COLORING GRAPHS WITH FORBIDDEN INDUCED SUBGRAPHS

MARIA CHUDNOVSKY (PRINCETON)

COLORING IS NPC

3-COLORING IS NPC

BEST KNOWN APPROX:
(THORUP & KAWARABAYASHI)

AN n-VERTEX 3-COLORABLE GRAPH CAN BE $n^{4/11}$-COLORED IN POLYNOMIAL TIME

COLORING GRAPHS WITH CERTAIN INDUCED S.G.'S EXCLUDED

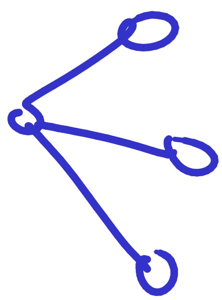
THM (KAMINSKI, LOZIN)
$\forall k \geq 3$, $g \geq 3$ THE k-COLORING PROBLEM IS NPC FOR GRAPHS OF GIRTH $\geq g$

COR $\forall k \geq 3$, k-COLORING IS NPC FOR H-FREE GRAPHS IF H HAS A CYCLE

THM (HOLYER)

K-EDGE-COLORING IS NPC FOR $K \geq 3$

COR K-COLORING CLAW-FREE GRAPHS IS NPC FOR $K \geq 3$



CLAW

REMANS OPEN:

- H IS THE DISJONT UNION OF PATHS
- G IS H-FREE
- $K \geq 3$

? K-COLOR G

FOCUS ON: H IS A PATH

$\boxed{P_t \quad \text{t-}\underline{\text{VERTEX}} \text{ PATH}}$

	$k=3$	$k=4$	$k=5$	$k \geq 6$
$t \leq 5$	P	P	P	P
$t=6$	P	?	NPC	NPC
$t=7$	P*	NPC	NPC	NPC
$t \geq 8$	?	NPC	NPC	NPC

FROM: THREE COMPLEXITY RESULTS ON COLORING P_k-FREE GRAPHS

(BROERSMA, FOMIN, GOLOVACH, PAULUSMA)

IMPROVED COMPLEXITY RESULTS ON k-COLORING P_t-FREE GRAPHS

(HUANG)

* (BONOMO, C., MACELI, SCHAUDT, STEIN, ZHONG)

OPEN CASES:

- k-COLORING P_t-FREE GRAPHS
  $k=3$, $t \geq 8$
- 4-COLORING P_6-FREE GRAPHS
  - LIST VERSION IS NPC
    (GOLOVACH, PAULUSMA, SONG)
  - 4-COLORING (P_6, C_5)-FREE GRAPHS IS POLY
    (C., MACELI, STACHO, ZHONG)
  - 4-COLORING (P_6, C_3)-FREE GRAPHS IS POLY
    (RANDERATH, SCHIERMEYER, TEWES)

RECENTLY: COMPLETE STRUCTURE THM
(C, SEYMOUR, SPIRKL, ZHONG)

THM (BONOMO, C., MACELI, SCHAUDT, STEIN, ZHONG)

THERE IS A POLY TIME ALG TO TEST IF A P_7-FREE GRAPH IS 3-COLORABLE

LIST COLORING

G GRAPHS

$L = \{L(v)\}_{v \in V(G)}$ LISTS OF COLOR

(G, L) IS COLORABLE IF $\exists$ COLORI

c OF $V(G)$, S.T. $\forall v$ $c(v) \in L(v)$

IDEA: REDUCE COLORING G

TO $\{(G_1, L_1), \ldots, (G_k, L_k)\}$ S.T.

