INDUCED SUBGRAPHS AND COLORING

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$G(V, E)$

VERTICES EDGES

$\chi(G)$ CHROMATIC NUMBER

$\omega(G)$ CLIQUE NUMBER

$\forall G \quad \chi(G) \geq \omega(G)$

WHAT ABOUT $\chi(G) \leq f(\omega(G))$?

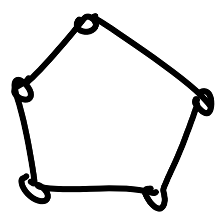
WHAT ABOUT $\chi(G) \leq f(\omega(G))$?

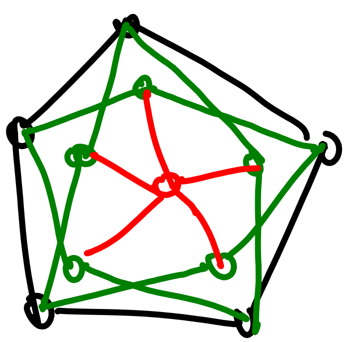
FALSE

THM (MYCIELSKI)

$\forall k \ \exists G$ s.t. $\omega(G)=2$ & $\chi(G) \geq k$

MYCIELSKI GRAPHS:

M_2  M_3 

M_4 

$\omega(M_k) = 2 \qquad \chi(M_k) = k$

STRONGER THEOREM (ERDŐS)

FOR EVERY g, k THERE IS A GRAPH WITH NO CYCLE OF LENGTH $< g$, AND WITH $\chi > k$.

(4)

WHEN $\exists f$ S.T. $\chi(G) \leq f(\omega(G))$?

MORE PRECISELY:

FOR WHICH INFINITE FAMILIES $\mathcal{F}$ OF GRAPHS, DOES THERE EXIST $f_{\mathcal{F}}$ S.T.

$$\chi(G) \leq f_{\mathcal{F}}(\omega(G)) \quad \forall G \in \mathcal{F}$$

QUESTION 1:

WHEN IS $\chi(G) = \omega(G)$?

BIPARTITE GRAPHS

COMPLEMENTS OF BIP

COMPARABILITY GRAPHS

$\vdots$

≤ 99 VERT

G IS **PERFECT** IF

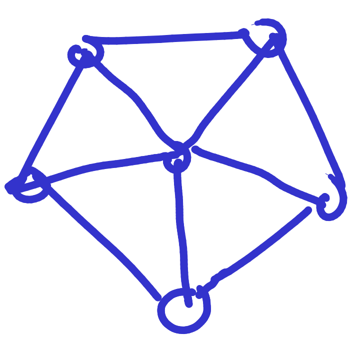
$\chi(H) = \omega(H)$ FOR EVERY INDUCED SUBGRAPH H OF G

(BERGE, 1961)

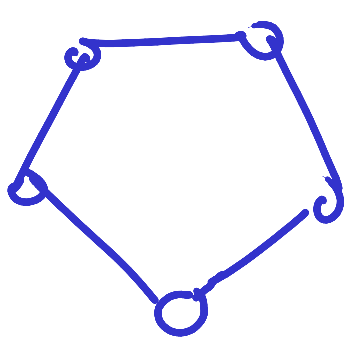
H IS AN **INDUCED SUBGRAPH** OF G IF

$V(H) \subseteq V(G)$, AND

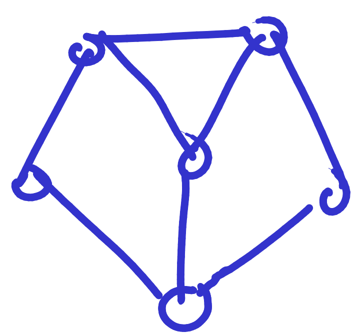
$uv \in E(H)$ IFF $uv \in E(G)$ & $u,v \in V(H)$



