Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **18** (2023), No. 4, pp. 451-469
DOI: 10.21915/BIMAS.2023404

WEISNER'S METHODIC STUDY OF MODIFIED JACOBI POLYNOMIAL

ASIT KUMAR CHONGDAR[1,a] AND AMARTYA CHONGDAR[2,3,b]

[1]Retired Professor, Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur; Howrah - 711103, West Bengal, India.

[a]E-mail: chongdarmath@yahoo.co.in

[2]Department of Mathematics, Bangabasi College, 19 Rajkumar Chakraborti Sarani, Kolkata - 700009, India.

[3]West Bengal State University, India.

[b]E-mail: acmath77@gmail.com

Abstract

In this paper we have obtained some novel generating functions of modified Jacobi polynomial $P_n^{(\alpha-n,\beta)}(x)$ by using Weisner's group theoretic method with the suitable interpretation of the index (n) of the polynomial under consideration. Moreover, it has been shown that the generating functions involving Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ found derived in [18] by means of the same technique of Weisner with the double interpretation of (n,α) can be easily obtained from our results by the mere replacement of α by $\alpha+n$.

Furthermore, we have made some comments on some recent works in the light of the present paper.

1. Introduction

The Jacobi polynomial, denoted by $P_n^{(\alpha,\,\beta)}(x)$ and defined by [27]:

$$P_n^{(\alpha,\,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n; \\ 1+\alpha; \qquad \frac{1-x}{2}\end{matrix}\right], \tag{1}$$

is a solution of the following ordinary differential equation:

$$(1-x^2)\frac{d^2u}{dx^2} + [\beta-\alpha-(2+\alpha+\beta)x]\frac{du}{dx} + n[1+\alpha+\beta+n]u = 0. \tag{2}$$

Received October 12, 2023.

AMS Subject Classification: 33C45.

Key words and phrases: Jacobi polynomial, generating function.

Replacing α by $\alpha - n$, we get

$$P_n^{(\alpha-n,\,\beta)}(x) = \frac{(1+\alpha-n)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n, & 1+\alpha+\beta; \\ & 1+\alpha-n; \end{matrix}\ \frac{1-x}{2}\right], \tag{3}$$

which satisfies the following ordinary differential equation:

$$(1-x^2)\frac{d^2u}{dx^2} + [\beta - \alpha + n - (2+\alpha+\beta-n)x]\frac{du}{dx} + n[1+\alpha+\beta]u = 0. \tag{4}$$

In this article we shall obtain some novel results on generating functions of $P_n^{(\alpha-n,\beta)}(x)$, a modification of Jacobi polynomial - $P_n^{(\alpha,\beta)}(x)$, by using Weisner's group theoretic method [35, 36, 37] which is lucidly presented in the monograph "Obtaining generating functions" written by E. B. McBride [23].

From eighties and onwards of the last century, L. Weisner's group-theoretic method has been mostly used in the investigation of generating functions for the various special functions, particularly for the classical orthogonal polynomials and their different modifications. Based on this method, a large number of generating functions of Jacobi polynomial and its different modifications have been obtained vide the works [5, 6, 7, 16, 17, 18, 19, 20, 21, 22, 25, 26, 30, 11, 28, 29, 2, 4].

In [18], Ghosh obtained some generating functions of Jacobi polynomial by Weisner's group theoretic method with the suitable interpretation of (n, α), but in Weisner's method double interpretation is a little bit cumbrous in comparison with single interpretation.

The aims of writing this article are

(1) to investigate the modified Jacobi polynomial - $P_n^{(\alpha-n,\beta)}(x)$ by means of Weisner's group-theoretic method with the single interpretation of the index (n) of the polynomial for obtaining some novel generating functions,

(2) to show that all the results of Jacobi polynomial found derived in [18] can be easily obtained from our results of this paper in a straightforward way and finally

(3) to make some valuable comments on some recent works in the light of the present paper.

In fact, while investigating $P_n^{(\alpha,\beta)}(x)$ for obtaining generating functions found derived in [18], one can easily observe that the double interpretation of (n,α) is not only cumbrous in comparison with single interpretation in Weisner's method but also quite unnecessary and may be replaced by the single interpretation of the index (n) while investigating $P_n^{(\alpha-n,\beta)}(x)$ by the same technique of Weisner for obtaining the same results found derived in [18] in a straightforward way - making the original problem simple and easier.

2. Group-Theoretic Discussion and Lie Algebra

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$ and u by $v(x,y)$ in (4), we get the following partial differential equation:

$$(1-x^2)\frac{\partial^2 v}{\partial x^2}+[\beta-\alpha-(2+\alpha+\beta)x]\frac{\partial v}{\partial x}+(1+x)y\frac{\partial^2 v}{\partial y\partial x}+[1+\alpha+\beta]y\frac{\partial v}{\partial y}=0. \quad (5)$$

Thus $v_1(x,y)=P_n^{(\alpha-n,\,\beta)}(x)y^n$ is a solution of (5) since $P_n^{(\alpha-n,\,\beta)}(x)$ is a solution of (4).

