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LARGE DEVIATIONS OF SUMS MAINLY DUE TO JUST ONE SUMMAND FOR SUPER-HEAVY TAILED DISTRIBUTIONS

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Abstract

Recently, Pinelis (2022) gave careful consideration that large deviations of the sums of independent and identically distributed random variables with power-like tails of index $\alpha \in (0, 2)$ are mainly due to just one of the summands. In this article, we show a corresponding result for $\alpha = 0$, which is constructed by *super-heavy* tailed distributions. The proof uses a property of *slow variation* investigated by Bojanic and Seneta (1971). It is applied to not only the *log-Pareto* distribution but also the distribution of the *super-Petersburg game*.

1. Introduction

1.1. Background

Throughout the article, let $X, X_1, X_2, \dots$ be independent and identically distributed random variables, whose common distribution function $F(x)$, the sum S_n, and the maximum M_n are defined by

$$F(x) = \mathrm{P}(X \leq x), \quad S_n = \sum_{i=1}^{n} X_i, \quad \text{and} \quad M_n = \max_{1 \leq i \leq n} X_i. \tag{1}$$

If the tail probability function $x \mapsto 1 - F(x) = \mathrm{P}(X > x)$ is *regularly varying of index* $\alpha \geq 0$ (see Embrechts et al. [2, Definition A3.1 (p. 564)]) then the

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tail of the maximum determines the tail of the sum, that is,

$$\mathrm{P}(S_n > x) \sim \mathrm{P}(M_n > x) \quad x \to \infty,$$

where $f(x) \sim g(x)$ denotes $\lim_{x\to\infty} f(x)/g(x) = 1$. This basic result is written in Embrechts et al. [2, Cor. 1.3.2 (p. 37)].

In this connection, Pinelis [11] gave careful consideration that large deviations of S_n are primarily due to just one of the summands when the tail function of X is power-like of index $\alpha \in (0, 2)$. He dealt with the case that X has a symmetric probability density function, however, this assumption is for clarity and not very important. A crucial point is that no assumption of regularly varying property is necessary. His discussion goes directly without the Tauberian theorem in the sense of Karamata (see Seneta [13, Theorem 2.3 (p. 59)]). Now, the distribution of the St. Petersburg game is a simple example in which the tail is heavy but does not satisfy the regularly varying property. For this game, the investigation of the maximum is still significant (see e.g. Fukker et al. [4] and [8]).

Pinelis [11] did not touch the case of $\alpha = 0$. In this case, the tail is very heavy, therefore it is called *super-heavy* in general. A concrete explanation for the super-heavy tailed distribution is written in Falk et al. [3, Section 2.7 (p. 80)], but it seems that there are not many studies on probability theory because it is not so easy to obtain interesting limit theorems. Recently, Nakata [9] provided a few results when truncating super-heavy tails. Besides, large deviations in the sense of Hu and Nyrhinen [7] were investigated by [10].

1.2. Our contributions

In this article, we show a corresponding result of Pinelis [11] for $\alpha = 0$. Our assumptions written in Section 2 below come from a property of *slow variation* (see [2, Definition A3.1 (p. 564)]) investigated by Bojanic and Seneta [1] (see Lemma 4.1 below). Although our assumptions may be strong, the assertion of Theorem 3.1, which is the main theorem, becomes simpler. The *log-Pareto* distribution is an obvious application (see Equation (9)), and the distribution of the *super-Petersburg game* (see Equation (10)) can be handled.

The plan of this article is as follows. Section 2 deals with the assumptions of Theorem 3.1, their remarks, and examples. In Section 3, we state the main theorem and the corollary, and give the proofs in Section 4.

2. Our assumptions

For a fixed $a > 0$ let $L : [a, \infty) \to \mathbb{R}$ be a continuously differentiable function such that

$$L(a) \geq 0, \quad L'(x) > 0 \quad \text{for } x \geq a, \quad \lim_{x\to\infty} L(x) = \infty, \tag{2}$$

and

$$\limsup_{x\to\infty} xL'(x) < \infty. \tag{3}$$

For example, functions $\log x, \log\log x$, and $\sqrt{\log x}$ fulfill conditions both (2) and (3). For a random variable X with (1) we assume (2), (3),

$$\mathrm{P}(X > x) \asymp \frac{1}{L(x)}, \quad \text{and} \quad \mathrm{P}(X \geq 0) = 1, \tag{4}$$

where $f(x) \asymp g(x)$ denotes $0 < \liminf_{x\to\infty} f(x)/g(x) \leq \limsup_{x\to\infty} f(x)/g(x) < \infty$ for positive functions $f(x)$ and $g(x)$.

