Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **18** (2023), No. 4, pp. 419-436
DOI: 10.21915/BIMAS.2023402

CODIMENSION ONE FOLIATIONS WITH COUPLING OF SADDLE SINGULARITIES

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Abstract

On closed manifolds, we investigate smooth foliations with Morse singularities in codimension one. In a dead branch, it has been investigated how to join two saddle singular points with complementary indices. We drive a description of the manifolds exhibiting c center and s saddle singular points in $\mathrm{sing}(\mathfrak{F})$ satisfying $\mathrm{c} \geq \mathrm{s} + 1$, alternatively, in the case where $\mathrm{c} > \mathrm{s} - 2k$, there are at least k pairs of saddle singular points that are in stable connection. These findings are extension of Camacho-Scardua results, which describe the topology of three and $n-$dimensional manifolds, which are in fact, an extension of earlier findings by Reeb for foliations with only centre singularities, result of Wagneur for foliations containing Morse singularities and Eells and Kuiper for manifolds containing three singularities for the Morse functions.

1. Introduction

Foliations with Morse singularities are a fascinating and important concept in the field of differential geometry and topology. They provide a powerful framework to study the geometry and topology of smooth manifolds, revealing intriguing patterns and structures within these spaces. Morse singularities play a crucial role in understanding the behavior of foliations. These singularities occur at critical points of a certain real-valued function, known as the Morse function, defined on the manifold. A critical point is a

Received 28 August, 2023.
AMS Subject Classification: Primary 57R30, 57R32; Secondary 53C12.
Key words and phrases: Compact foliations, stable singularities, Reeb stability theorem, Morse lemma.

location where the gradient of the function vanishes, and its behavior around this point resembles that of a paraboloid. The idea of using Morse functions in the context of foliations comes from Morse theory, which connects the topology of a manifold to the critical points of a smooth function defined on it. By employing Morse functions in the study of foliations, researchers can gain valuable insights into the global structure and topological properties of the manifold and its foliation.

Definition 1.1. Morse foliation is a transversely oriented codimension one foliation $\mathfrak{F}$ of class C^2 on $\mathbb{M}$ with isolated singularities if:

(i) The singularities in $\mathfrak{F}$ are of the Morse type.

(ii) There are no saddle connections in $\mathfrak{F}$.

Foliations with Morse singularities find applications in various branches of mathematics and theoretical physics. They have connections to the theory of dynamical systems, where they provide a geometric understanding of the flow of vector fields on manifolds. Additionally, they have been used in the study of low-dimensional topology, Riemannian geometry, and even in theoretical physics, particularly in the context of string theory and gauge theory.

Let $\mathfrak{F}_0$ is a regular codimension one foliation on an open non-singular manifold $\mathbb{M}\backslash sing(\mathfrak{F})$. The pair $(\mathfrak{F}_0, sing(\mathfrak{F}))$ is a codimension one foliation [1] exhibiting isolated singularities on a compact manifold $\mathbb{M}$ [6], where $sing(\mathfrak{F})$ is a subset of isolated singular points of manifold $\mathbb{M}$. Let $f : \mathcal{O} \longrightarrow \mathbb{R}$ be a C^∞ function, for $p \in \mathcal{O} \subset \mathbb{M}$ such that $sing(\mathfrak{F}) \cap \mathcal{O} = \{p\}$. Point p is said to be Morse singularity if $\mathfrak{F} \mid_{\mathcal{O}}$ is given by $df = 0$ for a critical point p at which Hessian matrix is non-singular. If the index of a critical point p for which Hessian matrix non-singular for a function of n variables is n or 0, then $p \in sing(\mathfrak{F})$ is a centre type singularity; otherwise, it is a saddle type singularity [6]. The number of negative eigenvalues in the Hessian matrix at the critical point p is the index of a critical point p for which Hessian matrix is non-singular for a function of n variables. In the neighbourhood of a centre singularity, the leaves of foliation $\mathfrak{F}$ are diffeomorphic to $\mathsf{S}^{(n-1)}$ spheres. Cone leaves, also known as separatrices of $\mathfrak{F}$ through p, exist in the neighbourhood of a saddle singularity $p \in sing(\mathfrak{F})$. Saddle connection refers to a mathematical leaf that contains the separatrices of two different mathematical saddles.

Understanding the topology and geometry of smooth manifolds relies significantly on Morse theory and its applications. The idea of Morse foliations is one fundamental idea that has resulted from this theory. We examine the pioneering works and innovative studies in the study of Morse foliations in this literature review. Gelbukh, I. [8] has considered the foliation of a Morse form ω on a closed manifold $\mathbb{M}$. Its maximal components (cylinders formed by compact leaves) form the foliation graph; the cycle rank of this graph is calculated. The number of minimal and maximal components is estimated in terms of characteristics of $\mathbb{M}$ and ω. Gelbukh, I. [9] has proved a criterion for compactness of the singular foliation $\mathfrak{F}_\omega$ to estimate the number of its minimal components, and to give an upper bound on the rank $rk\omega$, in terms of genus for a Morse form ω. Limón and Seade [10] have studied germ of holomorphic foliation at origin by looking at the intersection of its leaves with the level sets of real analytic function g, and the way in which these intersections change as the sphere gets smaller. Charitos has studied [11] Morse foliations of codimension 1 on the sphere S^3 and the existence of special components for these foliations is derived. Corro Diego [12] has shown that a singular Riemannian foliation of codimension two on a compact simply connected Riemannian $(n+2)$-manifold, with regular leaves homeomorphic to the $n-$ torus, is given by a smooth effective $n-$ torus action. Leon and Scardua [13] has introduced a notion of L-stability for the singularity, similar to Lyapunov stability. We prove that L-stability is equivalent to the existence of a holomorphic first integral, or the foliation is a real logarithmic foliation.

