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SEMI-CLASSICAL HEAT KERNEL ASYMPTOTICS AND MORSE INEQUALITIES

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Abstract

In this paper, we study the asymptotic behavior of the heat kernel with respect to the Witten Laplacian. We introduce the localization and the scaling technique in semi-classical analysis, and study the semi-classical asymptotic behavior of the family of the heat kernel, indexed by k, near the critical point p of a given Morse function, as $k \to \infty$. It is shown that this family is approximately close to the heat kernel with respect to a system of the harmonic oscillators attached to p. We also furnish some asymptotic results regarding heat kernels away from the critical points. These heat kernel asymptotic results lead to a novel proof of the Morse inequalities.

1. Introduction

In his marvellous paper [15], Witten gave a new proof of the Morse inequalities by considering the family of the so called Witten Laplacians $\Delta_k = \Delta + k^2 |df|^2 + kA$, where $k > 0$ is a parameter, f is a Morse function, and A is an operator of order 0. He proved that the spectral functions of Δ_k is approximately close to the spectral functions of a system of harmonic oscillators attached to the critical point of f, as $k \to \infty$. His idea of studying deformed operators indexed by k has led to several breakthrough in several fields.

In complex geometry, Demailly [6] discovered the holomorphic Morse inequalities that describe the k-large asymptotic upper bounds for the Morse-

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Witten sums of the Betti numbers of $\overline{\partial}$ on $L^{\otimes k}$ in terms of the Chern curvature of the Hermitian holomorphic line bundle L. The key of finding such inequalities is that he managed to localize the problem by converting the local frame of L to the local weight function that plays a role of a Morse function in [15]. This in turn, led to the consideration of the family of operators $\Box_k$ analogous to Δ_k. Meanwhile, Bismut gave the heat equation proofs of Morse inequalities in [2] and of Demailly's holomorphic Morse inequalities in [3], using the probability theory. Subsequently, Demailly [7] replaced Bismut's probabilistic argument and recovered his holomorphic Morse inequalities by investigating the heat kernel asymptotics. In view of the recent progress, we feel that it is important to investigate the heat kernel asymptotics in Witten's classical setting.

In this present paper, we study the asymptotic behavior of the heat kernel with respect to the Witten Laplacian Δ_k and in turn recover the Morse inequalities.

Let us briefly illustrate how we obtain our semi-classical heat kernel asymptotics (see Theorem 1.1). Our asymptotic behaviors of the heat kernel are achieved based on the two techniques: localization and scaling technique. We seek to localized the heat kernel near the critical point p of the Morse function by constructing a metric that is flat around p together with the Morse lemma. The localization of this kind was indeed motivated by Witten's work [15]. To obtain the asymptotics, we introduced the scaling technique in semi-classical analysis. This technique allows us to consider the family of the so-called scaled heat kernels near the critical point p, indexed by k, and study the asymptotic behavior of this family as $k \to \infty$. It turns out that this family is approximately close to the heat kernel with respect to a system of the harmonic oscillators attached to p that resembles Witten's finding. We would like to stress that these two techniques effectively tackle the asymptotic behavior of heat kernels in a computable way.

These techniques have been applied to several projects. In CR geometry, for example, Hsiao and Zhu [10] investigated the semi-classical asymptotics of the heat kernel with respect to Kohn Laplacian using these tricks and obtained the CR and $\mathbb{R}$-equivariant Morse inequalities. On the other hand, in complex geometry, Chiang [5] made use of these techniques and obtained the semi-classical asymptotic behaviors of Bergman kernels and spectral kernels. We refer the reader to [1], [13], [8], [9] for further related scaling techniques.

To recover the Morse inequalities from the heat kernel asymptotics, we use the classical trace integral formula (see [14]) in the local index theory. It turns out that, somewhat surprisingly, we need more delicate asymptotic results regarding the heat kernels away from the critical points (see Theorems 1.2 and 1.3). Perhaps one of the causes of the difference from Hsiao and Zhu's work [10] would be that they investigate their semi-classical heat kernel asymptotics for each point in the underlying manifold, while we only consider the asymptotics of this kind near the critical point. These delicate results are obtained based on certain Bochner-type estimates.

1.1. Statements of the main results

In this subsection, we state our results in detail. We refer the reader to Section 2 and Subsections 3.1 and 3.2 for the terminologies we use.

Let M be a compact smooth manifold of dimension n and let f be a Morse function on M. We equip M with a metric $g = \langle\cdot|\cdot\rangle$ such that for every critical point p of f, we can choose a coordinate chart such that $\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle = \delta^i_j$ and f is written as a quadratic form (according to Morse Lemma) near p, where δ^i_j is the Kronecker delta (see Theorem 3.1). For each $k > 0$, denote the Witten Laplacian (acting on r-forms) by

$$\Delta_k^{(r)} := d_k d_k^* + d_k^* d_k,$$

where $d_k := e^{-kf} d e^{kf}$ is the deformed exterior operator and d_k^* is its formal adjoint with respect to the induced L^2-inner product from the metric g.

