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ON THE SECOND FUNDAMENTAL FORM OF ALGEBRAIC HYPERSURFACES*

BY

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Abstract. Let M be a compact Kaehlerian hypersurface of CP^{n+1} with Fubini-Study metric of constant holomorphic sectional curvature 1, and A denotes the second fundamental form with respect to some unit normal vector. In this paper, we shall prove the following main results:

THEOREM 1. *If M contains a nonsingular complex projective line and A vanishes on its direction everywhere, then M must be a complex projective hyperplane.*

THEOREM 2. *If the (complex) rank of A is full, i.e., equal to $n=\dim_C M$ everywhere, then M must be biholomorphic to the complex hypersphere $Q^n=\{[z_0 \ldots, z_{n+1}]\in CP^{n+1}|(z_0)^2+\ldots+(z_{n+1})^2=0\}$ and the defining homogeneous polynomial is of degree 2.*

Note: In Theorem 1, the condition that A vanishes on the tangent space of some CP^1 is independent of associated unit normal vectors. In Theorem 2, we require that M is nonsingular itself.

0. **Introduction.** Let M be a Kaehlerian hypersurface of $CP^{n+1}(1)$ (which denotes the complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 1) and A denotes the second fundamental form with respect to some unit normal vector. So we can consider the rank of A. (The "rank" always means the "complex rank" throughout this paper.) In [7], it was shown that the rank of A at a point of M is determined by the curvature tensor of M at this point. Thus the rank of A is intrinsic, (this implies that rank A is independent of associated unit normal vectors) at each point and is also called the rank of M and denoted by rank M. In the same paper [7], K. Nomizu

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and B. Smyth also proved the following result: If M is compact, $n \geq 3$, then rank M cannot be identically equal to 1. And the case $n=2$ remained unsettled in their paper.

In 1973, K. Abe [2] obtained a very strong improvement: Let M be a Kaehlerian submanifold of complex n-dimension of $CP^{n+k}(1)$. If M is complete, then the complex index of relative nullity is either 0 or n. Specially, in hypersurface case, rank M cannot be identically equal to any constant except either 0 or n. This gives a motivation of the following result for compact case.

THEOREM 1. *Let M be a compact Kaehlerian hypersurface of $CP^{n+1}(1)$. If M contains a nonsingular complex projective line and the second fundamental form A vanishes on its direction everywhere, then M must be a complex projective hyperplane.*

Our proof of the above result is a direct computation just by using Chow's theorem and the exact formula of the second fundamental form A. Now, Abe's result for compact case becomes an easy application of our Theorem 1 for hypersurface case.

As above, constant rank of a compact Kaehlerian hypersurface M of $CP^{n+1}(1)$ is impossible unless it is either 0 or n and the rank $M \equiv 0$ case ("$\equiv$" means "identically equal") implies that the hypersurface M must be complete totally geodesic (i.e. $A \equiv 0$) and imbedded as a hyperplane. Therefore, we consider the rank $M \equiv n$ case and obtain the second main result:

THEOREM 2. *If the rank of a compact Kaehlerian hypersurface M of $CP^{n+1}(1)$ (= the rank of it's second fundamental form) is full, i.e., equal to $n = \dim_C M$ everywhere, then M must be biholomorphic to the complex hypersphere $Q^n = \{[z_0, \cdots, z_{n+1}] \in CP^{n+1} \mid (z_0)^2 + \cdots + (z_{n+1})^2 = 0\}$ and the defining homogeneous polynomial is of degree 2.*

1. **Preliminaries.** At first we give an outline of the "affine" representation to the complex projective space $CP^{n+1}(1)$ (cf. [5] or [7]) CP^{n+1} is obtained from $C^{n+2} - \{0\}$ by the familiar quotient process:

$$\begin{array}{ccccc} z \in C^{n+2} - \{0\} & \supset & S^{2n+3} & & \text{(unit sphere in } C^{n+2}) \\ \searrow & \pi\searrow & \downarrow \pi & & \\ [z] & & CP^{n+1} & & \end{array}$$

When we consider S^{2n+3} as a principal fibre bundle over CP^{n+1} with structure group S^1 acting as follows: $S^{2n+3} \times S^1 \ni (z, e^{it}) \rightarrow ze^{it} \in S^{2n+3}$, we use the Euclidean metric of $C^{n+2} - \{0\}$ restricted to S^{2n+3} as the Riemannian metric on S^{2n+3}. Let $T'_z = \{V \in C^{n+1} \mid V \cdot \bar{z} = 0\}$, T' defines a connection on $S^{2n+3} \xrightarrow{\pi} CP^{n+1}$ and so induces an isomorphism $\pi_{*_z} : T'_z \rightarrow TCP^{n+1}_{[z]}$ for any $z \in S^{2n+3}$ which is complex linear. Thus the complex structure on T'_z induces the complex structure on $TCP^{n+1}_{[z]}$.