We now define the infinitesimal operators A_i $(i=1,2,3)$:

$$A_i=A_i^{(1)}\frac{\partial}{\partial x}+A_i^{(2)}\frac{\partial}{\partial y}+A_i^{(0)},\ \ i=1,2,3$$

as follows:

$$\begin{aligned}
A_1 &= y\frac{\partial}{\partial y}\\
A_2 &= (1+x)y^{-1}\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\\
A_3 &= (1-x^2)y\frac{\partial}{\partial x}+2y^2\frac{\partial}{\partial y}-[(1+\alpha+\beta)(x-1)+2\alpha]y
\end{aligned}$$

such that

$$\begin{aligned}
A_1(P_n^{(\alpha-n,\,\beta)}(x)y^n) &= nP_n^{(\alpha-n,\,\beta)}(x)y^n\\
A_2(P_n^{(\alpha-n,\,\beta)}(x)y^n) &= (\beta+n)P_{(n-1)}^{(\alpha-n+1,\,\beta)}(x)y^{n-1}\\
A_3(P_n^{(\alpha-n,\,\beta)}(x)y^n) &= -2(n+1)P_{(n+1)}^{(\alpha-n-1,\,\beta)}(x)y^{n+1}.
\end{aligned}$$

Now we proceed to find the commutator relations. Using the notation:

$$[A,\, B]u = (AB - BA)u,$$

we have,

$$[A_1,\, A_2] = -A_2, \quad [A_1, A_3] = A_3, \quad [A_2, A_3] = -(4A_1 + 2(1+\beta)) \tag{6}$$

From the above commutator relations we can easily state the following theorem.

Theorem 2.1. *The set of operators $\{1, A_i : i = 1,2,3\}$, where 1 stands for the identity operator, generates a Lie algebra $\mathcal{L}$.*

It is easy to verify that the partial differential operator L, given by:

$$\begin{aligned} Lv &= (1-x^2)\frac{\partial^2 v}{\partial x^2} + [\beta - \alpha - (2+\alpha+\beta)x]\frac{\partial v}{\partial x} + (1+x)y\frac{\partial^2 v}{\partial y \partial x} \\ &\quad + (1+\alpha+\beta)y\frac{\partial v}{\partial y}, \end{aligned}$$

can be expressed in terms of A_i $(i = 1,2,3)$ as follows:

$$(1+x)L = A_3A_2 + 2A_1^2 + 2\beta A_1. \tag{7}$$

From (6) and (7), one can easily verify that $(x+1)L$ commutes with each A_i $(i = 1,2,3)$ i.e.,

$$[(1+x)L, A_i] = 0, \quad i = 1,2,3. \tag{8}$$

The extended form of the groups generated by A_i $(i = 1,2,3)$ are as follows:

$$\begin{aligned} e^{a_1A_1} f(x,y) &= f(x, e^{a_1}y) \\ e^{a_2A_2} f(x,y) &= f\left(\frac{xy + a_2}{y - a_2}, y - a_2\right) \\ e^{a_3A_3} f(x,y) &= (1-2a_3y)^{\alpha}\{1 - a_3y(1-x)\}^{-1-\alpha-\beta} \\ &\quad \times f\left(\frac{x + a_3(1-x)y}{1 - a_3(1-x)y}, \frac{y}{1-2a_3y}\right). \end{aligned}$$

Therefore, from above, we get

$$e^{a_3A_3}e^{a_2A_2}e^{a_1A_1} f(x,y) = (1-2a_3y)^{\alpha}\{1 - a_3y(1-x)\}^{-1-\alpha-\beta}$$

$$\times f\left(\frac{x+a_3(1-x)y+\frac{a_2}{y}(1-2a_3y)\{1-a_3(1-x)y\}}{\{1-a_3(1-x)y\}\{1-\frac{a_2}{y}(1-2a_3y)\}},\right.$$
$$\left. e^{a_1}\frac{y\{1-\frac{a_2}{y}(1-2a_3y)\}}{1-2a_3y}\right). \qquad (9)$$

3. Generating Functions

From (5), $v(x,y)=P_n^{(\alpha-n,\,\beta)}(x)y^n$ is a solution of the system:

$$\begin{aligned} Lv &= 0 \\ (A_1-n)v &= 0. \end{aligned}$$

From (8), we easily get

$$S(1+x)L\left(P_n^{(\alpha-n,\,\beta)}(x)y^n\right)=(1+x)LS\left(P_n^{(\alpha-n,\,\beta)}(x)y^n\right)=0,$$

where $S=e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}$.
Therefore the transformation $S\left(P_n^{(\alpha-n,\,\beta)}(x)y^n\right)$ is also annulled by $(1+x)L$. Putting $a_1=0$ and replacing $f(x,y)$ by $P_n^{(\alpha-n,\,\beta)}(x)y^n$ in (9), we get

$$\begin{aligned} &e^{a_3A_3}e^{a_2A_2}\left(P_n^{(\alpha-n,\,\beta)}(x)y^n\right)\\ &=(1-2a_3y)^{\alpha-n}\{1-a_3y(1-x)\}^{-1-\alpha-\beta}\{1-\frac{a_2}{y}(1-2a_3y)\}^ny^n\\ &\quad P_n^{(\alpha-n,\,\beta)}\left(\frac{x+a_3(1-x)y+\frac{a_2}{y}(1-2a_3y)\{1-a_3(1-x)y\}}{\{1-a_3(1-x)y\}\{1-\frac{a_2}{y}(1-2a_3y)\}}\right). \end{aligned} \qquad (10)$$

But,

$$\begin{aligned} &e^{a_3A_3}e^{a_2A_2}\left(P_n^{(\alpha-n,\,\beta)}(x)y^n\right)\\ &=\sum_{p=0}^{\infty}\sum_{k=0}^{n+p}\frac{(-2a_3y)^p}{p!}\frac{(-\frac{a_2}{y})^k}{k!}{}^{(-\beta-n)}k^{(n-k+1)_p}P_{n-k+p}^{(\alpha-n+k-p,\,\beta)}(x)y^n. \end{aligned} \qquad (11)$$