For each critical point $p \in \mathrm{Crit}\,(f)$, denote the scaled heat kernel by

$$A^r_{(k),p}\,(t,x,y) := k^{-\frac{n}{2}} e^{-\frac{t}{k}\Delta_k^{(r)}} \left(\frac{x}{\sqrt{k}}, \frac{y}{\sqrt{k}}\right),$$

where $e^{-t\Delta_k^{(r)}}\,(x,y)$ denotes the heat kernel with respect to $\Delta_k^{(r)}$ on r-forms. Moreover, denote by $e^{-t\Delta_{f,p}^{(r)}}\,(x,y)$ the heat kernel of the system of harmonic oscillators $\Delta_{f,p}^{(r)}$ attached to p (see (3.2)). Then our semi-classical heat kernel asymptotics is stated as follows:

Theorem 1.1. *For each critical point $p \in \mathrm{Crit}(f)$,*

$$\lim_{k\to\infty} A^r_{(k),p}(t,x,y) = e^{-t\Delta^{(r)}_{f,p}}(x,y).$$

in C^∞-topology in each compact subset of $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$. Consequently, we obtain the following pointwise asymptotic

$$\lim_{k\to\infty} k^{-\frac{n}{2}} e^{-\frac{t}{k}\Delta^{(r)}_k}(p,p) = e^{-t\Delta^{(r)}_{f,p}}(0,0).$$

Theorem 1.1 shows that the leading term of on-diagonal heat kernel expansion of $e^{-\frac{t}{k}\Delta^{(r)}_k}(x,y)$ at the critical point p as k large is given by $e^{-t\Delta^{(r)}_{f,p}}(0,0)$. We can further unwind $e^{-t\Delta^{(r)}_{f,p}}(x,x)$ by Mehler's formula and then capture the information of the critical point p (see Theorem 5.5).

Now, let us state the asymptotic results of heat kernels away from the critical points. Denote by $B_R(q) \subset \mathbb{R}^n$ an Euclidean ball centered at q with radius R. For each r-form ω (on $\mathbb{R}^n$), let $|\omega|$ be the norm of ω induced from the (flat) metric g (on $\mathbb{R}^n$). Define the norm $|\cdot|_x$ for the space of the linear transformations $\bigwedge^r T^*_x\mathbb{R}^n \otimes (\bigwedge^r T^*_x\mathbb{R}^n)^*$ by

$$|S|_x := \sup_{\omega_x \in \bigwedge^r T^*_x\mathbb{R}^n, \omega_x \neq 0} \frac{|S\omega_x|}{|\omega_x|}$$

for each $S \in \bigwedge^r T^*_x\mathbb{R}^n \otimes (\bigwedge^r T^*_x\mathbb{R}^n)^*$.

Theorem 1.2. *There exists $D > 0$ such that if k is large enough, then for each $p \in \mathrm{Crit}(f)$ and for each $N \in \mathbb{N}$,*

$$\left| A^r_{(k),p}(t,x,x) \right|_x \leq C(t,N)\,|x|^{-N},$$

where $C(t,N)$ depends on N and smoothly on t and is independent of D, x, k, for each $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$.

For each large k, set $\mathcal{U}^k = \bigcup_{p\in\mathrm{Crit}(f)} U^k_p$, where U^k_p is identified with $B_{k^{-\frac{1}{2}+\varepsilon}}(0)$, $\varepsilon \in \left(0,\frac{1}{2}\right)$, under the coordinate chart of $p \in \mathrm{Crit}(f)$.

Theorem 1.3. *If k is sufficiently large, then for each $t > 0$ and for each $N \in \mathbb{N}$,*

$$\left\| e^{-\frac{t}{k}\Delta^{(r)}_k}(x,x) \right\|_{\mathcal{C}^0\left(M\setminus\mathcal{U}^k\right)} \leq C(t,N)\,k^{-N},$$

where $C(t,N)$ depends on N and smoothly on t and is independent of k and the $\mathcal{C}^0$-norm is determined by a choice of partition of unity and orthonormal frame. This implies

$$\lim_{k\to\infty} e^{-\frac{t}{k}\Delta_k^{(r)}}(x,x)=0$$

for each $x\in M\setminus \mathrm{Crit}(f)$.

From Theorems 1.1, 1.2, and 1.3, we recover the Morse inequalities (See Section 5). In fact, from these three theorems, we can deduce

$$\lim_{t\to\infty}\lim_{k\to\infty}\int_M \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}}(x,x)\,dV=m_r,$$

where m_r is the number of the critical points of f with index r (see Theorem 5.6), which together with the local index theory, establishes the Morse inequalities. We refer the reader to [12] for a topological proof of the Morse inequalities.