The Fubini-Study metric $\tilde{g}$ on CP^{n+1} may also be defined as follows: For $\tilde{X}, \tilde{Y} \in TCP^{n+1}_{[z]}$, define $\tilde{g}_{[z]}(\tilde{X}, \tilde{Y}) = 4g_z(X, Y) = 4$ Real $(X \cdot \bar{Y})$ where $X, Y \in T'_z$ are the horizontal lifts of $\tilde{X}$ and $\tilde{Y}$ at $z \in \pi^{-1}([z]) \cap S^{2n+3}$.

The Hermitian connection $\tilde{\nabla}$ on CP^{n+1} corresponding to $\tilde{g}$ can also be described by using $T' : (\tilde{\nabla}_{\tilde{X}} \tilde{Y})_{[z]} = \pi_{*_z} \nabla'_X Y$ where X and Y are horizontal lifts of local vector fields $\tilde{X}$ and $\tilde{Y}$ respectively at $z \in \pi^{-1}([z]) \cap S^{2n+3}$ and where ∇' is the Riemannian connection on S^{2n+3}.

Since $\hat{\nabla}_X Y - \nabla'_X Y$ is tangent to the fibre of π ($\hat{\nabla}$ denotes the canonical connection on C^{n+2}), $\pi_{*_z}(\hat{\nabla}_X Y - \nabla'_X Y) = 0$. Hence $\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \pi_{*_z}(D_X Y_0, D_X Y_1, \cdots, D_X Y_{n+1})$ where D_X denotes the differentiation in C^{n+2} by the tangent vector X.

Let M be an algebraic hypersurface of $CP^{n+1}(1)$ i.e. $M = V(f) = \{[z] \in CP^{n+1} \mid f(z) = 0\}$ where f denotes a homogeneous polynomial. The complex structure, Kaehlerian metric, and Hermitian connection of M are induced from $CP^{n+1}(1)$ by restriction. Using the above scheme, we can describe them in terms of T'. Define $T_z = T\hat{M}_z \cap T'_z$, $z \in S^{2n+3} \cap \hat{M}$ where $\hat{M} = \pi^{-1}(M)$ and π induces a complex linear isomorphism $\pi_{*_z} : T_z \rightarrow TM_{[z]}$. The metric and complex structure on $TM_{[z]}$ are equal to those of T_z transferred to $TM_{[z]}$ via π_{*_z}. The connection ∇ on M satisfies $\nabla_{\tilde{X}} \tilde{Y} = \pi_{*_z} \pi_{T_z} \nabla'_X Y$ where $\pi_{T_z} : TC^{n+2}_z \rightarrow T_z$ is the orthogonal projection.

Altogether, let $[z] \in M$ with $z \in \pi^{-1}([z]) \cap S^{2n+3}$, $TM_{[z]}$ can be identified with $T_z = \{V \in C^{n+2} \mid V \cdot \bar{z} = 0 = V \cdot (\partial f/\partial z)(z)\}$, $TCP^{n+1}_{[z]}$ is identified with $T'_z = \{V \in C^{n+2} \mid V \cdot \bar{z} = 0\}$, and the unit normal field of M in CP^{n+1} is identified with $(1/2\,\|\partial f/\partial z\|)(\overline{\partial f/\partial z})$ where $\partial f/\partial z = (\partial f/\partial z_0, \cdots, \partial f/\partial z_{n+1})$. (Note that $\tilde{g}_{[z]}(\tilde{X}, \tilde{Y}) = 4g(X, Y)$).

Now, we derive the formula of the second fundamental form for projective hypersurfaces which, at first, appears in Albert Vitter's paper [9].