A manifold with singularities can be homeomorphic to a sphere under certain conditions. The precise conditions depend on the nature of the singularities and the dimension of the manifold. There are several important results and conjectures in this area, some of which are related to the Poincaré conjecture and its generalisations. We will examine the idea of pairing and removing two stable coupled saddle singular points [2] with complementary indices. We presume that at least k pairs of saddle singular points are connected in a stable connection (When two saddle singular points with complementary indices have stable and unstable manifolds respectively, they overlap transversely in a smooth connection). The aim of this work is to extend [2] the Camacho-Scardua result [3] by implementing the technique of coupling and eliminating pairs of complementary saddles, according to which a closed connected and orientated manifold of dimension three that admits Morse foliation and has more centre singularities than saddles is diffeomorphic to three spheres. Note that if there are more saddles than saddle singular points, or if $\mathrm{S} > \mathrm{C}$, the aforementioned condition in the assertion is automatically satisfied.

Theorem 1.2 (Reeb Sphere Theorem [15])**.** *Let M^n be a closed manifold of dimension $n \geq 3$ admitting transversely orientable Morse foliation with only centre type singularities, then M^n is homeomorphic to the n-sphere.*

This classical result of Reeb [7], [15] for Morse foliations with centre singular points was generalised by E. Wagneur [18] . He demonstrated that there is a limit on the number of centres corresponding to the number of saddles, particularly for the case of $\mathrm{C} \geq \mathrm{S} + 2$. So, there are only two situations in which $\mathrm{C} > \mathrm{S}$ occurs: either $\mathrm{C} = \mathrm{S} + 2$ or $\mathrm{C} = \mathrm{S} + 1$. With the use of singularities that satisfy $\mathrm{C} = \mathrm{S} + 2$, he was able to describe the manifold. Afterwards, Camacho and Scardua [4] argued about case (2), $\mathrm{C} = \mathrm{S} + 1$. It's interesting that in a few low dimensions, this is achievable.

We further develop the n-dimensional scenario [4], which is really just an extension of E. Wagner's [16], [17], [18], which characterises the topology of an n-dimensional manifold allowing a foliation with more centre singularities than saddle singular points through using approach of coupling and elimination saddle singularity.

The cancellation of a pair of centre and saddle singular points in a dead branch, a region in which there is a trivial pair of centre and saddle singular points, has been used to demonstrate the Camacho-Scardua results [3], [4]. In this study, we show that pair of saddle singular points exhibiting complementary indices can be combined in a dead branch, and by eliminating pair of saddle singular points of complementary indices that are connected in a stable connection (One saddle singularity's stable manifold meets smoothly and transversally with another saddle singularity's unstable manifold, both of which have complementary indices), we will generalise those findings [2].

2. Preliminaries

2.1. Dead Branches

Definition 2.1. Assume that $\mathbb{M}$ is a n-dimensional manifold that admits a foliation in codimension one with isolated singularities. Dead branch of $\mathfrak{F}$ is a region $\mathcal{R}$ of manifold $\mathbb{M}$ such that

$$\mathcal{R} \cong \mathbb{B}^{n-1} \times \mathbb{I}$$

where $\mathbb{B}^{n-1}$ is the closed unit ball and the interval $\mathbb{I} = [0, 1]$ is unit interval. So $\mathcal{R}$ is a manifold with boundary. Assume that the dead branch $\mathcal{R}$ contains some isolated singular points. The boundary of the region $\mathcal{R}$ is given as:

$$\partial\mathcal{R} = \partial\mathbb{B}^{n-1} \times \mathbb{I} \cup \mathbb{B}^{n-1} \times \partial\mathbb{I}$$

where $\partial\mathbb{B}^{n-1} \times \mathbb{I}$ segments transverse to $\mathfrak{F}$, say Σ_1 and Σ_2 and $\mathbb{B}^{n-1} \times \partial\mathbb{I}$ segments of leaves of $\mathfrak{F}$ say $\mathcal{L}_1$ and $\mathcal{L}_2$, which are invariant and connected components, as depicted in Figure 1, so

$$\partial\mathcal{R} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \Sigma_1 \cup \Sigma_2.$$

To restore the foliation within a particular dead branch, use the trivial foliation. The coupling of the centre and the saddle singularity is the trivial example of a dead branch. Moreover, while a diffeomorphism $h : \Sigma_1 \to \Sigma_2$ allows $\mathfrak{F}|_{\Sigma_1}$ and $\mathfrak{F}|_{\Sigma_2}$ to be conjugated such that $\mathcal{L}_{h(p)} = [\mathcal{L}_p, \forall p \in \Sigma_1$, as

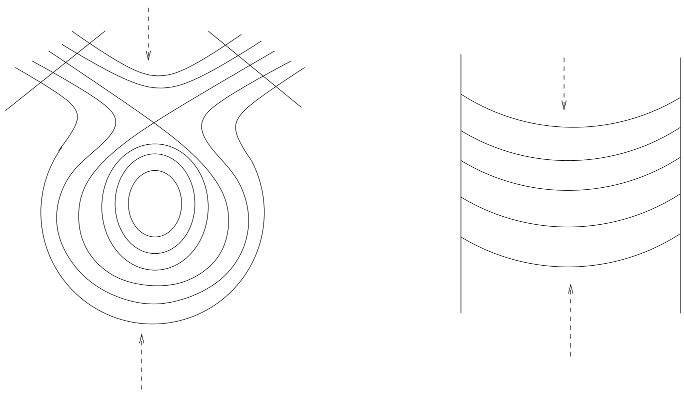

Figure 1: Dead Branch

long as p corresponds to a leaf carrying a separatrix in region $\mathcal{R}$ of some singularity of $\mathfrak{F}$, the holonomy from Σ_1 to Σ_2 is trivial.

Definition 2.2. Let region $\mathcal{R}$ be a dead branch of a foliation $\mathfrak{F}$ on $\mathbb{M}$. Trivial coupling occurs when pair of singularities p_1, p_2 of foliation $\mathfrak{F}$ on $\mathbb{M}$ are in a dead branch $\mathcal{R}$ and there are no other singularities of foliation $\mathfrak{F}$ in $\mathcal{R}$.

