right)^{1/2}B \le E\sum_{1}^{N}\theta_\alpha \hat{s}_\alpha - \Psi(G_N) \le N^{1/2}(6m)^{1/2}B\,.$$

If we modify Hannan's strategy by replacing $G_{\alpha-1}$ with $G^k_{\alpha-1}|\boldsymbol{\theta}^*_{\alpha-1}$ and $H_{\alpha-1}$ with $H'_{\alpha-1}=(6\|G^k_{\alpha-1}|\boldsymbol{\theta}^*_{\alpha-1}\|_1/m)^{1/2}$ we have $s_\alpha=\sigma(G^k_{\alpha-1}|\boldsymbol{\theta}^*_{\alpha-1}+H'_{\alpha-1}Z)$. Then

$$\sum_{1}^{N}\theta_\alpha s_\alpha - \Psi^k(G^k_N) = \sum_{\boldsymbol{\theta}\in\Theta^{k-1}}\left\{\left(\sum_{\alpha\ni\boldsymbol{\theta}^*_{\alpha-1}=\boldsymbol{\theta}}\theta_\alpha s_\alpha\right) - \Psi(G^k_N|\boldsymbol{\theta})\right\}.$$

Denote the term in brackets by $A(\boldsymbol{\theta})$ and the indices $\alpha$ for which $\boldsymbol{\theta}^*_{\alpha-1}=\boldsymbol{\theta}$ by $\alpha_1<\alpha_2<\cdots<\alpha_{N(\boldsymbol{\theta})}$ where $N(\boldsymbol{\theta})=\|G^k_N|\boldsymbol{\theta}\|_1$. Then

$$A(\boldsymbol{\theta}) = \sum_{j=1}^{N(\boldsymbol{\theta})}E\,\theta_{\alpha_j}\,\sigma\!\left(G^k_{\alpha_j-1}|\boldsymbol{\theta} + \left(\frac{6(j-1)}{m}\right)^{1/2}Z\right) - \Psi(G^k_N|\boldsymbol{\theta})\,.$$

The sequence $\{G^k_{\alpha_j}|\boldsymbol{\theta}\}$ is a sequence of $m$-vectors of successive cumulative frequencies of states with $G^k_{\alpha_j}|\boldsymbol{\theta}=\theta_{\alpha_j}+G^k_{\alpha_{(j-1)}}|\boldsymbol{\theta}$ and Hannan's result is applicable. So $-N^{1/2}(\boldsymbol{\theta})(\frac{3}{2}m)^{1/2}B\le A(\boldsymbol{\theta}) \le N^{1/2}(\boldsymbol{\theta})(6m)^{1/2}B$. Noting that $\sum_{\boldsymbol{\theta}}N(\boldsymbol{\theta})=N$, an application of the Schwarz inequality yields

$$(18)\quad -N^{1/2}\left(\frac{3}{2}m^k\right)^{1/2}B \le E\sum_{\alpha=1}^{n}\theta_\alpha s_\alpha - \Psi^k(G^k_N) \le N^{1/2}(6m^k)^{1/2}B\,.$$

Comparison of (17) and (18) shows the rate of convergence for the risk of the modified procedure to the extended envelope is the same as that for the risk of the original strategy to the unextended envelope. Indeed the bounds are $m^{(k-1)/2}$ times the original bounds.

5.4. **A modification of Blackwell's game theoretic strategy.** Hannan (1957) states that an unextended game theoretic strategy proposed by Blackwell (1956) has risk $\Psi(G_N)+O(N^{1/2})$. We introduce this strategy and show that a natural modification achieves risk $\Psi^k(G^k_N)+O(N^{1/2})$.