Since $\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \pi_{*_z} \hat{\nabla}_X Y = \pi_{*_z}(D_X Y_0, \cdots, D_X Y_{n+1})$ where $D_X \sigma$ is the derivative of the function σ by the tangent vector X and is equal to $d\sigma(X) = \sum_{j=0}^{n+1} (\partial\sigma/\partial z_j)\, X_j + \sum_{j=0}^{n+1} (\partial\sigma/\partial \bar{z}_j)\, \bar{X}_j$. Applying it to the normal vector field denoted by § identified with $(1/2\,\|\partial f/\partial z\|)(\overline{\partial f/\partial z})$ as above.

$$\hat{\nabla}_X \S = \frac{1}{2\,\|\partial f/\partial z\|} \hat{\nabla}_X \overline{\frac{\partial f}{\partial z}} + D_X\left(\frac{1}{2\,\|\partial f/\partial z\|}\right) \overline{\frac{\partial f}{\partial z}}\,.$$

$$\begin{aligned}\hat{\nabla}_X \overline{\frac{\partial f}{\partial z}} &= \left(\sum_{j=0}^{n+1} \overline{\frac{\partial^2 f}{\partial z_0\, \partial z_j}}\, \bar{X}_j, \cdots, \sum_{j=0}^{n+1} \overline{\frac{\partial^2 f}{\partial z_{n+1}\, \partial z_j}}\, \bar{X}_j\right) \\ &= \bar{X}\left(\overline{\frac{\partial^2 f}{\partial z_i\, \partial z_j}}\right)\end{aligned}$$

which means that the row vector $\bar{X}$ multiplies the square matrix $(\overline{\partial^2 f/\partial z_i\, \partial z_j})$. So

$$\begin{aligned}XA_{[z]} &= -\pi_{TM_{[z]}} \hat{\nabla}_X \S \\ &= -\pi_{TM_{[z]}}\left[\frac{1}{2\,\|\partial f/\partial z\|}\, \bar{X}\left(\overline{\frac{\partial^2 f}{\partial z_i\, \partial z_j}}\right)\right] \\ &= -\frac{1}{2\,\|\partial f/\partial z\|}\, \bar{X}\left(\overline{\frac{\partial^2 f}{\partial z_i\, \partial z_j}}\right) + n\,\overline{\frac{\partial f}{\partial z}}\,.\end{aligned}$$

Where $n = (1/2\,\|\partial f/\partial z\|^3)\, \bar{X}(\overline{\partial^2 f/\partial z_i\, \partial z_j})^t\, (\partial f/\partial z)$. (Notice that the component of $\bar{X}(\overline{\partial^2 f/\partial z_i\, \partial z_j})$ on z is zero since $(\partial^2 f/\partial z_i\, \partial z_j)^t z = (m-1)(\partial f/\partial z)$, $m = \deg f$). The above formula can have the following better form:

$$2\left\|\frac{\partial f}{\partial z}\right\| XA_{[z]} = \bar{X}\bar{B}({}^t N \bar{N} - I)\,.$$

Where B and N denote $(\partial^2 f/\partial z_i \partial z_j)$ and $(\partial f/\partial z)/\|\partial f/\partial z\|$, respectively, and I denotes the identity matrix of $(n+2)\times(n+2)$.

Next, we need a lemma. Let M^n be a Kaehlerian submanifold in a complex space form $\tilde{M}(c)$ with constant holomorphic sectional curvature c, the relative nullity space at x is defined to be the subspace $\{X \in TM_x : \alpha(X, Y) = 0$ for all $Y \in TM_x\}$ where α is the secone fundamental form as usual. (Note: we give α and A the same name.) Denote it by $RN(x)$. Suppose $\{\S_i, J\S_i\}_{1\le i\le k}$, $k =$ complex codimension of M, form an orthonormal basis with respect to $TM_x^{\perp}$, and $A_{\S_i}$ denotes the corresponding second fundamental form, then the following three subspaces are the same:

(a) $RN(x) = \{X \in TM_x : \alpha(X, Y) = 0,$ for all $Y \in TM_x\}$

(b) $\{X \in TM_x : A_\S(X) = 0,$ for all $\S \in TM_x^{\perp}\}$

(c) $\{X \in TM_x : A_{\S_i}(X) = 0,$ for all $i.$ $1 \le i \le k\}$.