G

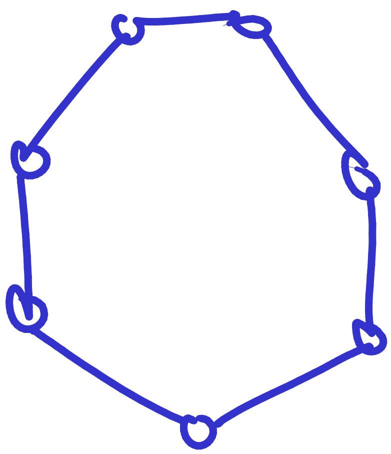


INDUCED S.G. OF G



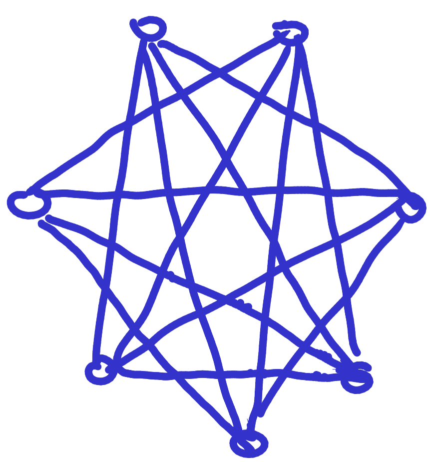
NOT AN INDUCED S.G. OF G

WHICH GRAPHS ARE NOT PERFECT?



C_{2n+1}

$n \geq 2$



C_{2n+1}^{c}

$n \geq 2$

THE STRONG PERFECT GRAPH THEOREM

(C., ROBERTSON, SEYMOUR, THOMAS)

G IS PERFECT IFF G HAS NO INDUCED SUBGRAPH ISOMORPHIC TO C_{2n+1} OR C_{2n+1}^{c} WITH $n \geq 2$

G IS BERGE IF NO INDUCED SUBGRAPH OF G IS ISOMORPHIC TO C_{2n+1} OR C_{2n+1}^{c} FOR $n \geq 2$

SPGT : BERGE = PERFECT

PROOF (IDEA): DESCRIBE ALL BERGE GRAPHS

"THM 1" EVERY BERGE GRAPH IS EITHER BASIC, OR ADMITS A "USEFUL" DECOMPOSITION

"THM 2" ALL BASIC GRAPHS ARE PERFECT

"THM 3" LET G BE BERGE, NOT PERFECT, WITH $|V(G)|$ MIN. THEN G DOES NOT ADMIT ANY OF THE DECOMPOSITIONS FROM "THM 1"

DEF LET $\mathcal{C}$ BE A CLASS OF GRAPHS CLOSED UNDER TAKING INDUCED SUBGRAPHS.

$\mathcal{C}$ IS χ-BOUNDED IF $\exists f$ S.T. $\chi(G) \leq f(\omega(G))$ $\forall G \in \mathcal{C}$

f IS A χ-BOUNDING FUNC FOR $\mathcal{C}$

FIX H.

$$\text{FORB}(H) = \{ G : H \not\leq G \}$$

IS $\text{FORB}(H)$ χ-BOUNDED FOR ALL H ?

NO

THM IF $\text{FORB}(H)$ IS χ-BOUNDED, THEN H HAS NO CYCLES

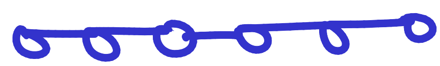
PROOF:

RECALL: $\forall g, k$ $\exists G$ WITH NO CYCLES OF LENGTH $< g$, AND WITH $\chi(G) > k$

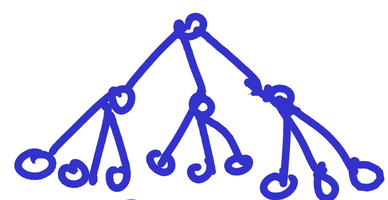
CONJ (GYARFAS; SUMNER)

IF H HAS NO CYCLES, THEN FORB(H) IS χ-BOUNDED

KNOWN CASES:

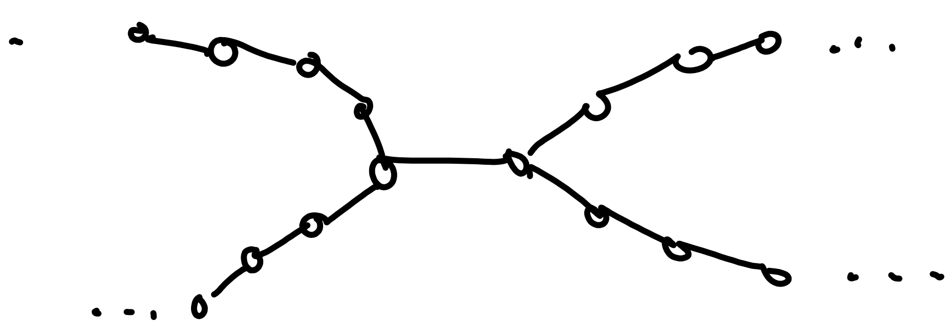
- H IS A PATH 

  (GYARFAS)

- H HAS RADIUS ≤ 2 

  (KIERSTEAD & PENRICE)

- TOPOLOGICAL VERSION (SCOTT)

- 

  (C., SCOTT, SEYMOUR, SPIRKL)

A HOLE IN G IS AN INDUCED CYCLE OF LENGTH ≥ 4

QUESTION: FORBID HOLES OF SOME LENGTHS. DO WE GET A χ-BOUNDED FAMILY?

OBSERVATION: NEED TO FORBID INFINITELY MANY LENGTHS

THM (ADDARIO-BERRY, C., HAVET, REED, SEYMOUR)

IF G HAS NO EVEN HOLES, THEN $\chi(G) \leq 2\,\omega(G)$.

(CONJECTURED BY REED)

THM (BONAMY, CHARBIT, THOMASSE)

IF G HAS NO HOLES OF LENGTH $0 \bmod 3$ AND NO Δ, THEN $\chi(G) \leq 10^6$.

(CONJECTURED BY KALAI & MESHULAM)

MORE CONJECTURES BY GYÁRFÁS:

CONJ 1 (GYÁRFÁS) THE FAMILY OF GRAPHS WITH NO ODD HOLE IS χ-BOUNDED

CONJ 2 (GYÁRFÁS)

$\forall \ell$ THE FAMILY OF GRAPHS WITH NO HOLE OF LENGTH $\geq \ell$ IS χ-BOUNDED

CONJ 3 (GYÁRFÁS)

$\forall \ell$ THE FAMILY OF GRAPHS WITH NO ODD HOLE OF LENGTH $\geq \ell$ IS χ-BOUNDED

THM 1 (SCOTT & SEYMOUR, 2014)

THE FAMILY OF ODD-HOLE-FREE GRAPHS IS χ-BOUNDED

(CONJ'D BY GYÁRFÁS 1987)

CONJ (HOANG MCDIARMID)

LET G BE AN ODD-HOLE-FREE GRAPH.

THEN $V(G) = V_1 \cup V_2$ S.T.

$\forall i = 1,2 \quad \omega(G|V_i) \leq \omega(G) - 1$

(AND THEREFORE $\chi(G) \leq 2^{\omega(G)}$)

THM (C., SIVARAMAN)

A VARIANT IS TRUE IF G IS BULLFREE

THM 2 (C., SCOTT, SEYMOUR, 2015)

FOR EVERY ℓ THE FAMILY OF GRAPHS WITH NO HOLE OF LENGTH $\geq \ell$ IS χ-BOUNDED

THM 3 (C., SCOTT, SEYMOUR, SPIRKL, 2016)

FOR EVERY ℓ THE FAMILY OF GRAPHS WITH NO ODD HOLE OF LENGTH $\geq \ell$ IS χ-BOUNDED

THM (GYÁRFÁS)

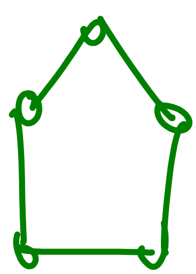
$\forall \ell$ THE FAMILY OF GRAPHS WITH NO INDUCED PATH OF LENGTH $\geq \ell$ IS χ BOUNDED

PF GROW A PATH INTO A COMPONENT WITH BIG χ

(GYÁRFÁS PATH)