Equating (10) and (11), we get

$$(1-2a_3y)^{\alpha-n}\{1-a_3y(1-x)\}^{-1-\alpha-\beta}\{1-\frac{a_2}{y}(1-2a_3y)\}^n$$

$$\times P_n^{(\alpha-n,\beta)}\left(\frac{x+a_3(1-x)y+\frac{a_2}{y}(1-2a_3y)\{1-a_3(1-x)y\}}{\{1-a_3(1-x)y\}\{1-\frac{a_2}{y}(1-2a_3y)\}}\right)$$
$$=\sum_{p=0}^{\infty}\sum_{k=0}^{n+p}\frac{(-2a_3y)^p}{p!}\frac{(-\frac{a_2}{y})^k}{k!}(-\beta-n)_k(n-k+1)_p P_{n-k+p}^{(\alpha-n+k-p,\beta)}(x). \tag{12}$$

The above generating function does not seem to have appeared in the earlier works. This, in turn, yields some particular novel generating functions by attributing different values to a_2 and a_3.

Before discussing the particular cases of (12), it may be pointed out that the operators A_2 and A_3 being non-commutative, the relation (12) will change if we change the order of the element $e^{a_3A_3}e^{a_2A_2}$. In fact, by this change, we get the following generating relation.

$$(1-\frac{a_2}{y})^n(1-2a_3y(1-\frac{a_2}{y}))^{\alpha-n}[1-a_3y\{(1-x)-\frac{2a_2}{y}\}]^{-1-\alpha-\beta}$$
$$\times P_n^{(\alpha-n,\beta)}\left(\frac{(x+\frac{a_2}{y})+a_3y(1-\frac{a_2}{y})[(1-x)-\frac{2a_2}{y}]}{(1-\frac{a_2}{y})[1-a_3y\{(1-x)-\frac{2a_2}{y}\}]}\right)$$
$$=\sum_{p=0}^{\infty}\sum_{k=0}^{n+p}\frac{(-2a_3y)^p}{p!}(n+1)_p\frac{(-\frac{a_2}{y})^k}{k!}(-\beta-n-p)_k P_{n+p-k}^{(\alpha-n-p+k,\beta)}(x). \tag{13}$$

The above pair of generating relations, (12) and (13) does not seem to appear before. We now consider the following particular cases of the relation (12).

Case 1. Putting $a_3=0$ and replacing $(\frac{a_2}{y})$ by t, we get

$$(1-t)^nP_n^{(\alpha-n,\beta)}\left(\frac{x+t}{1-t}\right)=\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}(-\beta-n)_kP_{n-k}^{(\alpha-n+k,\beta)}(x). \tag{14}$$

Case 2. Putting $a_2=0$ and replacing (a_3y) by t, we get

$$(1-2t)^{\alpha-n}\{1-t(1-x)\}^{-1-\alpha-\beta}P_n^{(\alpha-n,\beta)}\left(\frac{x+t(1-x)}{1-t(1-x)}\right)$$
$$=\sum_{p=0}^{\infty}\frac{(-2t)^p}{p!}(n+1)_pP_{n+p}^{(\alpha-n-p,\beta)}(x). \tag{15}$$

Case 3. Putting $a_2 = -\frac{1}{w}$, $a_3 = 1$ and replacing y by t, we get

$$
\begin{aligned}
&(1-2t)^{\alpha-n}\{1-t(1-x)\}^{-1-\alpha-\beta}\{1+\frac{1}{wt}(1-2t)\}^{n} \\
&\qquad \times P_n^{(\alpha-n,\,\beta)}\left(\frac{\{x+t(1-x)\}-\frac{1}{wt}(1-2t)\{1-t(1-x)\}}{\{1-t(1-x)\}\{1+\frac{1}{wt}(1-2t)\}}\right) \\
&= \sum_{p=0}^{\infty}\sum_{k=0}^{n+p}\frac{(-2t)^p}{p!}\frac{(\frac{1}{wt})^k}{k!}(n-k+1)_p(-\beta-n)_k P_{n+p-k}^{(\alpha-n-p+k,\,\beta)}(x). \qquad (16)
\end{aligned}
$$

It may be of interest to note that the mere replacement of α by $\alpha+n$ on both sides of $(14)-(16)$ above yields all the results obtained by B. Ghosh [18] while investigating generating functions of Jacobi polynomial using Weisner's group-theoretic method with the double interpretation of (n,α).

The above discussion reveals that the double interpretation of (n,α) for obtaining the generating functions found in [18] is not only cumbrous but also unnecessary and may be replaced by the single interpretation of the index (n) of the polynomial while investigating $P_n^{(\alpha-n,\beta)}(x)$ by the same technique of Weisner for obtaining the same results. This is what we wanted to prove.