G IS COLORABLE IFF

(G_i, L_i) IS COLORABLE FOR SOME i

IMPORTANT TOOL 1

THM (EDWARDS)

- G GRAPH
- $L = \{ L(v) \}_{v \in V(G)}$ LISTS
- $\forall v \quad |L(v)| \leq 2$

THERE IS A POLY-TIME ALG TO TEST IF (G, L) IS COLORABLE

PROOF: REDUCE TO 2-SAT

IMPORTANT TOOL 2

THM (CAMBI & SCHAUDT)

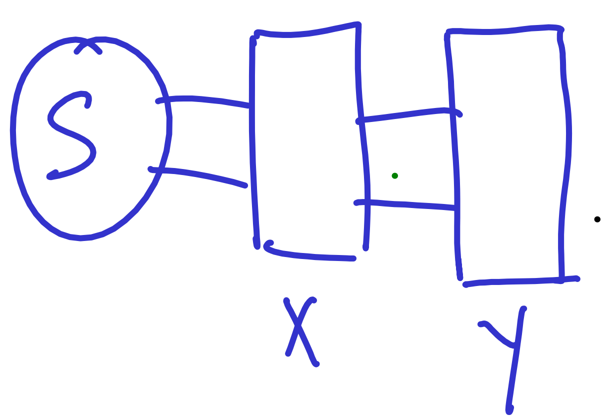
IF G IS P_t-FREE AND CONNECTED, THEN $\exists H < G$ S.T.

1. $V(H)$ IS DOMINATING G, AND
2. $H \approx P_{t-2}$ OR H IS P_{t-2}-FREE, AND
3. H IS CONNECTED.

USING THE TOOLS

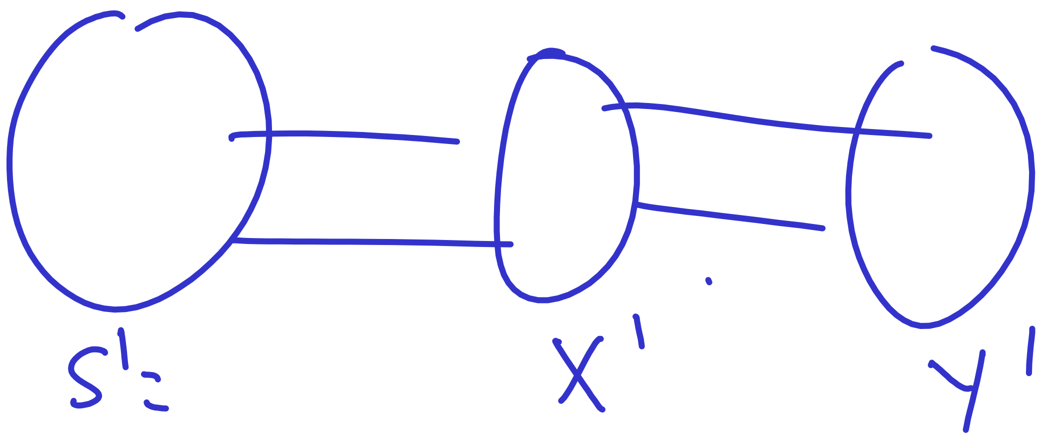
- IF $H \approx P_{t-2}$ TRY ALL POSSIBLE 3-COLORINGS OF H; REDUCE TO $3^{|V(H)|}$ 2-SAT PROBLEMS

- $t=7$
  H IS P_5-FREE
  $\Rightarrow$ H HAS A BOUNDED SIZE DOMINATING SUBGRAPH S

- $G =$ 

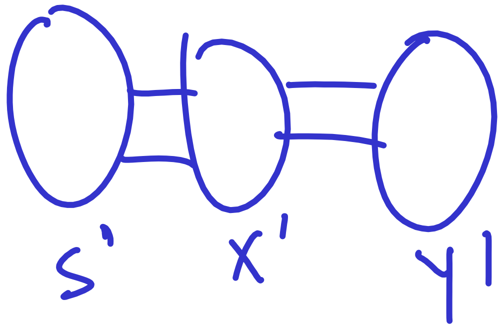
- "GUESS" (TRY ALL POSSIBILITIES)