For each $\alpha\ge 1$ we let $\phi_\alpha$ denote the $(m+1)$-vector $(\theta_\alpha,\theta_\alpha s_\alpha)$ and $\bar{\phi}_\alpha=(1/\alpha)\sum_{\beta=1}^{\alpha}\phi_\beta=(\bar{\theta}_\alpha,\bar{r}_\alpha)$. With $\Delta$ the convex subset of $R^{m+1}$ defined by $\Delta=\{(w,u)\in R^{m+1}|w\in R^m$ is a probability vector and $u\le\Psi(w)\}$ we let $\rho_\alpha$ be the Euclidean distance of $\bar{\phi}_\alpha$ from $\Delta$. Arbitrarily set $\rho_0=0$. For each $m$ dimensional probability vector $w$ let $\gamma(w)=(w,\Psi(w))$ and let $w_\alpha$ be the probability vector minimizing

$$\|\bar{\phi}_\alpha - \gamma(w)\|^2 = \|\bar{\theta}_\alpha - w\|^2 + (\bar{r}_\alpha - \Psi(w))^2 .$$

Blackwell's strategy $\check{s}$ is defined by

$$(19)\quad \check{s}_\alpha = \begin{cases} \text{any } s \in S, \text{ if } \rho_{\alpha-1} = 0 \\ \text{any } s \in S \text{ which minimizes} \\ \bigvee_{\theta \in \Theta} (\theta(\bar{\theta}_{\alpha-1} - w_{\alpha-1}) + \theta s(\bar{r}_{\alpha-1} - \Psi(w_{\alpha-1})), \text{ if } \rho_{\alpha-1} \neq 0 . \end{cases}$$

PROPOSITION. (Hannan) *If $S$ is convex, then*

$$\sum_{\alpha=1}^{N} \theta_\alpha \check{s}_\alpha - \Psi(G_N) \leq N^{1/2}((2 + B^2)(1 + mB^2))^{1/2} .$$

The convexity assumption on $S$ appears in order to allow an application of the Minimax Theorem in the proof.

Abbreviate $\|G_\alpha^k | \boldsymbol{\theta}\|_1$ as $n_\alpha(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \Theta^{k-1}$. With

$$\begin{aligned} \bar{\phi}_\alpha^k &= \frac{1}{n_\alpha(\boldsymbol{\theta}_\alpha^*)} \sum_{\beta \leq \alpha \ni \boldsymbol{\theta}_{\beta-1}^* = \boldsymbol{\theta}_\alpha^*} \phi_\beta \\ &= (\bar{\theta}_\alpha^k, \bar{r}_\alpha^k), \end{aligned}$$

(where we interpret 0/0 as 0), let $\rho_\alpha^k$ be the Euclidean distance of $\bar{\phi}_\alpha^k$ from $\Delta$ and $w_\alpha^k$ be the $m$ dimensional probability vector minimizing $\|\bar{\phi}_\alpha^k - \gamma(w)\|^2$. We consider a procedure $\boldsymbol{s}$ defined by

$$(20)\quad s_\alpha = \begin{cases} \text{any } s \in S, \text{ if } \rho_{\alpha-1}^k = 0 \\ \text{any } s \in S \text{ which minimizes} \\ \bigvee_{\theta \in \Theta} (\theta(\bar{\theta}_{\alpha-1}^k - w_{\alpha-1}^k) + \theta s(\bar{r}_{\alpha-1}^k - \Psi(w_{\alpha-1}^k)), \text{ if } \rho_{\alpha-1}^k \neq 0 . \end{cases}$$

As in §4.4,

$$(21)\quad \sum_{1}^{N} \theta_\alpha s_\alpha - \Psi^k(G_N^k) = \sum_{\boldsymbol{\theta} \in \Theta^{k-1}} \left\{ \left( \sum_{\alpha \ni \boldsymbol{\theta}_{\alpha-1}^* = \boldsymbol{\theta}} \theta_\alpha s_\alpha \right) - \Psi(G_N^k | \boldsymbol{\theta}) \right\},$$

we again denote the term in brackets by $A(\boldsymbol{\theta})$ and the indices $\alpha$ for which $\boldsymbol{\theta}_{\alpha-1}^* = \boldsymbol{\theta}$ by $\alpha_1 < \alpha_2 < \cdots < \alpha_{N(\boldsymbol{\theta})}$ with $N(\boldsymbol{\theta}) = n_N(\boldsymbol{\theta})$, and have

$$A(\boldsymbol{\theta}) = \sum_{j=1}^{N(\boldsymbol{\theta})} \theta_{\alpha_j} s_{\alpha_j} - \Psi\left( \sum_{j=1}^{N(\boldsymbol{\theta})} \theta_{\alpha_j} \right).$$