Remark 2.1.

(i) In virtue of (4), X does not necessarily have a density function, and may be discrete. Lemma 4.1 below claims that $L(x)$ is slowly varying, but the tail function $x \mapsto \mathrm{P}(X > x)$ itself need not be slowly varying. It is an example of the *O-subexponential distributions* studied in Shimura and Watanabe [14, Section 1 (p. 447)]. Also, according to [9, Definition 1.1], it is called *O-super-heavy tailed.*

(ii) The precise condition of (4) is that there exist $0 < C_- < C_+ < \infty$ and $A > a$ such that

$$\frac{C_-}{L(x)} \leq \mathrm{P}(X > x) \leq \frac{C_+}{L(x)} \quad \text{for } x > A. \tag{5}$$

Since $L(x)$ fulfills (2), there are constants $\widetilde{A_+} > 0$, $\widetilde{A_-} > 0$, and $0 <$

$\delta < 1$ with

$$0 < C_\pm/L(x) < \delta < 1 \quad \text{for } x > \widetilde{A_\pm}. \tag{6}$$

Hence we can construct random variables $\widetilde{X_+}$ and $\widetilde{X_-}$ whose distribution functions $\widetilde{F_+}(x)$ and $\widetilde{F_-}(x)$ satisfy

$$\widetilde{F_\pm}(x) = \mathrm{P}(\widetilde{X_\pm} \le x) = \begin{cases} 1 - C_\pm/L(x) & \text{for } x \ge \widetilde{A_\pm}, \\ 0 & \text{for } x < \widetilde{A_\pm}, \end{cases} \tag{7}$$

respectively. Consequently, we have the stochastic domination

$$\mathrm{P}(\widetilde{X_-} > x) \le \mathrm{P}(X > x) \le \mathrm{P}(\widetilde{X_+} > x) \quad \text{for } x > A_*, \tag{8}$$

where $A_* = \max\{A, \widetilde{A_+}, \widetilde{A_-}\}$.

Example 2.1. We give two examples of random variables satisfying (2), (3), and (4).

(i) (The log-Pareto distribution, [5, Example 2.6.1]) Let X^{lP} be a random variable with $\mathrm{P}(X^{\mathrm{lP}} \le x) = \begin{cases} 1 - 1/\log x & \text{for } x > e, \\ 0 & \text{otherwise.} \end{cases}$ Since

$$\mathrm{P}(X^{\mathrm{lP}} > x) = 1/\log x \quad \text{for } x > e, \tag{9}$$

we have $L(x) = \log x, C_- = C_+ = 1$, and $A_* = e$.

(ii) (The distribution of the super-Petersburg game, [9, Example 4.2]) Let X^{sP} be a random variable with

$$\mathrm{P}(X^{\mathrm{sP}} = 2^{2^k}) = 2^{-k} \quad \text{for } k = 1, 2, \dots. \tag{10}$$

Since

$$\mathrm{P}(X^{\mathrm{sP}} > x) = \frac{2^{\{\lg \lg x\}}}{\lg x} \quad \text{for } x > 4,$$

we have $L(x) = \lg x, C_- = 1, C_+ = 2$, and $A_* = 4$, where $\lg x = \log x/\log 2$ and $\{x\}$ denotes the fractional part of $x > 0$.

3. Results

Our main result is the following.

Theorem 3.1. *Let $\beta < 0$ be a negative real number. For a random variable X with* (1)*, we assume* (2)*,* (3)*, and* (4)*, and put $c = L^{\beta}(x)$. Then we have*

$$1 \prec cx \prec x \tag{11}$$

and

$$\mathrm{P}(S_n > x) \sim \mathrm{P}\Big(S_n > x, \bigcup_{i\in[n]} \{X_i > x, S_n - X_i \le cx\}\Big), \tag{12}$$

whenever $n \in \mathbb{N}$ and $x \in (0, \infty)$ vary in such a way that

$$n \prec L(x), \tag{13}$$

where $[n] = \{1, 2, \ldots, n\}$, and $E \prec F$ or $F \succ E$ denotes $\lim E/F = 0$ *with positive expressions E and F in terms of x and n.*

Remark 3.1.