Proposition 2.3 ([3])**.** *A foliation $\tilde{\mathfrak{F}}$ on manifold $\mathbb{M}$ can be created from a codimension one foliation $\mathfrak{F}$ exhibiting discrete singular points on the manifold $\mathbb{M}$ in a dead branch region $\mathcal{R} \subset \mathbb{M}$ such that:*

(i) *Initial and modified foliations $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ coincide on all of $\mathbb{M}$ except $\mathcal{R}$.*

(ii) *In the neighbourhood of $\mathcal{R}$, the modified foliation $\tilde{\mathfrak{F}}$ is not singular; in fact, modified foliation $\tilde{\mathfrak{F}}$ in the region $\mathcal{R}$ is equivalent to a trivial fibration.*

(iii) *In the following way, the holonomy for leaf $\tilde{\mathcal{L}}$ of transformed foliation $\tilde{\mathfrak{F}}$ is equivalent to the holonomy for the leaf $\mathcal{L}$ of initial foliation $\mathfrak{F}$.*

Definition 2.4. Foliation $\tilde{\mathfrak{F}}$ will be referred to as a modification of the initial foliation $\mathfrak{F}$ by removing the dead branch. Foliation $\mathfrak{F}$ is referred to as the inverse modification of $\tilde{\mathfrak{F}}$ if it is created by introducing a dead branch into a foliation $\tilde{\mathfrak{F}}$ [4].

An illustration of pairing of centre and saddle singular points, where the saddle singularity is made up of spherical leaves from a third centre singularity. We start with a foliation determined by a centre singularity, and using an inverse modification, we incorporate a pair of centre-saddles into a regular component, as shown in Figure 2.

A self connection is provided by the saddle's separatrix, which has the topology of S^2 with a single intersection point. The region enclosed by one internal leaf $\mathcal{L}_1$ and one exterior leaf $\mathcal{L}_2$ has non-trivial centre-saddle pairing since all leaves are diffeomorphic to spheres.

2.2. Topology of separatrices

Suppose that $\mathbb{M}^n, n \geq 3$ is a compact n-dimensional manifold that admits the Morse foliation $\mathfrak{F}$. We represent the set $\mathcal{C}(\mathfrak{F})$ as union of all leaves diffeomorphic to S^{n-1} and centre singularities in $\mathbb{M}^n$. We refer to the set

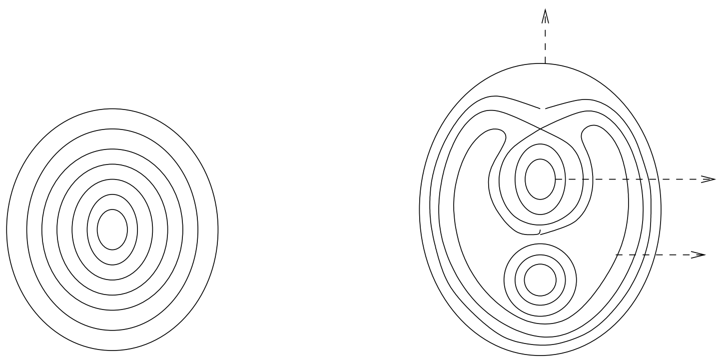

Figure 2: Inverse modification

$\mathcal{C}_{p_1}(\mathfrak{F})$ as the connected component of $\mathcal{C}(\mathfrak{F})$ that comprises singularity p_1 for a given centre singularity $p_1 \in \operatorname{sing}(\mathfrak{F})$.

Remark 2.5. Thus, $\mathcal{C}(\mathfrak{F})$ and $\mathcal{C}_{p_1}(\mathfrak{F})$ are open in $\mathbb{M}$ as a result of the Reeb local stability theorem. Intersection of $\mathcal{C}_{p_1}(\mathfrak{F})$ and $\mathcal{C}_{p_2}(\mathfrak{F})$ is not empty if and only if $\mathcal{C}_{p_1}(\mathfrak{F}) = \mathcal{C}_{p_2}(\mathfrak{F})$. Given that p_2 is a singular point in the foliation $\mathfrak{F}$, it follows that p_2 is a saddle singularity if $p_2 \in \partial\mathcal{C}(\mathfrak{F})$. $\mathcal{C}_{p_1}(\mathfrak{F})$ is open in $\mathbb{M}$. If we have $\partial\mathcal{C}_{p_1}(\mathfrak{F}) = \emptyset$ then $\mathcal{C}_{p_1}(\mathfrak{F}) = \mathbb{M}$ and vice versa. In this instance, $\mathfrak{F}$ creates a singular fibration $\mathbb{M} \to \mathsf{S}^1$ with fibres S^{n-1} because it is a foliation formed by leaves diffeomorphic to S^{n-1} and centre singularities. This particular foliation will be referred to as a singular Seifert fibration.

We first emphasis on the $\mathfrak{F}$ 3-dimensional compact manifold exhibiting Morse foliation. Singular point p_2 such that $p_2 \in \partial\mathcal{C}_{p_1}(\mathfrak{F})$ must be a saddle singularity if $p_1 \in \operatorname{sing}(\mathfrak{F})$ is a centre singularity such that $p_2 \in \partial\mathcal{C}_{p_1}(\mathfrak{F})$. The separatrix of $\mathfrak{F}$ through p_2, at which is spherical leaves from $\mathcal{C}_{p_1}(\mathfrak{F})$ accumulate, is included in the leaf of $\mathfrak{F}$ designated as Γ_{p_2}. There are two possibilities: either $\partial\mathcal{C}_{p_1}(\mathfrak{F}) \setminus \{p_2\}$ is connected or $\partial\mathcal{C}_{p_1}(\mathfrak{F}) \setminus \{p_2\}$ has pair of connected components. Furthermore, for the $\partial\mathcal{C}_{p_1}(\mathfrak{F}) \setminus \{p_2\}$, we have two options: either the $\partial\mathcal{C}_{p_1}(\mathfrak{F})$ is homeomorphic to S^2 with a pinch at p_2 and the couple of singularities $p_2 - p_1$ corresponds to a dead branch, showing that it may be changed to a trivial foliation; alternatively, $\partial\mathcal{C}_{p_1}(\mathfrak{F})$ and singular torus T created by holding a sphere at two points and connecting those points has the homeomorphism. Thus, the singular Reeb component is $\mathcal{C}_{p_1}(\mathfrak{F})$ with $\partial\mathcal{C}_{p_1}(\mathfrak{F})$.