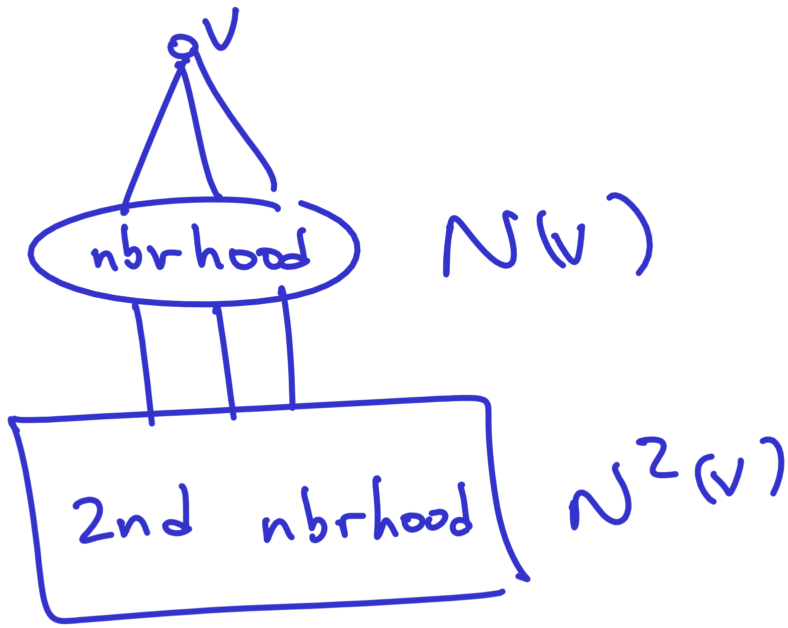
THM (SCOTT, ~1995)

$\forall \ell$ THE FAMILY OF GRAPHS WITH NO C_5 AND NO HOLE OF LENGTH $\geq \ell$ IS χ-BOUNDED