(i) In this article, both x and n are considered variables for asymptotic symbols. While Pinelis [11] treated not only x and n but also c as variables, c is put by $L^{\beta}(x)$ here.

(ii) We also use the notation $E = o(F)$ when $E \prec F$. Since $S_n - X_i \le cx \prec x$, Equation (12) means

$$\mathrm{P}\left(\begin{matrix}\text{Just only one } X_i \text{ satisfies } X_i > x,\\ \text{other than } X_i \text{ is } o(x)\end{matrix}\;\middle|\; S_n > x\right) \sim 1.$$

Corollary 3.1. *Under the assumptions of Theorem* 3.1*, we have*

$$\mathrm{P}(S_n > x) = o(1) \quad \textit{if and only if} \quad (13) \textit{ follows.} \tag{14}$$

Moreover, if (13) *holds then we have*

$$\mathrm{P}(S_n > x) \asymp \frac{n}{L(x)}. \tag{15}$$

4. Proofs

The outline of the proofs is the same as [11]. To prove Theorem 3.1, we use four lemmas.

Lemma 4.1. *Suppose that $L(x)$ satisfies* (2) *and* (3). *Then $x \mapsto L(x)$ is a slowly varying function, and*

$$\lim_{x\to\infty} \frac{L(xL^\gamma(x))}{L(x)} = 1 \quad \textit{for } \gamma \in \mathbb{R}. \tag{16}$$

Proof. Combining (2) and (3) leads to

$$\lim_{x\to\infty} \frac{xL'(x)}{L(x)} = 0,$$

which implies that $x \mapsto L(x)$ is a slowly varying function by Resnick [12, Proposition 2.5 (a) (p. 31)] (see also Seneta [13, Equation (1.11) (p. 7)]). Moreover, using (3) and $\lim_{x\to\infty} (\log L(x))/L(x) = \lim_{t\to\infty}(\log t)/t = 0$, we have

$$\lim_{x\to\infty} \frac{xL'(x)}{L(x)} \log L(x) = 0.$$

This assures us that (16) follows from Bojanic and Seneta [1, Theorem 1 (p. 304)]. □

Lemma 4.2. *Under the assumptions of Theorem* 3.1 *and* (13), *we have*

$$\mathrm{P}(S_n > x) \succcurlyeq n\mathrm{P}(X_1 > x) \asymp \frac{n}{L(x)}, \tag{17}$$

where $E \preccurlyeq F$ or $F \succcurlyeq E$ denotes $\limsup E/F < \infty$.

Proof. By (4) we obtain the asymptotic property $\asymp$ of (17). It follows from (6) that $0 < C_-/L(x) < \delta$ for $x > A_*$. Noting $X \geq 0$ and (5), we have for $x > A_*$

$$\begin{aligned} \mathrm{P}(S_n > x) &\geq \mathrm{P}(M_n > x) = 1 - \{1 - \mathrm{P}(X > x)\}^n \geq 1 - \left(1 - \frac{C_-}{L(x)}\right)^n \\ &\overset{(*)}{\geq} \frac{(1-\delta)C_- n}{L(x)}, \end{aligned}$$

which yields $\succcurlyeq$ of (17). The last inequality of $(*)$ follows by applying Gut [6, Lemma A.4.2 (p. 561)] to $n/L(x) = o(1)$ by (13). $\square$

Lemma 4.3. *For $c = L^{\beta}(x)$ with $\beta < 0$, we have*

$$1 \prec c^2 x \prec x. \tag{18}$$

In addition, if (13) holds then we obtain

$$n \prec \frac{(L(cx))^2}{L(x)} \tag{19}$$

and

$$n \prec \frac{(L(c^2x))^2}{c}. \tag{20}$$

Proof. Since $L(x)$ is slowly varying, so is $x \mapsto L^{2\beta}(x)$ from Seneta [13, Item 3° (p. 18)]. Therefore, Seneta [13, Item 1°] yields $1 \prec xL^{2\beta}(x) = c^2 x$. It turns out that $c^2 x \prec x$ because of $\beta < 0$ and (2). Thus we have (18), and (11) is also proved in the same manner. Using (11), (18), and (16), we obtain