Let's talk about the situation where a manifold $\mathbb{M}^n$ with dimension $n \geq 3$ admits a Morse foliation $\mathfrak{F}$. Given two centre singularities $p_1, p_2 \in \text{sing}(\mathfrak{F})$, Intersection of $\mathcal{C}_p(\mathfrak{F})$ and $\mathcal{C}_q(\mathfrak{F})$ is not empty if $\mathcal{C}_p(\mathfrak{F}) = \mathcal{C}_q(\mathfrak{F})$ and converse is also true.

In addition, if $\partial\overline{\mathcal{C}_p(\mathfrak{F})} = \emptyset$ then $\mathcal{C}_p(\mathfrak{F}) = \mathbb{M}$ and vice versa. In this instance, $\mathbb{M}^n$ is homomorphic to S^n assuming that $\mathbb{M}^n$ is orientable. In particular, either $\mathbb{M}^n$ and S^n are homomorphic, or some saddle singularity exists in $\partial\overline{\mathcal{C}_p(\mathfrak{F})}$. Assume that $p_1, p_2 \in \text{sing}(\mathfrak{F})$ are two distinct centre singular points such that the intersection of $\partial\overline{\mathcal{C}_{p_1}(\mathfrak{F})}$ and $\partial\overline{\mathcal{C}_{p_2}(\mathfrak{F})}$ is not empty. Then the following possibilities are mutually incompatible for the aforementioned two centres when the intersection of their basins is not empty:

(i) If closure of the boundary of $\mathcal{C}_{p_1}(\mathfrak{F})$ is equal to the closure of the boundary of $\mathcal{C}_{p_2}(\mathfrak{F})$ then the manifold $\mathbb{M}$ is equal to the union $\overline{\mathcal{C}_{p_1}(\mathfrak{F})}$ and $\overline{\mathcal{C}_{p_2}(\mathfrak{F})}$

(ii) There exists a saddle point $q \in \partial\overline{\mathcal{C}_{p_1}(\mathfrak{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathfrak{F})}$ with 1 or $m-1$ as a Morse index, admitting no self-connection for $\partial\overline{\mathcal{C}_{p_1}(\mathfrak{F})} \neq \partial\overline{\mathcal{C}_{p_2}(\mathfrak{F})}$.

Proposition 2.6 (Eells-Kuiper Manifold [4]). *Consider a closed connected manifold $\mathbb{M}^n, n \geq 3$ admitting Morse foliation $\mathfrak{F}$. Assume that $\mathfrak{F}$ has one saddle singularity and precisely two centre singularities. Then an Eells-Kuiper manifold is homeomorphic to $\mathbb{M}^n$.*

3. Examples of Morse Foliations

Example 3.1 (Singular Reeb foliation). Similar to the Reeb foliation [7], [15] on the solid torus, singular Reeb foliation displays two Morse singularities in a centre-saddle configuration. A saddle at the north pole and a centre in the south pole are introduced after starting with a central sphere in $\mathbb{R}^3$. The inner leaves and the leaves at the boundary of this foliation $\mathfrak{F}$ on the solid torus are the leaves. The interior leaves are diffeomorphic to sphere, while the leaf on the boundary, which forms the boundary of a singular solid torus, has two separatrices that are connected to one another for saddle singularity. The leaves are diffeomorphic to the $S^1 \times S^1$ possessing trivial outside holonomy by extending the foliation $\mathfrak{F}$ to the outside of solid torus containing singularities. The saddle's separatrices are contained in a single solid torus that is diffeomorphic to a sphere. That torus is formed by squeezing the sphere at two points and connecting them.

Example 3.2 (Eells-Kuiper manifold and foliations [17]). Assume that $\mathbb{M}^n$ is a connected closed manifold. Suppose that $\mathbb{M}^n$ admits a class C^3 real valued Morse function f with precisely three singular points. Then, $\mathbb{M}^n$ is a topological compactification of the $\mathbb{R}^n$ by the $\mathsf{S}^{\frac{n}{2}}$ sphere for $n \in \{2,4,8,16\}$, often known as the Eells-Kuiper manifold. Let $\mathbb{M}^n$, $n \in \{2,4,8,16\}$ be an Eells-Kuiper manifold and suppose f be a real valued Morse function on $\mathbb{M}^n$. Then the level sets of the Morse function f on Eells-Kuiper manifold $\mathbb{M}^n$ define the foliation of codimension one, with precisely three singular points of Morse type (two centre singularities and one saddle).

Example 3.3 (Singular Seifert fibration). We designate $\mathcal{C}(\mathfrak{F})$ as the set of all leaves diffeomorphic to S^{n-1} in $\mathbb{M}^n$ and centre singularities for foliation exhibits Morse singularities on $\mathbb{M}^n, n \geq 3$. Considering centre singularity $p \in \mathrm{sing}(\mathfrak{F})$, we write $\mathcal{C}_p(\mathfrak{F})$ signify the connected component of $\mathcal{C}(\mathfrak{F})$ which contains p. We have $\mathcal{C}_p(\mathfrak{F}) = \mathbb{M}$ if and only if $\partial\mathcal{C}_p(\mathfrak{F}) = \emptyset$ because $\mathcal{C}_p(\mathfrak{F})$ is open in $\mathbb{M}$. So singular points of $\mathfrak{F}$ comprise centres and leaves that are diffeomorphic to S^{n-1}. This codimension one foliation with only centre sigularities will be referred to as a singular Seifert fibration.