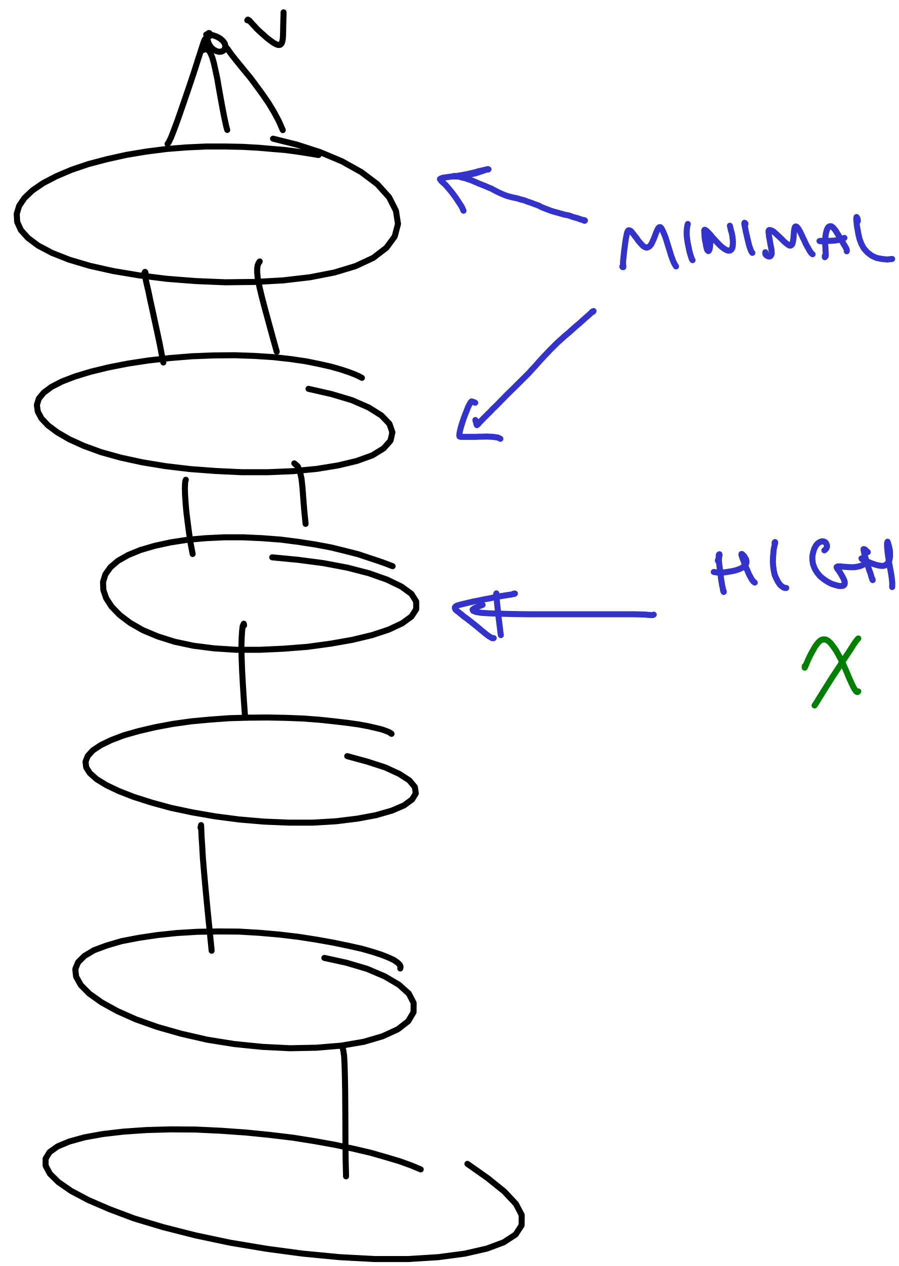
C_5

PROOF INDUCTION ON $\omega(G)$



IDEA : LAYERING

ASSUME: $\chi(G)$ BIG



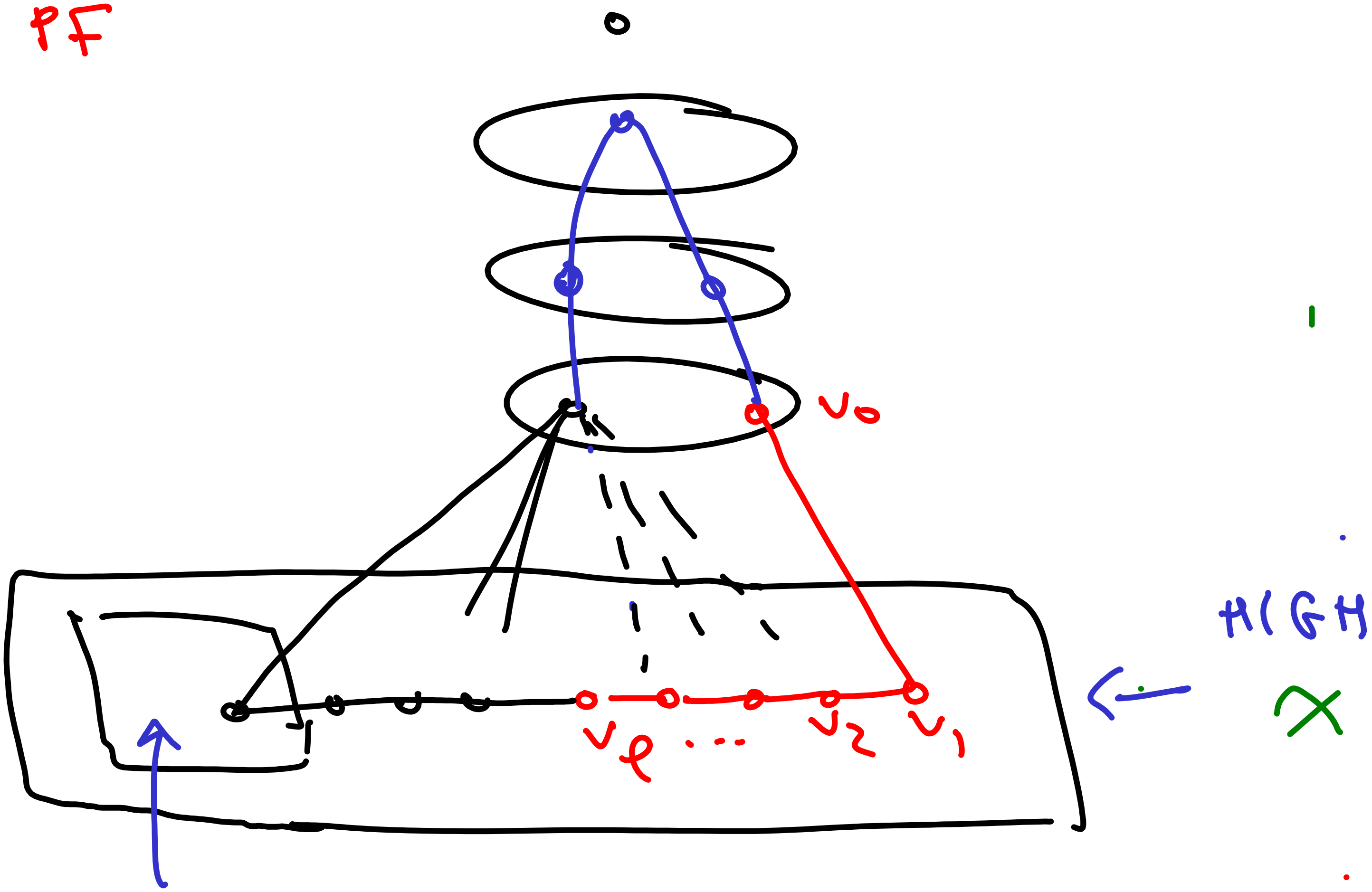
STEP 1

THM (SCOTT) LET $k, \ell \in \mathbb{Z}^+$, LET $\mathcal{F}$ BE A FAMILY OF GRAPHS WITH NO HOLE OF LENGTH $\geq \ell$, AND S.T.