~20 VERTICES WITH CERTAIN PROPERTIES TO ARRANGE



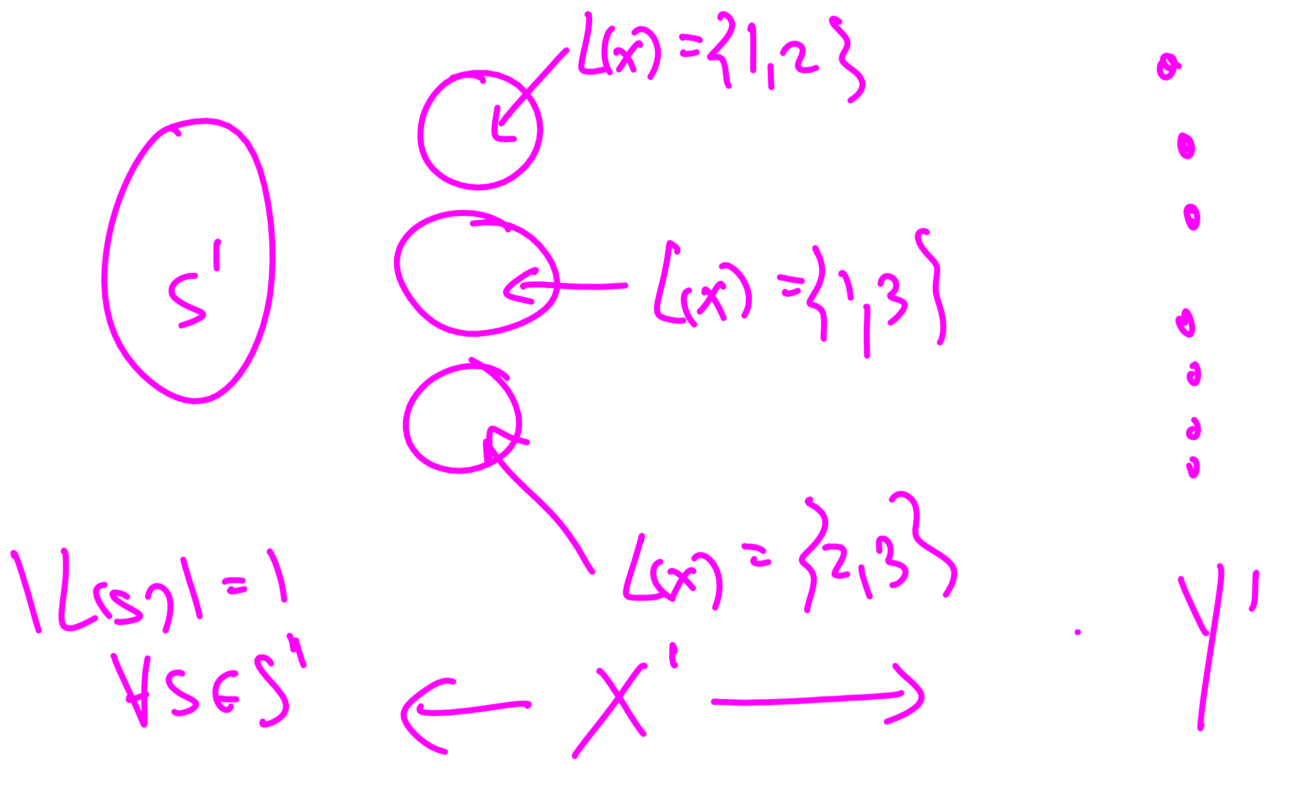
S' = S + 20 VERTICES

Y' STABLE



TRY ALL POSSIBLE COLORING OF S'

- FIX A COLORING OF S', AND UPDATE LISTS
- $|L(x)| \leq 2 \quad \forall x \in X'$



- GUESS THE COLORS OF ~ 5 MORE VERTICES TO ARRANGE THAT $|L(y)| \leq 2 \quad \forall y \in Y'$

NOW WIN BY EDWARD'S THM (2-SAT)

RELATED QUESTION (SEYMOUR)

IS THERE A BOUNDED NUMBER OF P_t-FREE OBSTRUCTIONS TO 3-COLORING?