Now, define $\tilde{\upsilon}(x) =$ complex dimension of $RN(x)$ and call it the relative nullity at x. The minimum among $\tilde{\upsilon}(x)$ over M is called the index of relative nullity of the submanifold M. Let $G = \{x \in M : \tilde{\upsilon}(x) = \upsilon\}$ where υ denotes the index, then by upper semicontinuity of $\tilde{\upsilon}$, G is open. Define the relative nullity distribution RN by assigning $RN(x)$ to x in G, and call it the relative nullity distribution of M. Now, we give a description of well-known properties of the relative nullity distribution in the following lemma. The proof of all details is referred to [2] or [4].

LEMMA 0. *Under the notation mentioned above, the relative nullity distribution is involutive on* G, *and in addition, if* M *is complete, then it's leaves are imbedded as* CP^υ, C^υ *or* D^υ *(complex hyperbolic space of complex dimension* υ*) with respect to* $c > 0$, $c = 0$ *or* $c < 0$, *respectively.*

2. **Proof of Theorem 1.** By a theorem of Chow, M is defined by a homogeneous polynomial $f = 0$, and we can choose coordinates $(z_0, z_1, \cdots, z_{n+1})$ such that CP^1 is given by $(0, 0, \cdots, 0, z_n, z_{n+1})$. We write f in the form

$$f(z_0, z_1, \cdots, z_{n+1}) = F(z_n, z_{n+1}) + z_0 f_0(z_n, z_{n+1}) + z_1 f_1(z_n, z_{n+1}) + \cdots + z_{n-1} f_{n-1}(z_n, z_{n+1}) + \sum_{l_0+l_1+\cdots+l_{n-1}\geq 2} z_0^{l_0} z_1^{l_1} \cdots z_{n-1}^{l_{n-1}} \cdot f_{l_0 l_1 \cdot l_{n-1}}(z_n, z_{n+1}) .$$

That $CP^1 \subset M$ implies $F \equiv 0$. At $[0, 0, \cdots, 0, z_n, z_{n+1}] \in CP^1$, $\partial f/\partial z_0 = f_0$, $\partial f/\partial z_1 = f_1, \cdots$, $\partial f/\partial z_{n-1} = f_{n-1}$, $\partial f/\partial z_n = 0$, $\partial f/\partial z_{n+1} = 0$ (1). Now, substitute (1) into the formula of the second fundamental form A as in section 1, we get

$$(\bar{X}_0, \bar{X}_1, \cdots, \bar{X}_{n+1}) \begin{bmatrix} 0_{n\times n} & {}^tQ \\ Q & 0_{2\times 2} \end{bmatrix} \left\{ \frac{1}{r} \begin{pmatrix} {}^tK \\ 0 \\ 0 \end{pmatrix} (\bar{K}, 0, 0) - I \right\} = \left((\bar{X}_n, \bar{X}_{n+1}) \bar{Q} \left[\frac{1}{r} {}^tK\bar{K} - I \right], - (\bar{X}_0, \bar{X}_1, \cdots, \bar{X}_{n-1}) {}^t\bar{Q} \right).$$

Where Q and K denote $\begin{bmatrix} f_{0,n} & f_{1,n} \cdots f_{n-1,n} \\ f_{0,n+1} & f_{1,n+1} \cdots f_{n-1,n+1} \end{bmatrix}$ and $(f_0, \cdots, f_{n-1})$ resp. and $r = |f_0|^2 + \cdots + |f_{n-1}|^2$, I denotes the suitable identity matrix. Take $(\bar{X}_0, \bar{X}_1, \cdots, \bar{X}_{n-1}, \bar{X}_n, \bar{X}_{n+1}) = (0, 0, \cdots, 0, z_{n+1}, -z_n)$ $\in \overline{TCP^1}_{[0,\ldots,0,z_n,z_{n+1}]}$, so get $(z_{n+1}, -z_n) \bar{Q}[(1/r)\, {}^tK\bar{K} - I] = 0(2)$ by the assumption.