Example 3.4 (Codimension one smooth foliation in the 4-dimensional closed ball $\overline{\mathbb{B}^4}$). Here, we present an illustration of a closed ball of radius one, centred at the origin of $\mathbb{R}^4$, with codimension one C^∞ foliation in which there is only one $2-2$ saddle type singularity, located at $0 \in \mathbb{B}^4$ and intersects the boundary transversely $\mathsf{S}^3 = \partial\mathbb{B}^4$. Consider a function

$$f(x) = -x_1^2 - x_2^2 + x_3^2 + x_4^2$$

in $\mathbb{R}^4$. This function has level zero as follows:

$$\mathsf{C} = f^{-1}(0),$$

is a cone over T^2. When looking at an intersection, it is simple to observe this,

$$\mathsf{C} \cap \mathsf{S}^3 = \mathsf{T}$$

which is obviously a T^2, where the cylinders intersect,

$$x_1^2 + x_2^2 = \tfrac{1}{2} \text{ and } x_3^2 + x_4^2 = \tfrac{1}{2}.$$

For given $\delta > 0, f^{-1}([-\delta, \delta])$ is a neighbourhood of C and,

$$\mathbb{R}^4 \setminus f^{-1}([-\delta, \delta]) = \mathsf{R}_1 \cup \mathsf{R}_2,$$

where R_1 and R_2 are two connected components diffeomorphic to $\mathbb{B}^4 \times \mathsf{S}^1$. Observe that intersection of R_1 with $\{x_3 = x_4 = 0\}$ is non-empty, and the intersection of R_1 with $\{x_1 = x_2 = 0\}$ is also non-empty. For $\delta > 0$ small enough,

$$S^3 \setminus f^{-1}([-\delta, \delta]) = \mathsf{T}_1^2 \cup \mathsf{T}_2^2,$$

where T_1^2 and T_2^2 are two solid tori, diffeomorphic to $\mathbb{B}^2 \times \mathsf{S}^1$. We define a new domain,

$$\mathbb{D} = f^{-1}([-\delta, \delta]) \cup A_1 \cup A_2,$$

where A_1 is a subset of region R_1 and A_2 is a subset of region R_2 both are diffeomorphic to $\mathbb{B}^3 \times \mathsf{S}^1$ such that,

$$\partial A_1 \cap \mathbb{B}^4 = \partial \mathsf{R}_1 \cap \mathbb{B}^4 \text{ and } \partial A_2 \cap \mathbb{B}^4 = \partial \mathsf{R}_2 \cap \mathbb{B}^4.$$

We define foliation $\mathfrak{F}$ on $\mathbb{D}$, where levels of f are the leaves of $\mathfrak{F}$ on $f^{-1}((-\delta, \delta))$. We establish a Reeb component on A_1 whose axis is circular $(x_3 = x_4 = 0) \cap \mathsf{S}^3$ and which has a foliation of spheres S^2 as sections on each $\mathbb{B}^3 \times \{\theta\}$. Similarly on A_2 Reeb component on $\mathbb{B}^3 \times \mathsf{S}^1$ whose axis is circle $(x_1 = x_2 = 0) \cap \mathsf{S}^3$ has been introduced, leaves of $\mathfrak{F}$ are transverse to S^3. The restriction $\mathfrak{F}|_{\overline{\mathbb{B}^4}}$ is eventually admitted.

4. Trivial Pairing of Singularities

4.1. Coupling and elimination of centre-saddle

Let's start by looking at the reduction of specific singularity combinations in given dimension two. This elimination process is depicted in Figure 3. Vector field is represented by $Z_\delta = (x^2 - \delta)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $\delta > 0$

This vector field has:

- For $\delta > 0$, there exists a saddle and a source.
- For $\delta = 0$, there exists saddle singularity.
- For $\delta < 0$, there is no singular point.

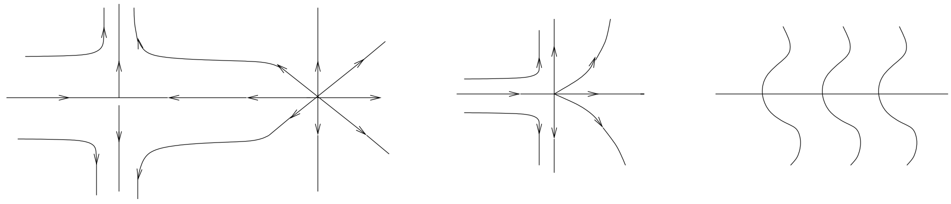

Figure 3: Modification of foliation.

By solving the equation,

$$grad(\Omega_\delta) = Z_\delta$$

we obtain dual foliation. This gives $\Omega_\delta = df_\delta$ for the function,

$$f_\delta = (\tfrac{x^3}{3} - \delta x) + \tfrac{y^2}{2},$$

whose level curves may be illustrated using Figure 3.

Thus, by passing via a saddle-node, the initial non-trivial centre-saddle pairing foliation can be changed by the trivial vertical foliation. A trivial foliation replaces a pair of centre-saddle singular points that we have.

Remark 4.1. The substitution of a centre-saddle pairing as shown above has no effect on the foliation's holonomy.

4.2. Coupling and elimination of saddle-saddle

Now, we'll show you how to merge pair of saddle singular points with complementary indices into a dead branch. Considering the function,

$$f_\delta : \mathbb{R}^3 \to \mathbb{R}, \text{ given by}$$
$$f_\delta = -\tfrac{x^2}{2} + (\tfrac{y^3}{3} - \delta y) + \tfrac{z^2}{2},\ \delta \in \mathbb{R}$$

and take into account the foliation modification indicated by $df_\delta = 0$ from $\delta > 0$ to $\delta < 0$ passing through $\delta = 0$. We restrict our attention to the examination of the function,

$$f = f_0 = -\tfrac{x^2}{2} + \tfrac{y^3}{3} + \tfrac{z^2}{2}.$$

It results in an origin saddle-node singularity. We consider leaves $\mathcal{L}_1$ and $\mathcal{L}_2$ as in Figure 4, where $\mathcal{L}_1 = \mathcal{L}_{(0,-1,0)}$ and $\mathcal{L}_2 = \mathcal{L}_{(0,1,0)}$ are planes. By deforming the initial foliation for $\delta > 0$ by a regular trivial foliation for