THM (C., GOEDGEBEUR, SCHAUDT, ZHONG)
YES IF $t \le 6$

THM (POKROVSKIY)
NO IF $t \geq 7$

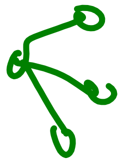
MORE GENERALLY

(GOLOVACH, JOHNSON, PAULUSMA, SONG)

FOR WHICH H ARE THERE FINITELY MANY H-FREE 4-CRITICAL GRAPHS?

H IS POWERFUL IF THERE ARE FINITELY MANY H-FREE VERTEX-4-CRITICAL GRAPHS

GRAPHS THAT ARE NOT POWERFUL

- IF  $\leq H$ OR $C_k \leq H$

  3-COLORING IS NPC

  (HOLYER; KAMINSKI & LOZIN)

$\Downarrow$

H IS NOT POWERFUL

A CONSTRUCTION:

INFINITE FAMILY OF  -FREE 4-CRITICAL GRAPHS (POKROVSKIY)

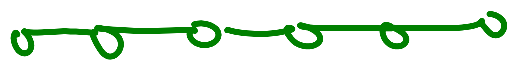
$\Downarrow$

 NOT POWERFUL

P_7 NOT POWERFUL

THM (C., GOEDGEBEUR, SCHAUDT, ZHONG)

H IS POWERFUL IFF

$H \leq$  , OR

 , OR

A CLOSELY RELATED PROBLEM:

LIST - 3 - COLORING

(G, L) $\quad L(v) \subseteq \{1,2,3\}$ $\quad \forall v$

FIND A PROPER COLORING c OF G S.T. $c(v) \in L(v)$ $\forall v$

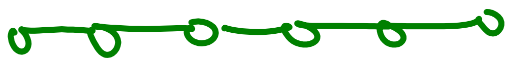
FOR WHICH H ARE THERE FINITELY MANY H-FREE MINIMAL OBSTRUCTIONS TO LIST-3-COLORING?

THE ANSWER IS DIFFERENT!

LIST-POWERFUL

THM (C., GOEDGEBEUR, SCHAUDT, ZHONG)

H IS LIST-POWERFUL IFF

H $\leq$  , OR

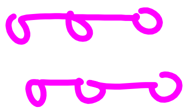
A CONSTRUCTION:

INFINITE FAMILY OF

 -FREE OBSTRUCTIONS

WITH $|L(v)|=2$ $\forall v$

(SCHAUDT)

SO  IS POWERFUL,

BUT NOT LIST-POWERFUL

PROOFS

ENOUGH TO SHOW:

P_6, $2P_3$, $P_4 + kP_1$

ARE POWERFUL

STEP 1 SHOW THAT IF G IS A MINIMAL OBSTRUCTION THEN

$\exists\, G', L'$ S.T.

- $|V(G)| \le 100^{|V(G')|}$
- $|L'(v)| \le 2 \quad \forall v \in V(G')$
- * IF $i \notin L'(v)$, THEN $\exists u$ S.T. $uv \in E(G')$ & $L(u) = \{i\}$
- (G', L') IS A MINIMAL OBSTRUCTION

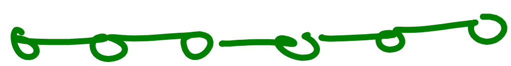
STEP 2

SHOW THAT IF G' IS H-FREE AND

- $|L'(v)| \leq 2 \ \forall v \in V(G')$.
- (G', L') IS A MINIMAL OBSTRUCTION

THEN $|V(G')| \leq 10^6$

- WORKS FOR

$H =$ 

- FOR H : 

WORKS UNDER ASSUMPTION*

MORE DETAILS FOR STEP 1

STRUCTURAL RESULT:

LET G BE A MINIMAL H-FREE OBSTRUCTION. THEN $\exists R \leq G$ S.T.