$$\chi(N^2(v)) \leq k \quad \forall v \in V(G) \quad \forall G \in \mathcal{F}.$$

THEN $\chi(G) \leq f(k, \ell, \omega(G)) \quad \forall G \in \mathcal{F}$.

PF



HIGH χ S.G. OF $G \setminus \bigcup_{i=0}^{\ell} N^2(v_i)$

STEP 1 IMPLIES:

IF

$G' < G$ AND $\chi(G')$ IS HUGE,

THEN

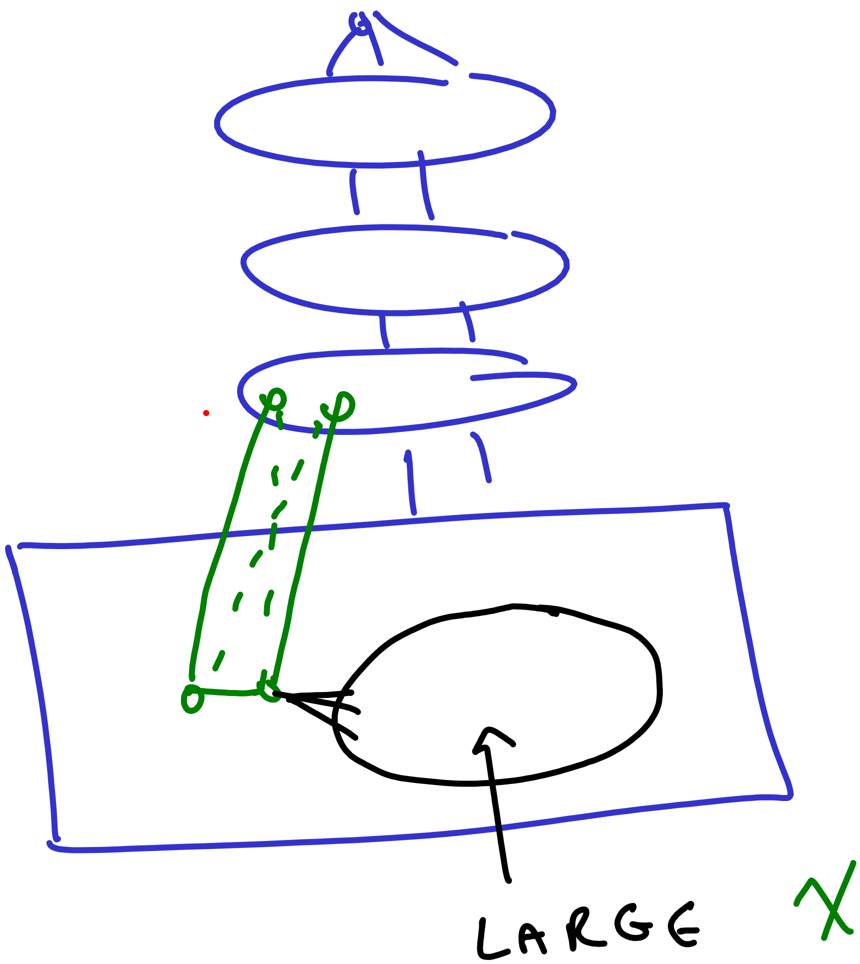
$\exists v \in V(G')$ S.T. $\chi(N^2(v))$ IS LARGE