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = \lim_{x\to\infty} \frac{L(c^2x)}{L(cx)} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{L(x)}{c} = \infty. \tag{21}$$

Hence it follows from (13) that

$$n \prec L(x) \sim L(cx) \sim \frac{L(cx)}{L(x)} L(cx),$$

which gives (19). Similarly, since

$$n \prec L(x) \prec L(x) \left(\frac{L(c^2x)}{L(cx)}\right)^2 \left(\frac{L(cx)}{L(x)}\right)^2 \frac{L(x)}{c} = \frac{(L(c^2x))^2}{c},$$

Equation (20) holds. $\square$

To prove Theorem 3.1, we define the following probabilities

$$p_0(n,x) = \mathrm{P}\left(S_n > x, \max_{j\in[n]} X_j \le cx\right), \tag{22}$$

$$p_{\ge 2}(n,x) = \mathrm{P}\Big(S_n > x, \bigcup_{i\in[n]} \bigcup_{j\in[n]\setminus\{i\}} \{X_i > cx, X_j > cx\}\Big), \tag{23}$$

$$p_{1,0}(n,x) = \mathrm{P}\Big(S_n > x, \bigcup_{i\in[n]} \Big\{cx < X_i \le x, \max_{j\in[n]\setminus\{i\}} X_j \le cx\Big\}\Big), \quad (24)$$

$$p_{1,1}(n,x) = \mathrm{P}\Big(S_n > x, \bigcup_{i\in[n]} \Big\{X_i > x, \max_{j\in[n]\setminus\{i\}} X_j \le cx\Big\}\Big), \quad (25)$$

and the random variables

$$Y_i = X_i \mathbb{I}\{|X_i| \le cx\} \quad \text{for } i = 1,\dots,n, \quad \text{and} \quad T_n = \sum_{i=1}^{n} Y_i, \quad (26)$$

where $\mathbb{I}\{B\}$ denotes the indicator random variable for the event B.

Lemma 4.4. *Under the assumptions of Theorem 3.1, we have*

$$p_0(n,x) = o\left(\frac{n}{L(x)}\right), \quad (27)$$

$$p_{\ge 2}(n,x) = o\left(\frac{n}{L(x)}\right), \quad (28)$$

$$p_{1,0}(n,x) = o\left(\frac{n}{L(x)}\right). \quad (29)$$

4.1. Proof of Lemma 4.4

In the following proofs, let $x > 0$ be taken sufficiently large for A_*.

(i) **Proof of (27):** We show

$$\mathrm{E}Y_1^2 \preccurlyeq \left(\frac{cx}{L(cx)}\right)^2 \quad (30)$$

and

$$\mathrm{P}\left(T_n > x\right) \preccurlyeq n \left(\frac{c}{L(cx)}\right)^2. \quad (31)$$

Since $\int_0^{A_*} u^2 dF(u) < \infty$ and (8), we have

$$\mathrm{E}Y_1^2 = \int_0^{A_*} u^2 dF(u) + \int_{A_*}^{cx} u^2 dF(u) \le 2 \int_{A_*}^{cx} u^2 d\widetilde{F_+}(u).$$

Using integration by parts (see Gut [6, Theorem 2.12.1 (ii) (p. 75)]) and (3), we get

$$\begin{aligned}
\int_{A_*}^{cx} u^2 d\widetilde{F_+}(u) &= \left[-u^2(1-\widetilde{F_+}(u))\right]_{A_*}^{cx} + \int_{A_*}^{cx} (u^2)'(1-\widetilde{F_+}(u))du \\
&= C_+ \left\{ -\frac{(cx)^2}{L(cx)} + \frac{A_*^2}{L(A_*)} + \int_{A_*}^{cx} \frac{2udu}{L(u)} \right\} \\
&\asymp -\frac{(cx)^2}{L(cx)} + \int_{A_*}^{cx} \frac{(u^2)'du}{L(u)} \asymp \int_{A_*}^{cx} \frac{u^2 L'(u)du}{(L(u))^2} \\
&= \int_{A_*}^{cx} uL'(u) \times \frac{u}{(L(u))^2} du \overset{(3)}{\preccurlyeq} \int_{A_*}^{cx} \frac{u}{(L(u))^2} du \\
&\sim \frac{1}{2}\left(\frac{cx}{L(cx)}\right)^2,
\end{aligned}$$