(1) R HAS ≤ 1000 3-COLORINGS

(2) PRECOLORING R & UPDATING 3 TIMES GIVES $|L(v)| \leq 2$ $\forall v \in V(G)$

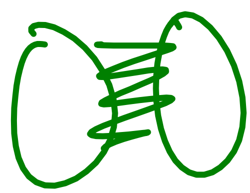
"PROOFS"

- $H = P_4 + kP_1$

  TAKE $R = P_4 + (k-1)P_1$

- $H = P_6$

  CAMBI-SCHAUDT + TRICKS

  R IS 

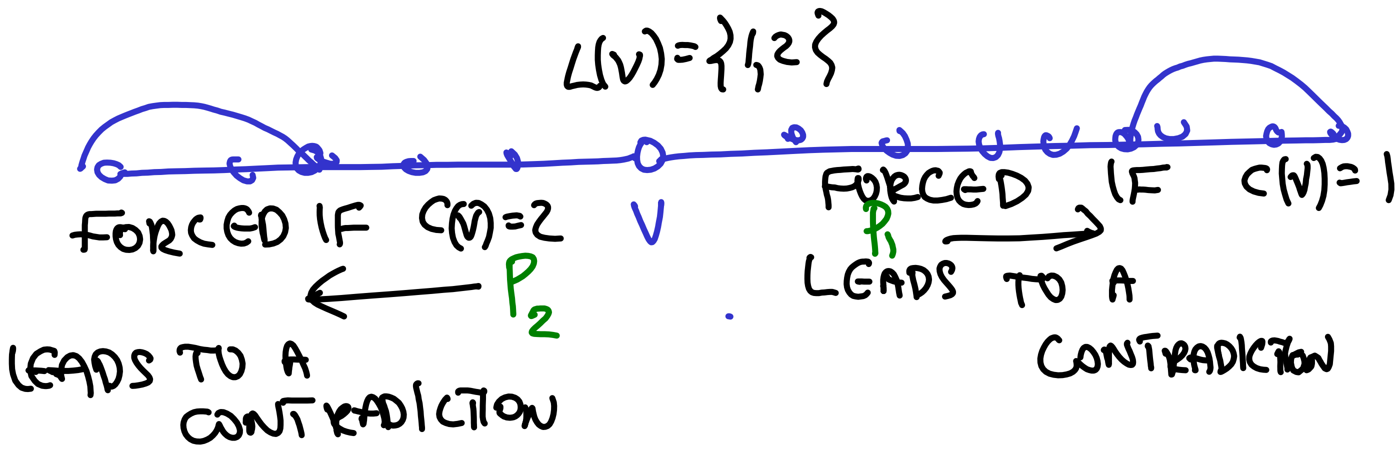
- $H = 2P_3$ HARD

- $\forall$ PRECOLORING c_i OF R
  G CONTAINS AN OBSTRUCTION
  (G_i, L_i)
- BY STEP 2 $|V(G_i)| \leq 10^6$
- $G' = R \cup \bigcup_i G_i \cup$ updating paths
  IS NOT 3-COLORABLE
- $G' = G$ (BY MINIMALITY OF G)
- $|V(G') \setminus R|$ BOUNDED
  $\Downarrow$
  BOUNDED # "TYPES" OF
  VERTICES IN R
- USE STRUCTURE OF R TO
  DEDUCE THAT $|V(R)|$ BOUNDED

MORE DETAILS OF STEP 2

ORIGINALLY DONE BY COMPUTER, NOW HAVE A COMPUTER-FREE PROOF

PROPAGATION PATH:



$$V(G) = V(P_1) \cup V(P_2)$$

STUDY THE STRUCTURE OF P_1, P_2; FIND H