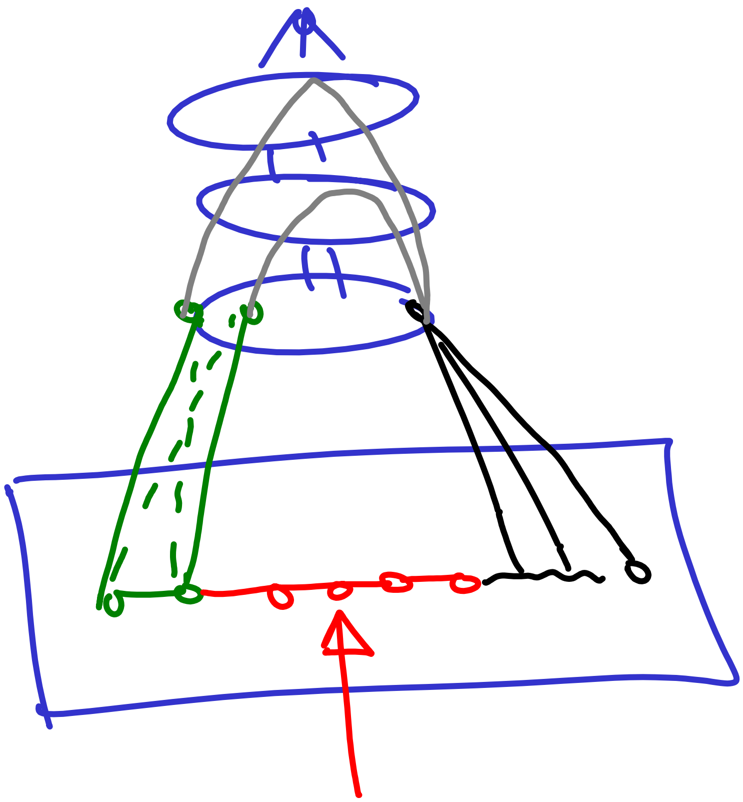
STEP2 $\forall v \; \chi(N^2(v)) \leq K_{e,w}$

THM $\forall \ell$ THE FAMILY OF GRAPHS WITH NO ODD HOLE OF LENGTH $\geq \ell$ IS χ-BOUNDED