Now we would like to mention that the above three results $(14)-(16)$, with the help of the symmetry relation:

$$
P_n^{(\alpha,\,\beta)}(-x) = (-1)^n P_n^{(\beta,\,\alpha)}(x), \qquad (17)
$$

give rise to the following generating relations,

$$
(1-t)^n P_n^{(\alpha,\,\beta-n)}\left(\frac{x-t}{1-t}\right) = \sum_{k=0}^{n}\frac{(-\alpha-n)_k}{k!}P_{n-k}^{(\alpha,\,\beta-n+k)}(x)\; t^k, \qquad (18)
$$

$$
\begin{aligned}
&(1+2t)^{\beta-n}\{1+t(1+x)\}^{-1-\alpha-\beta}P_n^{(\alpha,\,\beta-n)}\left(\frac{x+t(1+x)}{1+t(1+x)}\right) \\
&= \sum_{p=0}^{\infty}\frac{(-2t)^p}{p!}(n+1)_p P_{n+p}^{(\alpha,\,\beta-n-p)}(x), \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&(1+2t)^{\beta-n}\{1+t(1+x)\}^{-1-\alpha-\beta}\{1+\frac{1}{wt}(1+2t)\}^{n} \\
&\qquad \times P_n^{(\alpha,\,\beta-n)}\left(\frac{\{x+t(1+x)\}+(\frac{1}{wt})(1+2t)\{1+t(1+x)\}}{\{1+t(1+x)\}\{1+(\frac{1}{wt})(1+2t)\}}\right)
\end{aligned}
$$

$$= \sum_{p=0}^{\infty} \sum_{k=0}^{n+p} \frac{(-2t)^p}{p!} \frac{(-\frac{1}{wt})^k}{k!} (n-k+1)_p (-\alpha-n)_k P_{n+p-k}^{(\alpha,\,\beta-n-p+k)}(x). \quad (20)$$

Now one can easily note that the mere replacement of β by $\beta+n$ on both sides of (18) - (20) gives rise to the results found derived in [5] while investigating the generating functions of Jacobi polynomial $P_n^{(\alpha,\,\beta)}(x)$ by using Weisner's group-theoretic method with the double interpretation of (n,β).

Here also, for the same reasons as before, we see that the double interpretation of (n,β) may be replaced by the single interpretation of the index (n) while investigating the polynomial $P_n^{(\alpha,\beta-n)}(x)$ by the same technique of Weisner for obtaining the same results found derived in [5] in a straight-forward way - making the original problem simple and easier.

4. Derivation of Some Interesting Results

We now discuss the generating relation (13) to obtain some interesting results given below.

(a) Now replacing x by $(-x)$ on both sides of (13) and then using the symmetry relation (17),
we get

$$\left(1-\frac{a_2}{y}\right)^n \{1-2a_3y\left(1-\frac{a_2}{y}\right)\}^{\alpha-n}[1-a_3y\{(1+x)-2\frac{a_2}{y}\}]^{-1-\alpha-\beta}$$
$$\times P_n^{(\beta,\,\alpha-n)}\left(1-\frac{1-x}{(1-\frac{a_2}{y})[1-a_3y\{(1+x)-\frac{2a_2}{y}\}]}\right)$$
$$= \sum_{p=0}^{\infty} \sum_{k=0}^{n+p} (2a_3y)^p(-\frac{a_2}{y})^k \binom{n+p}{p} \binom{\alpha+n+p}{k} P_{n+p-k}^{(\beta,\,\alpha-n-p+k)}(x). \quad (21)$$

Finally, interchanging α, β and writing u, v in place of $(-\frac{a_2}{y})$ and $(2a_3y)$, we get

$$(1+u)^n\{1-v(1+u)\}^{\beta-n}[1-\frac{v}{2}\{(1+x)+2u\}]^{-1-\alpha-\beta}$$
$$\times P_n^{(\alpha,\,\beta-n)}\left(1-\frac{1-x}{(1+u)[1-\frac{v}{2}\{(1+x)+2u\}]}\right)$$

$$=\sum_{p=0}^{\infty}\sum_{k=0}^{n+p}\binom{n+p}{p}v^p\binom{\alpha+n+p}{k}u^k\times P_{n+p-k}^{(\alpha,\,\beta-n-p+k)}(x), \tag{22}$$

which is found derived in [13].

(b) Now putting $a_2 = 0$ in (21) and then replacing $2a_3y$ by u and finally interchanging α, β we get,

$$(1-u)^{\beta-n}\{1-\frac{u}{2}(1+x)\}^{-1-\alpha-\beta}P_n^{(\alpha,\,\beta-n)}\left(\frac{x-\frac{u}{2}(1+x)}{1-\frac{u}{2}(1+x)}\right)$$
$$=\sum_{p=0}^{\infty}u^p\binom{n+p}{p}P_{n+p}^{(\alpha,\,\beta-n-p)}(x), \tag{23}$$

which is found derived in [32, 33].

Again if we put $n = 0$ in (23), we get

$$(1-u)^{\beta}\{1-\frac{u}{2}(1+x)\}^{-1-\alpha-\beta}=\sum_{p=0}^{\infty}P_p^{(\alpha,\,\beta-p)}(x)u^p, \tag{24}$$

which was also derived by E. Feldheim [15], W. A. Al-salam [1] and V. K. Verma [34].

5. Comments on Some Recent Works

Recently, some results on generating functions involving Jacobi polynomials are obtained by P. K. Maiti and A. K. Chongdar [22], B. Samanta and K. P. Samanta [28], and K. P. Samanta [29] while investigating

(i) $P_n^{(\alpha+n,\beta)}(x)$ - a modification of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$,

(ii) ${}_2F_1(-n,\alpha,;\gamma+n;x)$ - a modification of Hypergeometric polynomial, ${}_2F_1(-n,\alpha;\gamma;x)$ and

(iii) $F_n(\alpha,\beta+n,x)$ - a modification of extended Jacobi polynomial, $F_n(\alpha,\beta;x)$

by Weisner's method with the double interpretation of (n,α) in case of (i), (n,γ) in case of (ii) and (n,β) in case of (iii) respectively.