which proves (30). Here the last $\sim$ follows from the Karamata theorem (see [2, Theorem A3.4 (p. 567)]). Thus, it turns out that

$$\mathrm{P}\left(T_n > x\right) \leq \frac{\mathrm{E}T_n^2}{x^2} = \frac{n\mathrm{E}Y_1^2}{x^2} \preccurlyeq n\left(\frac{c}{L(cx)}\right)^2,$$

which yields (31). Hence we have

$$\begin{aligned}
p_0(n,x) &= \mathrm{P}\left(S_n > x, \max_{j\in[n]} |X_j| \leq cx\right) \leq \mathrm{P}\left(T_n > x\right) \\
&\preccurlyeq n\left(\frac{c}{L(cx)}\right)^2 = \frac{nL(x)}{(L(cx))^2} \cdot \frac{c^2}{n}\frac{n}{L(x)} \overset{(19)}{\prec} \frac{n}{L(x)}.
\end{aligned} \tag{32}$$

(ii) <u>**Proof of (28):**</u> This is followed by

$$\begin{aligned}
p_{\geq 2}(n,x) &\leq \mathrm{P}\left(\bigcup_{1\leq i\neq j\leq n} \{X_i > cx, X_j > cx\}\right) \\
&\leq \binom{n}{2}\mathrm{P}(X_1 > cx, X_2 > cx) \asymp \left(\frac{n}{L(cx)}\right)^2 \\
&= \frac{nL(x)}{(L(cx))^2}\frac{n}{L(x)} \overset{(19)}{\prec} 1 \times \frac{n}{L(x)}.
\end{aligned}$$

(iii) **Proof of (29):** The probability is estimated by

$$\begin{aligned} p_{1,0}(n,x) &\le n\mathrm{P}\left(cx < X_1 \le x, X_1 + \sum_{j=2}^{n} Y_j > x \right) \\ &\asymp \int_{cx}^{x} n\mathrm{P}\left(T_{n-1} > x-u\right) dF(u) \preccurlyeq \int_{cx}^{x} g(u) dF(u), \end{aligned} \tag{33}$$

where

$$g(u) = n \min\left\{ 1, n\left(\frac{c}{L(c(x-u))}\right)^2 \right\} \quad \text{for } cx < u < x.$$

The last $\preccurlyeq$ of (33) follows from (31). We define

$$u_x = x - \frac{L^{-1}(\sqrt{cn})}{c}, \tag{34}$$

where $L^{-1}(x)$ is the inverse function of $L(x)$. Note that $L^{-1}(x)$ for $x > 0$ exists because of the strict monotonicity and continuity of $L(x)$. We obtain

$$u_x \sim x. \tag{35}$$

Since $\sqrt{cn} \le L(c^2x)$ by (20), we have $L^{-1}(\sqrt{cn}) \le c^2x$. Therefore (35) holds. From this and (11), we get

$$cx < x/2 < u_x < x. \tag{36}$$

Dividing the right-hand side of (33) into three parts

$$\int_{cx}^{x} g(u)dF(u) = \int_{cx}^{x/2} + \int_{x/2}^{u_x} + \int_{u_x}^{x} = I_1 + I_2 + I_3,$$

we show $I_i \prec n/L(x)$ for $i = 1, 2$, and 3.

- I_1 : Since $u \mapsto 1/L(c(x-u))$ is monotone increasing on $cx < u < x/2$, and $\lim_{cx\to\infty} L(cx/2)/L(cx) = 1$, we have

$$I_1 = \int_{cx}^{x/2} \left(\frac{cn}{L(c(x-u))}\right)^2 dF(u)$$

$$
\begin{aligned}
&\le \left(\frac{cn}{L(cx/2)}\right)^2 \int_{cx}^{x/2} dF(u) \le \left(\frac{cn}{L(cx/2)}\right)^2 \mathrm{P}(X > cx) \\
&\preccurlyeq \left(\frac{cn}{L(cx/2)}\right)^2 \frac{1}{L(cx)} = \left(\frac{c}{\frac{L(cx/2)}{L(cx)}}\right)^2 \frac{nL(x)}{(L(cx))^2} \frac{1}{L(cx)} \frac{n}{L(x)} \\
&\overset{(19)}{\prec} \frac{n}{L(x)}.
\end{aligned}
$$