Denote $N = K/\sqrt{r}$, then $(1/r)\, {}^tK\bar{K} - I = {}^tN\bar{N} - I$, $\bar{N}({}^tN\bar{N} - I) = (\bar{N}\, {}^tN)\bar{N} - \bar{N} = 0$. Since $N \neq 0$, some $f_i \neq 0$, we can assume $f_0 \neq 0$. Let ${}^t\sigma = {}^t(f_1, \cdots, f_{n-1})/\sqrt{r}$, $\|\sigma\| < 1$ and $\langle x, y\rangle \overset{\text{def.}}{=} \bar{x}\, {}^ty$, $\bar{x}({}^t\sigma\bar{\sigma} - I)^t x = \langle x, \sigma\rangle\langle\sigma, x\rangle - \langle x, x\rangle = |\langle x, \sigma\rangle|^2 - \|x\|^2$ < 0 so ${}^t\sigma\bar{\sigma} - I$ is negative definite. The above computation shows the dimension of the kernel of ${}^tN\bar{N} - I$ is equal to 1 and $\bar{N}$ spans its kernel. Therefore, by (2), $(z_{n+1}, -z_n)\bar{Q} = \lambda\bar{N}$, where λ is a complex-valued function. That's to say,

$$(z_{n+1}, -z_n) \begin{bmatrix} \bar{f}_{0,n} & \bar{f}_{1,n} \cdots \bar{f}_{n-1,n} \\ \bar{f}_{0,n+1} & \bar{f}_{1,n+1} \cdots \bar{f}_{n-1,n+1} \end{bmatrix} = \lambda(\bar{f}_0, \bar{f}_1, \cdots, \bar{f}_{n-1}).$$

So $(z_{n+1}, -z_n) \begin{bmatrix} \bar{f}_{0,n} & \bar{f}_{1,n} \\ \bar{f}_{0,n+1} & \bar{f}_{1,n+1} \end{bmatrix} = \lambda(\bar{f}_0, \bar{f}_1)$ implies

$$\begin{vmatrix} z_{n+1}\bar{f}_{0,n} - z_n\bar{f}_{0,n+1} & z_{n+1}\bar{f}_{1,n} - z_n\bar{f}_{1,n+1} \\ \bar{f}_0 & \bar{f}_1 \end{vmatrix} = 0 .$$

Let $z_n = z$, $z_{n+1} = 1$, $g(z) = f_0(z, 1)$, $h(z) = f_1(z, 1)$, by using Euler's identities: $zg' + f_{0,n+1}(z, 1) = lg$, $zh' + f_{1,n+1}(z, 1) = lh$, $l = \deg g = \deg h$, we can reduce the above equation to $g'h - gh'$ and $(g/h)' = (g'h - gh')/h^2 = 0$ implies $g = ch$ for some constant c which tells us that f_0 and f_1 are proportional. The same argument shows that $f_0, f_1, \cdots, f_{n-1}$ are proportional which guarantees they have a common zero, a contradiction unless all are constants, which implies M is a projective hyperplane. Q. E. D.

REMARK 1. In 1968, K. Nomizu proved the following result: Let M be a compact complex hypersurface of CP^{n+1}. If M contains a certain projective subspace CP^k, $k \geq (n+1)/2$, then M is a projective hyperplane [6].

REMARK 2. The condition that A vanishes on the direction of CP^1 can't be removed. We give an example to explain it here.

EXAMPLE. Let $M = \{[z_0, z_1, z_2, z_3] \in CP^3 \mid z_0 z_2 + z_1 z_3 = 0\} \subset CP^3$. Take $CP^1 = \{[0, 0, z_2, z_3] \in CP^3 \mid z_2, z_3 \in C\} \subset M$, and $X = (0, 0, \bar{z}_3, -\bar{z}_2) \in TCP^1_{[0,0,z_2,z_3]}$, $\|X\| = |z_2|^2 + |z_3|^2 = 1$. A direct computation gives

$$XA = \frac{1}{2}(-z_3, z_2, 0, 0) .$$

So A cannot vanish on the tangent direction of CP^1, and M isn't CP^2, of course. In fact, M is holomorphically isometric to Q^2 if we take the unitary transform $[z_0, z_1, z_2, z_3] \leftrightarrow [\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3] \in Q^2$ by $z_0 = \tilde{z}_0 - i\tilde{z}_2$, $z_2 = \tilde{z}_0 + i\tilde{z}_2$, $z_1 \tilde{z}_1 - i\tilde{z}_3$, $\tilde{z}_3 = \tilde{z}_1 + i\tilde{z}_3$. (also cf. Prop. in section 3).