In this section, we shall show that all the results involving Jacobi polynomials obtained in (i-iii) above can be easily obtained from our results (14)−(19) as follows.

If we replace

(a) α by $\alpha + 2n$ in the relations (14) - (16) and then simplify, we get the results found derived in [22],

(b) α by $\alpha + n$ in the relations (14) - (16) and then simplify, we get the results found derived in [28] and

(c) α by $\alpha + 2n$ in (14)-(15) and β by $\beta + 2n$ in (18)-(19) and simplify, we get all the results obtained in [29].

Thus we see that double interpretation in each of the above three cases (i-iii) is quite unnecessary for obtaining the generating functions of Jacobi polynomials found in [22, 28, 29] and may be replaced by the single interpretation of the index (n) while investigating the modified Jacobi polynomial, $P_n^{(\alpha-n,\beta)}(x)$ by the same technique of Weisner for obtaining the same results involving Jacobi polynomial found in [22, 28, 29] in a straightforward way.

6. Applications

In this section, we shall discuss some useful applications of the generating relation (15).

In fact, here we shall show that a number of theorems and results on bilateral, mixed trilateral and trilateral generating functions, which are of general interest and very much useful for generalizing the known results, can be easily obtained in course of application of our result (15). The following theorems and corollaries are the main results of this section.

Theorem 6.1. *If there exists a unilateral generating relation of the form:*

$$G(x,t) = \sum_{n=0}^{\infty} a_n P_{n+r}^{(\alpha-n,\beta)}(x)\, t^n, \tag{25}$$

then

$$\sum_{n=0}^{\infty} P_{n+r}^{(\alpha-n,\beta)}(x)\, \sigma_n(y)\, t^n$$

$$= (1+t)^{\alpha} \left(1+\frac{t}{2}(1-x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}, \frac{yt}{1+t}\right), \qquad (26)$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \binom{n+r}{p+r} y^p. \qquad (27)$$

Theorem 6.2. *If there exists a bilateral generating relation of the form:*

$$G(x,u,t) = \sum_{n=0}^{\infty} a_n \, P_{n+r}^{(\alpha-n,\beta)}(x) \, g_n(u) \, t^n, \qquad (28)$$

where $g_n(u)$ is an arbitrary polynomial of degree n, then

$$\sum_{n=0}^{\infty} P_{n+r}^{(\alpha-n,\beta)}(x) \, g_n(u,y) \, t^n$$
$$= (1+t)^{\alpha} \left(1+\frac{t}{2}(1-x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}, u, \frac{yt}{1+t}\right), \qquad (29)$$

where

$$g_n(u,y) = \sum_{p=0}^{n} a_p \binom{r+n}{r+p} g_p(u) \, y^p. \qquad (30)$$

Theorem 6.3. *If there exists a unilateral generating relation of the form:*

$$G(x,t) = \sum_{n=0}^{\infty} a_n \, P_{n+r}^{(\alpha-n,\beta)}(x) \, t^n, \qquad (31)$$

then

$$\sum_{n=0}^{\infty} P_{n+r}^{(\alpha-n,\beta)}(x) \, \sigma_n(z) \, T_n(u) \, w^n$$
$$= \frac{1}{2}(1+\rho_1)^{\alpha} \left(1+\frac{\rho_1}{2}(1-x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{\rho_1}{2}(1-x)}{1+\frac{\rho_1}{2}(1-x)}, \frac{z\rho_1}{1+\rho_1}\right)$$
$$+ \frac{1}{2}(1+\rho_2)^{\alpha} \left(1+\frac{\rho_2}{2}(1-x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{\rho_2}{2}(1-x)}{1+\frac{\rho_2}{2}(1-x)}, \frac{z\rho_2}{1+\rho_2}\right), \qquad (32)$$

where $T_n(u)$ is the Tchebycheff polynomial of degree n, $\rho_1 = t(u+\sqrt{u^2-1})$,

$\rho_2 = t(u - \sqrt{u^2-1})$, *and*

$$\sigma_n(z) = \sum_{p=0}^{n} a_p \binom{r+n}{r+p} z^p. \tag{33}$$

We now proceed to prove the above theorems serially.

To prove the theorems, we shall use the generating relation (15) which, on simplification, reduces to:

$$\begin{aligned}(1+t)^{\alpha}\left(1+\frac{t}{2}(1-x)\right)^{-1-\alpha-\beta-n} & P_n^{(\alpha,\beta)}\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}\right)\\ &=\sum_{p=0}^{\infty}\frac{(n+1)_p}{p!}P_{n+p}^{(\alpha-p,\beta)}(x)\, t^p.\end{aligned} \tag{34}$$

Again, (34), with the help of symmetry relation (17), becomes

$$\begin{aligned}(1-t)^{\beta}\left(1-\frac{t}{2}(1+x)\right)^{-1-\alpha-\beta-n} & P_n^{(\alpha,\beta)}\left(\frac{x-\frac{t}{2}(1+x)}{1-\frac{t}{2}(1+x)}\right)\\ &=\sum_{p=0}^{\infty}\frac{(n+1)_p}{p!}P_{n+p}^{(\alpha,\beta-p)}(x)\, t^p.\end{aligned} \tag{35}$$