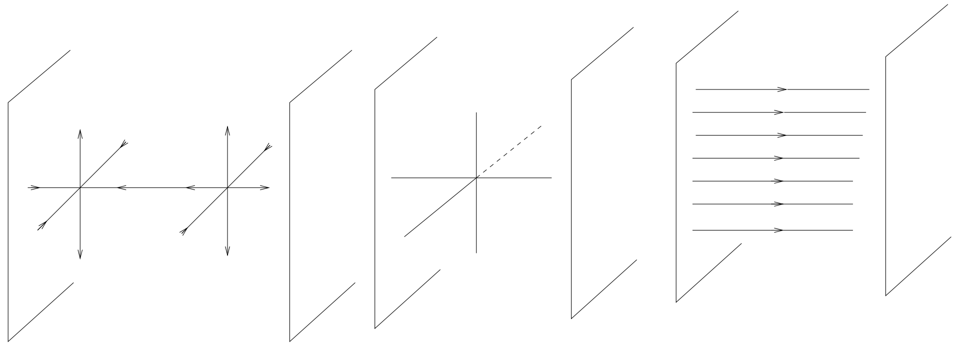

Figure 4: Designularisation

$\delta < 0$ as the end of the process, the deformation occurs in the region laterally limited by planes $\mathcal{L}_1$ and $\mathcal{L}_2$, going through the saddle-node for $\delta = 0$ in Figure 4. As we have already seen, the two-dimensional situation lends itself to a clearer understanding of this method. In the case of three dimensions, we modify this initial graphic by adding one transverse axis, then follow the same process. Our arrows-scheme may be observed in Figure 5 to produce an elimination technique given pair of saddle singular points exhibiting complementary indices in dimension 3. Observe that the $\mathrm{Index}(\Omega, p)$ is $+1$, $\mathrm{Index}(\Omega, q)$ is -1 and their sum is zero.

We will establish a natural hypothesis to explain the connection between two saddle singular points p and q with complementary indices: Assume that we have two complementary indices saddle singular points such that the stable manifold of saddle p joins the unstable manifold of saddle q transversally through a smooth connection (curve) $\gamma_{p,q}$. These connections are referred to as stable connections between p and q.

In general, the aforementioned construction leads to the proposition:

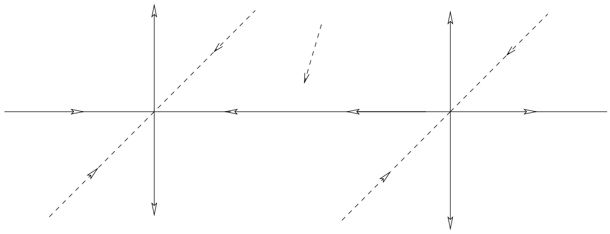

Figure 5: Arrow Scheme

Proposition 4.2 (Coupling of saddle singular points 6.4 [14])**.** *Suppose there exists a foliation $\mathfrak{F}$ on a manifold of dimension n $\mathbb{M}^n$. Then, a modification of $\mathfrak{F}$ on $\mathbb{M}^n$ exists called $\tilde{\mathfrak{F}}$, admitting a pair of saddle singular points with complementary indices that are strongly stable connected [14].*

Below is a construction that is the counterpart of the one mentioned above. We begin by pointing out the following observation:

Remark 4.3. A neighbourhood U of γ, p_1, p_2 in $\mathbb{M}^3$ and a coordinate system given a foliation $\mathfrak{F}$ on $\mathbb{M}^3$ admits couple of saddle singular points $p_1, p_2 \in sing(\mathfrak{F})$ with complementary indices possessing a stable connection γ.

$$\begin{gathered}\varphi : U \to \mathbb{R}^3, \quad \text{such that} \\ \varphi(p_1) = (0,0,0) \\ \varphi(p_2) = (0,1,0)\end{gathered}$$

Putting γ on the y-axis where $\{x = z = 0\}$, furthermore, the unstable manifold of saddle p_2 is tangent to $\varphi^{-1}(\{x = 0\})$ at p_2 and the stable manifold of saddle p_1 is tangent to $\varphi^{-1}(\{z = 0\})$ at p_1.

We have two leaves in dimension 2, $\mathcal{L}_1$ and $\mathcal{L}_2$ and segments Σ_1, Σ_2, then $\mathcal{L}1, \mathcal{L}2, \Sigma_1, \Sigma_2$ represent the space of leaves in the area that they enclose. To create a fibration admitting no singularities using discs that is coherent with the leaves $\mathcal{L}_1$ and $\mathcal{L}_2$, we can first integrate the singularities into a saddle-node. In dimension three, coupling of pair of saddle singualrities will be accomplished using the same concept. We go back to the three dimensional scenario when there are two saddle singular points with complementary indices and a stable connection. We may infer that we are on $\mathbb{R}^3$ by using the chart from the remark, $\varphi : U \to \mathbb{R}^3$. We discover a dynamical behaviour that is similar to the 2-dimensional case using the restriction of $\mathfrak{F}$ to the stable manifold of p_1 (diffeomorphic to $\mathbb{R}^2$). In the same manner as before, we obtain leaves $\mathcal{L}_1, \mathcal{L}'_1, \mathcal{L}_2, \mathcal{L}'_2$ and segments Σ_1, Σ_2 representing the leaf space of $\mathfrak{F}$. As a result, inside the cylinder the foliation can be altered to produce a non-singular fibration by discs that are transverse to the axis of the stable connection.