COROLLARY 1. *If M^n is a compact hypersurface of $CP^{n+1}(1)$, then $M^n = CP^n$ or there exists a point such that* rank $A = n$ *at that point.*

Proof. By Lemma 0 and Theorem 1, note that rank A is equal to dimension minus the index of relative nullity. Q. E. D.

COROLLARY 2. *With the same assumption as in Corollary* 1 *that the rank of the second fundamental form A is constant everywhere is impossible unless it is full, i.e.* rank $A \equiv \dim M$ *or zero.*

Now, we give a sufficient condition on the sectional curvature for rank A to be full.

PROPOSITION. *Let M^n be a compact Kaehlerian hypersurface of $CP^{n+1}(1)$, $n \geq 2$, and suppose the sectional curvature $K < \frac{1}{4}$ at some point x. Then* rank $A = n$ *at x.*

Proof. There is an orthonormal basis in $T_x M$ of the form $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ (J denotes the complex structure) such that $Ae_i = \lambda_i e_i$ and $AJe_i = -\lambda_i Je_i$, $1 < i < n$. Where we may assume $\lambda_1 > \lambda_2 \geq \cdots \lambda_n \geq 0$. (See Lemma 1 of Smyth's paper [8]). By Corollary 1 of the same paper, the sectional curvature $K(e_j, Je_k)$, $j \neq k$, is equal to $\frac{1}{4} - \lambda_j \lambda_k$. So by the assumption, $\lambda_j \neq 0$, i.e. rank $A = n$. Q. E. D.

REMARK. In [6], K. Nomizu also proved that $n \geq 3$, if M is compact and if the sectional curvature of M is $\geq \frac{1}{4}$ for every tangent 2-plane, then M is imbedded as a projective hyperplane. The case $n = 2$ is also true since Corollary 1 holds when $n = 2$ specially.

3. **Proof of Theorem 2.** $M = V(f)$. Suppose $\deg f \geq 3$, then $\det(B) = 0$ and $f = 0$ have a nontrivial common zero by Lemma 1 as below. So B is singular at this point, and by Lemma 2 (also see below), rank $A = n$ implies B must be nonsingular, a contradiction. Therefore, def $f = 1$ or 2, but $\deg f = 1$ implies $M = CP^n$ and $A \equiv 0$, so $\deg f$ must be equal to 2. It's well known that a quadratic nonsingular hypersurface of CP^{n+1} must be biholomorphic to Q^n by a projective transformation and complete the proof. Q.E.D.

LEMMA 1. *Let $p, q \in C[X_0, \cdots, X_r]$ $(r \geq 2)$, the polynomial ring over C, be nonconstant homogeneous polynomials. Then p and q have a common zero other than* $(0, \cdots, 0)$.

Proof. It's well known. Cf. textbooks on algebraic geometry. Q. E. D.

LEMMA 2. *Under the same notations as before and assumptions as in Theorem 2, in addition, if* rank $A = n$ *at $[z] \in M$, then B is nonsingular at z. Moreover, if XB' is defined to be $\overline{XB}$, $X \in C^{n+2}$ which is identified with $T_z C^{n+2}$,* $\|z\| = 1$, *then* $zB' = l$

$(\overline{\partial f/\partial z}) = l\,\|\partial f/\partial z\|\,\bar{N}$, $\langle \bar{N}B', z\rangle = l\,\|\partial f/\partial z\|$ *where* $l = \deg(\partial f/\partial z) \geq 1$, $\bar{N} = (\overline{\partial f/\partial z})/\|\partial f/\partial z\|$, $\langle\ ,\ \rangle$ *denote the standard inner product in* C^{n+2} *i.e.* $\langle X, Y\rangle = X^t\,\bar{Y}$.

Proof. Firstly, observe that nonsingularity of B is equivalent to that of B'. So consider $B' : C^{n+2} \to C^{n+2}$ which is antilinear, i.e. $(\alpha X)\,B' = \bar{\alpha}(XB')$, α a scalar. Note that $\{z, \bar{N}, T_{[z]}M\}$ are orthogonal with respect to the standard inner product $\langle\ ,\ \rangle$ in C^{n+2} where $\bar{N} = (\overline{\partial f/\partial z})/\|\partial f/2z\|$.