***Proof of Theorem* 6.1.**

$$\begin{aligned}\text{L.H.S. of (26)} &= \sum_{n=0}^{\infty} P_{n+r}^{(\alpha-n,\beta)}(x)\,\sigma_n(y)\, t^n\\ &= \sum_{p=0}^{\infty} a_p (yt)^p \sum_{n=0}^{\infty}\frac{(r+p+1)_n}{n!}\, P_{n+r+p}^{(\alpha-n-p,\beta)}(x)\, t^n\\ &= (1+t)^{\alpha}\left(1+\frac{t}{2}(1-x)\right)^{-1-\alpha-\beta-r}\sum_{p=0}^{\infty} a_p\left(\frac{yt}{1+t}\right)^p P_{p+r}^{(\alpha-p,\beta)}\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}\right)\\ &\qquad\qquad\text{(using the relation (34))}\\ &= (1+t)^{\alpha}\left(1+\frac{t}{2}(1-x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}, \frac{yt}{1+t}\right)\\ &= \text{R.H.S. of (26)},\end{aligned}$$

which is found derived in [8, 12]. ☐

Corollary 6.4. *Putting $r = 0$ in the above theorem, we get the following result. If*

$$G(x,t) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha-n,\beta)}(x)\ t^n \tag{36}$$

then

$$\begin{aligned}&\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(x)\ \sigma_n(y)t^n \\ &\quad =(1+t)^{\alpha}\left(1+\frac{t}{2}(1-x)\right)^{-1-\alpha-\beta}\ G\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}, \frac{yt}{1+t}\right)\end{aligned} \tag{37}$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \binom{n}{p}\ y^p, \tag{38}$$

which is the correct version of the result found derived in [14].

***Proof of Theorem* 6.2.** The proof being exactly similar to that of Theorem 6.1, is left for the readers. This theorem is also found derived in [31] while unifying a class of trilateral generating functions for certain special functions. □

Corollary 6.5. *Putting $r = 0$ in the above thoerem, we get the result found derived in* [9, 24].

***Proof of Theorem* 6.3.**

$$\begin{aligned}\text{L.H.S. of (32)} &= \sum_{n=0}^{\infty} P_{n+r}^{(\alpha-n,\beta)}(x)\ \sigma_n(z)T_n(u)\ w^n \\ &= \sum_{n=0}^{\infty} P_{n+r}^{(\alpha-n,\beta)}(x)\ \sigma_n(z) \times \frac{1}{2}\left((u+\sqrt{u^2-1})^n + (u-\sqrt{u^2-1})^n\right)\ w^n \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty} P_{n+r}^{(\alpha-n,\beta)}(x)\ \sigma_n(z)\rho_1^n + \sum_{n=0}^{\infty} P_{n+r}^{(\alpha-n,\beta)}(x)\ \sigma_n(z)\rho_2^n\right) \\ &= \frac{1}{2}(1+\rho_1)^{\alpha}\left(1+\frac{\rho_1}{2}(1-x)\right)^{-1-\alpha-\beta-r}\ G\left(\frac{x-\frac{\rho_1}{2}(1-x)}{1+\frac{\rho_1}{2}(1-x)}, \frac{z\rho_1}{1+\rho_1}\right) \\ &+ \frac{1}{2}(1+\rho_2)^{\alpha}\left(1+\frac{\rho_2}{2}(1-x)\right)^{-1-\alpha-\beta-r}\ G\left(\frac{x-\frac{\rho_2}{2}(1-x)}{1+\frac{\rho_2}{2}(1-x)}, \frac{z\rho_2}{1+\rho_2}\right),\end{aligned}$$

by Theorem 6.1

$= \text{R.H.S. of (32)},$

where

$$\rho_1 = w(u + \sqrt{u^2 - 1}), \rho_2 = w(u - \sqrt{u^2 - 1}),$$

and

$$\sigma_n(z) = \sum_{p=0}^{n} a_p \binom{n+r}{p+r} z^p,$$

which does not seem to have appeared in the earlier works. □

Corollary 6.6. *Putting $r = 0$ in the above theorem, we get the following result:*

If

$$G(x,t) = \sum_{n=0}^{\infty} a_n \, P_n^{(\alpha-n,\beta)}(x) w^n, \tag{39}$$

then

$$\begin{aligned}
&\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(x) \, \sigma_n(z) T_n(u) \, w^n \\
&= \frac{1}{2}(1+\rho_1)^{\alpha} \left(1 + \frac{\rho_1}{2}(1-x)\right)^{-1-\alpha-\beta} G\left(\frac{x - \frac{\rho_1}{2}(1-x)}{1 + \frac{\rho_1}{2}(1-x)}, \frac{z\rho_1}{1+\rho_1}\right) \\
&\quad + \frac{1}{2}(1+\rho_2)^{\alpha} \left(1 + \frac{\rho_2}{2}(1-x)\right)^{-1-\alpha-\beta} G\left(\frac{x - \frac{\rho_2}{2}(1-x)}{1 + \frac{\rho_2}{2}(1-x)}, \frac{z\rho_2}{1+\rho_2}\right), \quad (40)
\end{aligned}$$

where

$$\rho_1 = w(u + \sqrt{u^2 - 1}), \rho_2 = w(u - \sqrt{u^2 - 1}),$$

and

$$\sigma_n(z) = \sum_{p=0}^{n} a_p \binom{n}{p} z^p, \tag{41}$$

which also does not seem to have appeared before.

6.1. Observation

Here it is easy to observe that the above theorems and the corollaries are not only very much important but also of general interest for their usefulness in generalizing the known results.