With this notation $\bar{\phi}_{\alpha_{j-1}}^k = (1/(j-1)) \sum_{l=1}^{j-1} \phi_{\alpha_l}$, $\rho_{\alpha_{j-1}}^k$ is the Euclidean distance from $\Delta$ to $(1/(j-1)) \sum_{l=1}^{j-1} \phi_{\alpha_l}$, and $w_{\alpha_{j-1}}^k$ is the $m$ dimensional probability vector which minimizes $\|(1/(j-1)) \sum_{l=1}^{j-1} \phi_{\alpha_l} - \gamma(w)\|^2$.

So that comparing (19) and (20) and applying the proposition we see that if $S$ is convex $A(\boldsymbol{\theta}) \leq N^{1/2}(\boldsymbol{\theta})((2+B^2)(1+mB^2))^{1/2}$. Applying the Schwarz inequality, the $lhs(21) \leq N^{1/2}\, m^{(k-1)/2}((2+B^2)(1+mB^2))^{1/2}$, and the modification of Blackwell's strategy provides another solution of the $k$-extended game theoretic problem.

5.5. **A comment on the effect of play against a random perturbation of $G^k_{\alpha-1}$ in the $k$-extended setting.** Recall the $k$-extended procedure suggested in §3.2 had the form

$$\hat{s} = s^*(\tilde{T}_{\alpha-k} + H_{\alpha-k}Z,\ X^*_{\alpha-1})\,. \tag{22}$$

The proof in §4.3 depends heavily on the fact that $\tilde{T}_{\alpha-k} + H_{\alpha-k}Z$ is independent of $X^*_{\alpha-1}$. However, because of the degeneracy of the $P_\theta$ in the game theoretic situation, it is possible to replace $\tilde{T}_{\alpha-k} + H_{\alpha-k}Z$ by $\tilde{T}_{\alpha-1} + H_{\alpha-1}Z$, invoke unextended results for a sequence compound problem with $\Gamma^k$ construct as the component problem, and improve on the bound of §4.3.

That is, redefine $\hat{s}$ by

$$\hat{s}_\alpha = s^*(T_{\alpha-1} + H_{\alpha-1}Z,\ X^*_{\alpha-1})\,.$$

Then almost everywhere $P_N$, $\hat{s}_\alpha = s^*(G^k_{\alpha-1} + H_{\alpha-1}Z,\ \boldsymbol{\theta}^*_{\alpha-1})$, so that

$$\sum_{\alpha=1}^{N} E\theta_\alpha\, s_\alpha = \sum_{\alpha=1}^{N} \mu(\theta_{\alpha-k+1},\cdots,\theta_\alpha)\,\tilde{\sigma}(G^k_{\alpha-1} + H_{\alpha-1}z)\,. \tag{23}$$

The unextended version of Theorem 1 applied to a compound problem with $\Gamma^k$ *component* implies that (23) is bounded above by

$$\Psi^k(G^k_N) + \frac{1}{2}BH_N\, m^k + B\left(1 + 2\sum_{\alpha=1}^{N}\frac{1}{H_\alpha}\right).$$

In fact, with the choice $H_\alpha = (6\alpha)^{1/2}\, m^{-k/2}$ application of Hannan's result quoted in §5.3 gives the bounds of (18) for (23).

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STATISTICAL LABORATORY AND DEPARTMENT OF STATISTICS, IOWA STATE UNIVERSITY, 303 SNEDECOR HALL, AMES, IOWA 50011, U. S. A.