Now, $zB' = \overline{l(\partial f/\partial z)}$ is obvious since $\partial f/\partial z$ is homogeneous and then use Euler's identity. $\langle \bar{N}B', z\rangle = N\bar{B}^t\,\bar{z} = l\,\|\partial f/\partial z\| > 0$, and $(T_{[z]}M)\,B'$ misses z component since $z({}^tN\bar{N} - I) = -z$ and $X({}^tN\bar{N} - I) = -X$, $X \in T_{[z]}M$ is identified with $\{X \in C^{n+2} \mid \langle X, z\rangle = 0,\ \langle X, \bar{N}\rangle = 0\}$ as described in Section 1. By the above argument, we know both $\bar{N}$ and z must belong to the image of B'. On the other hand, $T_{[z]}M$ must also be included in the image of B' since $X({}^tN\bar{N} - I) = -X$, $X \in T_{[z]}M$ and $\operatorname{rank} A = n$. So we have proved nonsingularity of B'. Q. E. D.

By the way, we give a sufficient and necessary condition for a hypersurface to be isometric to the hypersphere.

PROPOSITION. *Suppose the hyperquadric* M *of* $CP^{n+1}(1)$ *defined by the equation* $\sum_{A,B} b_{AB}\, z_A\, z_B = 0$, $b_{AB} = b_{BA}$, *then* M *is holomorphically isometric to the hypersphere* Q^n *if and only if* $B\bar{B} = cI$, *where* c *is some positive constant,* I *denotes the identity matrix, and* $B = (b_{AB})$.

Proof. Suppose $B\bar{B} = cI$, $c > 0$, by a lemma of Chern [3], there exists a unitary matrix U such that ${}^tUBU$ is diagonal and every diagonal entry is nonnegative. Let $B^* = {}^tUBU$, then $B^*\,\bar{B}^* = {}^tUB\bar{B}\bar{U} = cI$. Denote

$$B^* = \begin{bmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_{n+1} \end{bmatrix}$$

we get $\lambda_i^2 = c$ which implies $\lambda_i = \sqrt{c}$, $0 \leq i \leq n+1$. It follows that the equation of the hyperquadric can be brought to the normal form $\sqrt{c}\,(z_0^2 + \cdots + z_{n+1}^2) = 0$ by the unitary transformation U.

And $U \in I(CP^{n+1}(1))$, the isometry group of $CP^{n+1}(1)$, so M is holomorphically isometric to Q^n.

Conversely, if M is holomorphically isometric to Q^n, by rigidity of Kaehlerian hypersurface [7], there exists an isometry V in $I(CP^{n+1}(1))$ which maps Q^n onto M. But $I(CP^{n+1}(1)) = I_0(CP^{n+1}(1)) \cup \alpha \cdot I_0(CP^{n+1}(1))$, $n \geq 1$, where $I_0(CP^{n+1}(1)) = U(n+2)/\{\text{scalars}\}$, $U(n+2)$ denotes the unitary group for the standard inner product in C^{n+2}, α: complex conjugation. (cf. Wolf [10]). So $V = \mu U$ or $\mu\alpha \cdot U$, which is impossible because α is not holomorphic, μ: nonzero constant, $U \in U(n+2)$. Take $[\tilde{z}] \in Q^n$, $(\tilde{z}, \tilde{z}) = 0$ implies $(\tilde{z}VB, \tilde{z}V) = 0$ where (z, w) is defined to be $z^t w$. Hence $(\tilde{z}VB^tV, \tilde{z}) = 0$. Now, $(\tilde{z}UB^tU, \tilde{z}) = 0$ if $V = \mu U$. Since $\tilde{z}$ runs on $(\tilde{z}, \tilde{z}) = 0$, it's easily obtained that $UB^tU = \lambda I$, λ nonzero constant, I: identity matrix. Hence $B\bar{B} = |\lambda|^2 I$, complete the proof. Q. E. D.

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Note Added in Proof. The author was informed that the young Russian geometer, F. L. Zak, also proved Theorem 1 (in an equivalent form) and many similar results in a letter of December 1979 to W. Fulton (see Fulton's report to the Arbeitstagung at Bonn in June, 1981), by a different approach.

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