In fact, the importance of the above theorems (Theorems 6.1–6.3) lies in the fact that whenever one knows a generating relation of type: (a) (25) or

(b) (28) or (c) (31), the corresponding bilateral or mixed trilateral generating relation or trilateral generating relation with Tchebycheff polynomial can at once be written down from (i) (26) or (ii) (29) or (iii) (32). Thus, one can get a large number of bilateral or mixed trilateral or trilateral generating relations with Tchebycheff polynomial from (i) (26) or (ii) (29) or (iii) (32) by attributing different values to a_n in (a) (25) or (b) (28) or (c) (31).

Finally, we would like to point out that the above theorems, with the help of the symmetry relation (17), yield the following results analogous to the theorems 6.1 − 6.3.

Theorem 6.7. *If*

$$G(x,t)=\sum_{n=0}^{\infty} a_n P_{n+r}^{(\alpha,\beta-n)}(x)\, t^n, \tag{42}$$

then

$$\begin{aligned}\sum_{n=0}^{\infty} & P_{n+r}^{(\alpha,\beta-n)}(x)\,\sigma_n(y)\,t^n \\ &= (1-t)^{\beta}\left(1-\frac{t}{2}(1+x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{t}{2}(1+x)}{1-\frac{t}{2}(1+x)}, \frac{ty}{1-t}\right)\end{aligned} \tag{43}$$

where

$$\sigma_n(y)=\sum_{p=0}^{n} a_p \binom{n+r}{p+r} y^p, \tag{44}$$

which is found derived in [8, 12].

Theorem 6.8. *If*

$$G(x,u,t)=\sum_{n=0}^{\infty} a_n\, P_{n+r}^{(\alpha,\beta-n)}(x) g_n(u)\, t^n, \tag{45}$$

then

$$\begin{aligned}\sum_{n=0}^{\infty} & P_{n+r}^{(\alpha,\beta-n)}(x) g_n(u,z)\, t^n \\ &= (1-t)^{\beta}\left(1-\frac{t}{2}(1+x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{t}{2}(1+x)}{1-\frac{t}{2}(1+x)}, u, \frac{tz}{1-t}\right)\end{aligned} \tag{46}$$

where

$$g_n(u,z)=\sum_{p=0}^{n} a_p \binom{n+r}{p+r}\, g_p(z) z^p, \tag{47}$$

which is found derived in [31, 9].

Theorem 6.9. *If*

$$G(x,t)=\sum_{n=0}^{\infty} a_n P_{n+r}^{(\alpha,\beta-n)}(x)\, t^n, \tag{48}$$

then

$$\begin{aligned}\sum_{n=0}^{\infty} & P_{n+r}^{(\alpha,\beta-n)}(x)\sigma_n(z)\, T_n(u)t^n \\ =&\frac{1}{2}(1-\rho_1)^{\beta}\left(1-\frac{\rho_1}{2}(1+x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{\rho_1}{2}(1+x)}{1-\frac{\rho_1}{2}(1+x)}, \frac{z\rho_1}{1-\rho_1}\right) \\ &+\frac{1}{2}(1-\rho_2)^{\beta}\left(1-\frac{\rho_2}{2}(1+x)\right)^{-1-\alpha-\beta-r} G\left(\frac{x-\frac{\rho_2}{2}(1+x)}{1-\frac{\rho_2}{2}(1+x)}, \frac{z\rho_2}{1-\rho_2}\right),\end{aligned} \tag{49}$$

where $T_n(u)$ is Tchebycheff polynomial of degree n in u,

$$\rho_1=t(u+\sqrt{u^2-1}),\quad \rho_2=t(u-\sqrt{u^2-1})$$

and

$$\sigma_n(z)=\sum_{p=0}^{n} a_p\binom{n+r}{p+r}z^p. \tag{50}$$

Here we would like to mention that without taking recourse to symmetry relation (17), one can easily prove Theorems 6.7 − 6.9 by proceeding exactly in the same way as done in Theorems 6.1 − 6.3 and using the generating relation (35) in place of (34).

Corollary 6.10. *Putting $r=0$ in Theorem 6.7, we get the correct version of the result found derived in [10].*

Corollary 6.11. *Putting $r=0$ in Theorem 6.8, we get the result found derived in [24].*

Corollary 6.12. *Putting $r=0$ in Theorem 6.9, we get the result found derived in [3].*

The above three Theorems 6.7 − 6.9 are of same importance as that of the earlier three (Theorems 6.1 − 6.3), as mentioned in the Observation at the start of Subsection 6.1.

6.2. Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflicts of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by the authors.

References

1. W. A. Al-salam, Operational representation for the Laguerre and other polynomials, *Duke Math. Jour.*, **31** (1964), 127-142.

2. C. S. Bera, Lie algebra and modified Jacobi polynomials, *Journal of Science and Arts*, Year 13, No. 2 (23), 125-130, 2013.

3. N. C. Biswas and A. Chongdar, On a class of trilateral generating functions with Tchebycheff polynomials, *IJRAR*, **6** (2019), No. 2, 616-622.

4. N. C. Biswas and K. P. Samanta, On generating functions of modified Hyper-geometric polynomial by group theoretic method, *Bull. Cal. Math. Soc.*, **111** (2019), No. 2, 137-142.

5. A. B. Chakraborti, Group-theoretic study of certain generating functions of Jacobi polynomials, *J. Ind. Inst. Sci.*, **64**(B)(1983), 97-103.

6. A. K. Chongdar, Group-theoretic study of certain generating functions, *Bull. Cal. Math. Soc.*, **77** (1985), 151-157.