The inverse path is now taken. Here, we'll look at how to get a singular foliation from a 3-dimensional manifold with a non-singular Reeb component. For given foliation $\mathfrak{F}$ without singularities, consider a transverse circle,

$$\gamma : S^1 \to \mathbb{M}^3, \gamma \pitchfork \mathfrak{F}$$

We can establish a standard Reeb component in a small tubular region of this curve. After that, by applying the usual modifications, we may swap out this Reeb component for a single Reeb foliation with γ as its central axis. This demonstrates that couplings of centre-saddle singular points can be introduced once we have made the transverse circles to the foliation in $\mathbb{M}^3$, we can swap out a saddle singularity with a centre singularity of the same index by applying the method described above. Consequently, let $W^{ss}(p_1) \subset \mathbb{M}^3$ be a stable two-dimensional manifold with a saddle singularity $p_1 \in \mathbb{M}^3 3$ in $\mathfrak{F}$. Consider a circle.

$$\gamma : \mathrm{S}^1 \to \mathrm{W}^{ss}(p_1)$$

which is transverse to $\mathfrak{F}$. Then, using the method outlined above, by creating a singular Reeb foliation with $\gamma(S^1)$ as its axis and a $\mathfrak{F}$-compatible structure, we can achieve a coupling of centre-saddle. By above construction, the stable manifold $\mathrm{W}^{ss}(p_1)$ meets transversely the unstable manifold $\mathrm{W}^{uu}(p_2)$ of p_2 along a stable connection, resulting in saddle singularity p_2 and a centre singularity p_3 in $\mathrm{W}^{ss}(p_1)$. With grad(Ω), which is a fixed one form that defines $\mathfrak{F}$, we get a vector field. The leaves of the foliation $\mathfrak{F}$ on $W^{ss}(p_1)$, are indicated by the curves of the vector field grad(Ω). Since the saddles p_1 and p_2 are connected in a stable manner, we may remove them and obtain a modification of $\mathfrak{F}$ in which the saddle p_1 has been substituted for a compatible centre p_3 of the same index.

We come to the conclusion that in a particular scenario of the formulation above, if the restriction $\mathfrak{F}|_{W^{ss}(p_1)}$ has some limit cycle, we can replace p_1 with a centre singular point which has the same index. In fact, if we possess a limit cycle on the two dimensional manifold, we can build a transverse circle by using the radial component of the related vector fields.

Observation. Consider a foliation in a 3-dimensional manifold denoted $\mathfrak{F}$ possessing a saddle singularity in $\mathbb{M}^3$ with a stable manifold $W^{ss}(p_1) \subset \mathbb{M}^3$.

A foliation in the 3-dimensional manifold $\mathbb{M}^3$ with a saddle singularity p_1 and stable manifold $W^{ss}(p_1) \subset M^3$ is denoted by $\mathfrak{F}$. Suppose there is a circle

$$\gamma : \mathrm{S}^1 \to \mathrm{W}^{ss}(p_1) \text{ such that } \gamma(\mathrm{S}^1) \pitchfork \mathfrak{F},$$

We may introduce coupling of the centre p_3 and saddle p_2 having $\gamma(S^1)$ as the axis because we have a circle that is transverse to the foliation. We can get rid of saddles p_1 and p_2 by connecting them where the stable manifold of p_1, $W^{ss}(p_1)$, meets transversely the unstable manifold of p_2, $W^{uu}(p_2)$. Therefore, using a modification process, the saddle singularity p_1 can be replaced by a centre singularity p_3 of the same index. We are now in a situation to demonstrate the key theorems.

Theorem 4.4. *If a closed connected, oriented, three dimensional manifold* $\mathbb{M}^3$ *admits an oriented Morse foliation* $\mathfrak{F}$ *satisfying* $\mathrm{C} > \mathrm{S} - 2k$*, where* C *is the number of centre singularities and* C *is the even number of saddle singular points in sing*($\mathfrak{F}$) *fulfilling* $\mathrm{C} \geq \mathrm{S} + 1$*. Assume that there exist at least* k *pairs of stable saddle connections. Then* $\mathbb{M}^3$ *is diffeomorphic to 3-sphere* S^3*.*

An isotopy to a Morse foliation with only two centres acting as singularities is, in fact, permitted by $\mathfrak{F}$*.*

Proof. Assuming that $\mathrm{C} \geq \mathrm{S} + 1$, we have a centre C and an even number of saddle singular points S. By using induction on saddle singular points, we will continue. If $\mathrm{S} = 0$, then the result is given by Reeb's thesis and we only have centre singularities. Assume that $\mathrm{S} \geq 2$ is valid and that the result holds for foliations with a maximum number of $\mathrm{S} - 2$ saddle singular points. The outcome for S saddles must now be proven. Because, according to our hypothesis, the foliation $\mathfrak{F}$ has S saddles and C centres that satisfy $\mathrm{C} \geq \mathrm{S} + 1$. Assuming that $\mathrm{C} > \mathrm{S} - 2k$, there are at least k pairs of saddles that are in stable connection. The modified foliation $\tilde{\mathfrak{F}}_1$ of $\mathfrak{F}$ on $\mathbb{M}^n$ that results from proposition 3 has two saddle singular points, q_1 and q_2, of complementary indices that are in a strong stable connection, i.e. the stable manifold of q_1 intersects the unstable manifold of q_2 transversally. By substituting $\tilde{\mathfrak{F}}_1$ with a modified Morse foliation $\mathfrak{F}_1$ on $\mathbb{M}^n$ with a number s_1 of saddles provided by $\mathrm{S}_1 = \mathrm{S} - 2$, we may remove saddles q_1 and q_2 which correspond to a dead branch. There are now $\mathrm{S} - 2$ saddle singular points. According to the induction hypothesis, the conclusion holds true for saddle-type $\mathrm{S} - 2$ singularities. For S saddles, this has been demonstrated. It validates our finding.