PROOF INDUCTION ON ω

STEP 1 ASSUME $\forall v \; \chi(N^2(v)) \leq K$





GYÁRFÁS PATHS

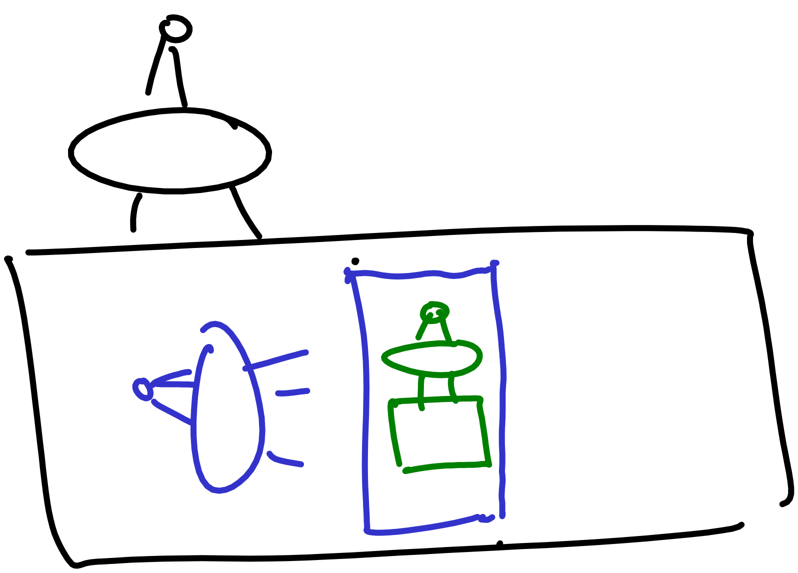
STEP 2

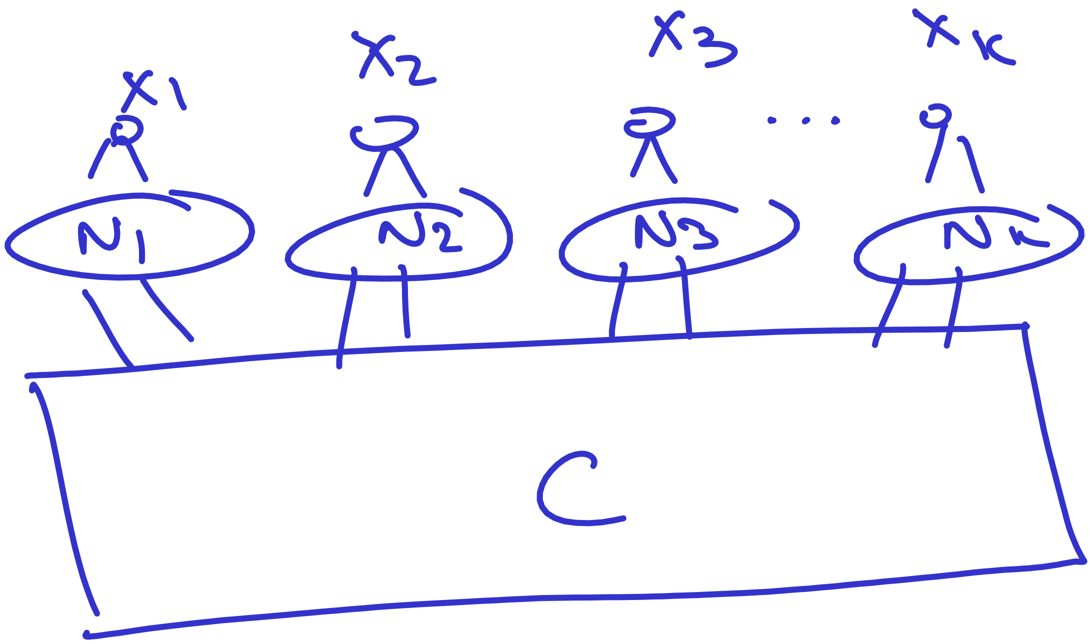
KNOW: $\forall G' < G$ WITH $\chi(G')$ HUGE

$\exists v \in V(G')$ WITH $\chi(N^2(v))$ LARGE

PROVE: $\exists$ ODD HOLE OF LENGTH $\geq \ell$

ITERATED COVERS:

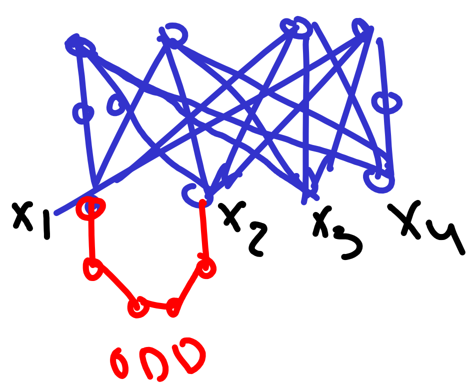




$\chi(C)$ BIG

$\forall c \in C$ $\forall i \in \{1,..,k\}$ c HAS A NBR IN N_i

GET



OR LONG ODD HOLE IN $\bigcup_i N_i \cup \{x_i\}$.