7. A. K. Chongdar, On Jacobi polynomials, *Bull. Cal. Math. Soc.*, **78**(1986), 363-374.

8. A. K. Chongdar, On a class of bilateral generating functions for certain special functions, *Proc. Indian Acad. Sci.* (Math. Sci.), **95** (1986), No. 2, 133-140.

9. A. K. Chongdar, On the extension of trilateral generating functions of extended Jacobi polynomials, *Rev. Academia de Ciencias, Zaragoza*, **50** (1995), 27-34.

10. A. K. Chongdar and S. K. Chatterjea, On a class of trilateral generating relations with Tchebycheff polynomials from the viewpoint of one parameter group of continuous transformations, *Bull. Cal. Math. Soc.*, **73** (1981), 127-140.

11. A. K. Chongdar and N. K. Majumdar, Some novel generating functions of extended Jacobi polynomials by group theoretic method, *Czechoslovak Mathematical Journal*, **46** (1996), No.121, 29-33.

12. A. K. Chongdar and M. C. Mukherjee, On a class of bilateral generating relations of certain special functions, *Bull. Inst. Math. Acad. Sinica*, **17** (1989), No. 1, 67-73.

13. M. K. Das, Some properties of special functions derived from theory of continuous transformation groups, *Proceedings of the Amer. Math. Soc.*, **35** (1972), No. 2, 565-573.

14. S. Das, On partial differential operators for Jacobi polynomials, *Pure Math. Manuscript*, **3** (1984), 95-104.

15. E. Feldheim, Relations entre les polynomes de Jacobi, Laguerre et Hermite, *Acta Math.*, **75** (1943), 117-138.

16. B. Ghosh, Group-theoretic study of certain generating functions of Jacobi polynomials, *Bull. Cal. Math. Soc.*, **75** (1983), No. 4, 227-231.

17. B. Ghosh, Group-theoretic origins of certain generating functions of Jacobi polynomials-I, *Pure Math. Manuscript*, **3**(1984), 139-144.

18. B. Ghosh, Group-theoretic origins of certain generating functions of Jacobi polynomials-II, *Pure Math. Manuscript*, **5** (1986), 21-27.

19. B. Ghosh, Group-theoretic origins of certain generating functions of Jacobi polynomials-III, *Pure Math. Manuscript*, **4** (1985), 5-10.

20. B. Ghosh, Group-theoretic study of certain generating functions involving modified Jacobi polynomials, *Bull. Cal. Math. Soc.*, **78** (1986), No. 3, 187.

21. B. K. Guhathakurta, Some generating functions of Jacobi polynomials, *Proc. Indian Acad. Sci.* (Math. Sci.), **95** (1986), No. 1, 53-59.

22. P. K. Maity and A. K. Chongdar, Some generating functions of modified Jacobi polynomials by Lie-algebraic method, *Ultra Scientist*, **24** (2012), No. 1B, 1-6.

23. E. B. McBride, *Obtaining Generating Functions*, Springer-Verlag, Berlin, 1971.

24. M. C. Mukherjee and A. K. Chongdar, On the unification of trilateral generating functions for certain special functions, *Tamkang Jour. Math.* **19** (1988), No. 3, 41-48.

25. S. K. Pan and A. K. Chongdar, On generating functions of certain orthogonal polynomials by group-theoretic method, *Sains Malaysiana*, **22** (1993), No. 4, 69-78.

26. S. K. Pan and A. K. Chongdar, Group-theoretic study of certain generating functions of modified extended Jacobi polynomials, *Indian Jour. Theoretical Phys.*, **42** (1994), No. 3, 203-209.

27. E. D. Rainville, *Special Functions*, Chelsea publishing company, Bronx, New york, 1960.

28. K. P. Samanta and B. Samanta, *On generating functions of Hypergeometric polynomials by group-theoretic method, J. of Ramannujan Society of Math. and Math. Sc.*, **4** (2015), No. 2, 51-60.

29. K. P. Samanta, Some generating functions of extended Jacobi polynomials by Lie algebraic method, *Journal of the Calcutta Mathematical Society*, **9** (2013), No. 2, 75-86.

30. R. Sharma and A. K. Chongdar, On generating functions of Jacobi polynomials by group-theoretic method, *Bull. Inst. Math. Acad. Sinica*, **20** (1992), No. 3, 253-259.

31. R. Sharma and A. K. Chongdar, Unification of a class of trilateral generating functions, *Bull. Cal. Math. Soc.*, **83** (1991), 481-490.

32. H. M. Srivastava, J. L. Lavoie and R. Tremblay, The Rodrigues type representations for a certain class of special functions, *Annali di Matematica Pura and Applicata*, **IV CXIX** (1979), 9.

33. H. M. Srivastava, Some bilateral generating functions for a certain class of special functions, *Proceedings of the Koninklijke Nederlandse Akademic Van Wetenschappen*, Series A, **83** (1980), No. 2, 221.

34. V. K. Verma, Double Hypergeometric functions as generating functions of the Jacobi and Laguerre polynomials, *Jour. Indian Math. Soc.*, **32** (1968), 1.

35. L. Weisner, Group - theoretic origin of certain generating functions, *Pacific J. Math.*, **5** (1955), 1033-1039.

36. L. Weisner, Generating functions for Hermite functions, *Canad. J. Math.*, **11** (1959), 141-147.

37. L. Weisner, Generating functions for Bessel functions, *Canad. J. Math.*, **11** (1959), 148-155.