- I_2 : Since $u \mapsto 1/L(c(x-u))$ is also monotone increasing on $x/2 < u < u_x$, and $L(c(x-u_x)) = \sqrt{cn}$, we obtain

$$
\begin{aligned}
I_2 &= \int_{x/2}^{u_x} g(u) dF(u) \le \left(\frac{cn}{L(c(x-u_x))}\right)^2 \int_{x/2}^{u_x} dF(u) \\
&\le cn\mathrm{P}(X > x/2) \asymp \frac{cn}{L(x/2)} \prec \frac{n}{L(x)}.
\end{aligned}
$$

- I_3 : Equation (35) yields $1 \le L(x)/L(u_x) \le L(x)/L(x/2) \overset{x\to\infty}{\to} 1$. Hence it follows that

$$
\begin{aligned}
I_3 &= \int_{u_x}^{x} g(u) dF(u) \le n \int_{u_x}^{x} dF(u) \asymp n\left(\frac{1}{L(u_x)} - \frac{1}{L(x)}\right) \\
&= \left(\frac{L(x)}{L(u_x)} - 1\right) \frac{n}{L(x)} \prec \frac{n}{L(x)}.
\end{aligned}
$$

4.2. Proofs of Theorem 3.1 and Corollary 3.1

4.2.1. Proof of Theorem 3.1

Equation (18) is shown in Lemma 4.3. By Lemmas 4.2 and 4.4, we obtain

$$
\mathrm{P}(S_n > x) \sim p_{1,1}(n,x) \succcurlyeq \frac{n}{L(x)}. \tag{37}
$$

Lemma 4.4 yields

$$
\begin{aligned}
\frac{n}{L(x)} &\overset{(17)}{\preccurlyeq} \mathrm{P}(S_n > x) = p_0(n,x) + p_{\ge 2}(n,x) + p_{1,0}(n,x) + p_{1,1}(n,x) \\
&= p_{1,1}(n,x) + o\left(\frac{n}{L(x)}\right) \sim p_{1,1}(n,x),
\end{aligned}
$$

which proves (37). Moreover, using (26) we have

$$\begin{aligned}
& |\text{RHS of } (12) - p_{1,1}(n,x)| \\
\leq\ & n\mathrm{P}\left(X_1 > x, S_n - X_1 > cx, \max_{j\in[n]\setminus\{1\}} X_j \leq cx\right) \\
\leq\ & n\mathrm{P}\left(X_1 > x, \sum_{j=2}^{n} Y_j > cx\right) = n\mathrm{P}\left(X_1 > x\right)\cdot \mathrm{P}\left(T_{n-1} > cx\right) \\
\overset{(17),(31)}{\preccurlyeq}\ & \mathrm{P}\left(S_n > x\right)\cdot n\left(\frac{c}{L(c^2x)}\right)^2 = \mathrm{P}\left(S_n > x\right)\cdot c\cdot\frac{nc}{(L(c^2x))^2} \\
\overset{(20)}{\prec}\ & \mathrm{P}(S_n > x),
\end{aligned}$$

which yields (12).

4.2.2. Proof of Corollary 3.1

Let us assume (13). Then we have

$$\mathrm{P}(S_n > x) \sim p_{1,1}(n,x) \leq n\mathrm{P}\left(X_1 > x\right) \overset{(5)}{\leq} C_+\frac{n}{L(x)} = o(1). \tag{38}$$

Next, we assume $\mathrm{P}(S_n > x) = o(1)$. It follows from [6, Equation (2.1) (p. 270)] that there exists $C > 0$ such that

$$o(1) = \mathrm{P}(S_n > x) \geq \mathrm{P}(M_n > x) \geq Cn\mathrm{P}(X_1 > x) \asymp \frac{n}{L(x)}, \tag{39}$$

which proves (13). Finally, if (13) holds then $\mathrm{P}(S_n > x) \overset{(38)}{\preccurlyeq} n/L(x) = o(1)$. Therefore $\mathrm{P}(S_n > x) \succcurlyeq n/L(x)$, since (39) is true. Hence we have (15).

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