Now, taking into account the situation when $\mathrm{S} > \mathrm{C}$. given that we have saddle singular points and that $\mathrm{C} > \mathrm{S} - 2k$, there must be at least k pairs of saddles are in stable connection. The modified foliation $\tilde{\mathfrak{F}}_2$ of $\mathfrak{F}$ on $\mathbb{M}^n$

contains two saddle singular points, q_1 and q_2, with complementary indices that are in a strong stable connection, i.e. the stable manifold of q_1 meets the unstable manifold of q_2 transversely. By substituting $\tilde{\mathfrak{F}}_2$ with a modified Morse foliation $\mathfrak{F}_2$ on $\mathbb{M}^n$ with a number s_2 of saddles provided by $\mathrm{s}_2 = \mathrm{s}-2$, we may remove saddles q_1 and q_2 that are part of a dead branch. A Morse foliation that satisfies the conditions $\{\#centres\} \geq \{\#saddles + 1\}$ can be produced by repeatedly modifying the foliation by elimination. We are once again concerned with the already established case. $\square$

Theorem 4.5. *Suppose that $\mathfrak{F}$ is a Morse foliation on a compact connected manifold M^n with c centres and s saddles fulfilling $c \geq s+1$, or in general, where $\mathrm{c} > \mathrm{s} - 2k$, there exist k or more pairs of saddles that are connected in a stable connection. Then we have following possibilities:*

(a) *Manifold $\mathbb{M}^n$ is homeomorphic to the n-Sphere S^n if number of centre singularities c are equal to $\mathrm{s}+2$.*

(b) *Manifold $\mathbb{M}^n$ is homeomorphic to the Eells-Kuiper manifold if number of centre singularities c are equal to $\mathrm{s}+1$.*

Proof. By using induction on the number of saddle singular points s, we will go forward for the case $\mathrm{c} \geq \mathrm{s}+1$. According to Reeb's theorem, $\mathbb{M}^n$ is homeomorphic to S^n if $\mathrm{s} = 0$. Assume that the result holds true for a maximum of $\mathrm{s}-1$ saddles and that $\mathrm{s} \geq 1$. We will now display the outcome for s saddles. Since $\mathrm{c} \geq \mathrm{s}+1$, our hypothesis is correct. Hence, $\mathrm{c} \geq 2$. Assume that $\mathbb{M}^n$ and S^n are not homomorphic. The saddle $q_{(p)} \in \partial\overline{\mathcal{C}_p(\mathfrak{F})}$ must then exist for each centre $p \in \operatorname{sing}(\mathfrak{F})$. Since there are two centres p_1, p_2 such that $q_{p_1} = q_{p_2}$, i.e. there is a saddle q such that $q \in \partial\overline{\mathcal{C}_{p_1}(\mathfrak{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathfrak{F})}$, then, we are left with two options:

(i) Closure of the boundary of $\mathcal{C}_{p_1}(\mathfrak{F})$ is equal to the closure of the boundary of $\mathcal{C}_{p_2}(\mathfrak{F})$. In this case $\partial\overline{\mathcal{C}_{p_1}(\mathfrak{F})} = \partial\overline{\mathcal{C}_{p_2}(\mathfrak{F})}$ we have $\mathbb{M} = \overline{\mathcal{C}_{p_1}(\mathfrak{F})} \cup \overline{\mathcal{C}_{p_2}(\mathfrak{F})}$, so clearly $\mathcal{C}_{p_i} \cap \operatorname{sing}(\mathfrak{F}) = \{p_i\}, i = 1, 2$. In light of the fact that $\operatorname{sing}(\mathfrak{F}) = \{p_1, p_2, q\}$ meets the assumption $\mathrm{c} = \mathrm{s}+1$, $mathrmM$ is an Eells-Kuiper manifold.

(ii) Closure of the boundary of $\mathcal{C}_{p_1}(\mathfrak{F})$ is not equal to the closure of the boundary of $\mathcal{C}_{p_2}(\mathfrak{F})$. If $\partial\overline{\mathcal{C}_{p_1}(\mathfrak{F})} \neq \partial\overline{\mathcal{C}_{p_2}(\mathfrak{F})}$, saddle singular points fulfill $sgeq2$, then assuming that $\mathrm{c} > \mathrm{s}-2k$, there are at least k pairs of saddles that are in stable connection. The modification $\tilde{\mathfrak{F}}$ of $\mathfrak{F}$ on Mn contains two saddle singular points of complementary indices, say q_3 and q_4, which are in stable connection. Then, by proposition 3, we obtain this modification. By substituting a Morse foliation $\mathfrak{F}_1$ on $\mathbb{M}^n$ with the same number of centres C and two less saddles $\mathrm{S}-2$ as determined by $\mathrm{S}_1 = \mathrm{S}-2$, we may get rid of these two saddles. In account of this, $\mathrm{C} > \mathrm{S}_1$ and $\mathrm{S} > \mathrm{S}_1 \geq 0$. According to the induction principle, $\mathbb{M}^n$ is homomorphic to S^n or an Eells-Kuiper manifold.

Now, according to the hypothesis, if $\mathrm{S} > \mathrm{C}$, there are at least k pairs of saddles that are connected in a stable manner, where $\mathrm{C} > \mathrm{S} - 2k$. Then, by eliminating complementary saddles in accordance with proposition 3, the foliation $\mathfrak{F}$ is modified repeatedly. As a consequence, we obtain a Morse foliation that satisfies either first scenario, which is $\mathrm{C} = \mathrm{S} + 2$, or second scenario, which is $\mathrm{C} = \mathrm{S} + 1$, and once more, the demonstration above yields the result. □

Conclusion

The interaction of combinatorics of a C^2 real valued function defined on the manifold and closed manifold topology is a well-known Morse theory phenomenon. It is plausible to foresee a similar relationship for foliated manifolds. G. Reeb's results, which proceeded from his stability theorem, revealed this for the first time. The classical Reeb result was improved by E. Wagneur to include Morse foliations with saddle singular points. He demonstrated that there is a limit to the number of centre singularities in relation to the number of saddles. We have examined the idea of coupling and eliminating two stable coupled saddle singular points with complementary indices. The number of saddle singular points in stable connection is taken to be at least r pairs. By coupling and detaching a pair of complementary saddles, the result which states that a closed connected and orientated manifold of dimension three possessing Morse foliation and exhibiting more centre singular points than saddle singular points is diffeomorphic to three sphere is generalised in this study. We have also generalised the n-dimensional case, which is essentially an extension of E. Wagner's work and defines the topology of an n-dimensional manifold that admits a foliation with more centre singularities than saddle singular points using the coupling and eliminating technique of saddle singularity.

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