This paper is organized as follows: In Section 2, we set up some notations and review some essential notions and theorems. In Section 3, we introduce the localization and scaling technique, and prove Theorem 1.1. Later on, in Section 4, we introduce the Bochner-type estimate and prove Theorems 1.2 and 1.3. Finally, in Section 5, we investigate the model kernel and present the new heat kernel proof of the Morse inequalities.

2. Preliminaries

2.1. Notations and terminologies

Let $\alpha=(\alpha_1,\ldots,\alpha_n)$ be a multi-index and we set $|\alpha|=\sum_{i=1}^n\alpha_i$ and put

$$\partial_x^\alpha=\frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}},$$

where $x=\left(x^1,\ldots,x^n\right)\in\mathbb{R}^n$.

Let M be a smooth manifold of dimension n. We denote by TM the tangent bundle of M and by T^*M the cotangent bundle of M. We say a (differential) r-form ω to be a section (not necessarily smooth) of the exterior bundle $\bigwedge^r(T^*M)$. Choosing a local coordinate chart with the coordinates

$x^1, \ldots, x^n$, we can locally write ω as $\omega = \sum_I' \omega_I dx^I$, where the primed sum $\sum_I'$ refers to the one that runs over all multi-indices $I = (i_1, \ldots, i_r)$ with $|I| = r$ arranged in increasing order (namely, $1 \le i_1 < \cdots < i_r \le n$), and where ω_I are component functions of ω and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_r}$. In particular, let $U \subset M$ be an open subset, and we denote by $\Omega^r(U) = \mathcal{C}^\infty(U, \bigwedge^r(T^*M))$ the space of smooth r-forms and by $\Omega_c^r(U)$ the space of all smooth r-forms compactly supported in U. We denote by $H_{dR}^r(M)$ the r-th de Rham cohomology group on M.

Let (M, g) be an orientable Riemannian manifold endowed with the metric $g(\cdot,\cdot) = \langle \cdot | \cdot \rangle$ and let dV be the volume form induced by g. We denote by $|\cdot|$ the norm associated to the metric g. Note that g induces the metric on $\bigwedge^r(T^*M)$ (still denoted as g and its associated norm is also denoted as $|\cdot|$), and thus the L^2-inner product $(\cdot|\cdot)$ on $\Omega_c^r(M)$ with respect to dV. The completion of $\Omega_c^r(M)$ with respect to $(\cdot|\cdot)$ is denoted by $L_r^2(M)$. We denote the associated L^2-norm on M by $\|\cdot\|_{L^2(M)}$, and we will omit the subscript $L^2(M)$ if there is no ambiguity. Let d be the exterior derivative and denote by d^* the formal adjoint of d with respect to $(\cdot|\cdot)$.

Let f be a Morse function. The set consisting of all critical points of f is denoted by $\mathrm{Crit}(f)$. The index of f at $p \in \mathrm{Crit}(f)$ is denoted by $\mathrm{Ind}_f p$. Besides, the j-th Morse number m_j is defined to be the number of the set $\{p \in \mathrm{Crit}(f) : \mathrm{Ind}_f p = j\}$.

We give a final remark on the appearance of constants. Throughout this paper, $C(\cdot)$ denotes a constant that depends on what appears within the parenthesis.

2.2. Witten Laplacians and heat kernels

Let M be a compact Riemannian manifold of dimension n and let f be a Morse function on M. For each $k > 0$, define the deformed exterior derivative $d_k : \Omega^r(M) \to \Omega^{r+1}(M)$ by

$$d_k := e^{-kf} d e^{kf} = d + k df \wedge .$$

It is evident to see that $d_k^2 = 0$. Define the deformed r-th de Rham cohomology group on M by

$$H_k^r(M) = \frac{\mathrm{Ker}\left(d_k : \Omega^r(M) \to \Omega^{r+1}(M)\right)}{\mathrm{Im}\left(d_k : \Omega^{r-1}(M) \to \Omega^r(M)\right)}.$$

Proposition 2.1 ([11], Theorem 2.3)**.** *For each $k > 0$, $H_k^r(M)$ is isomorphic to $H_{dR}^r(M)$.*

Let $d_k^* : \Omega^r(M) \to \Omega^{r-1}(M)$ be the formal adjoint of d_k with respect to the L^2-inner product $(\cdot|\cdot)$. Then

$$d_k^* = e^{kf} d^* e^{-kf} = d^* + k\iota_{\nabla f},$$

where $d^* = (-1)^{n(r+1)+1} * d*$, $*$ is the Hodge star operator, and ∇f is the gradient of f.

For each $k > 0$, the Witten Laplacian on $\Omega^r(M)$ is defined to be

$$\Delta_k^{(r)} := d_k^* d_k + d_k d_k^* : \Omega^r(M) \to \Omega^r(M).$$

By direct computation, we obtain

$$\Delta_k^{(r)} = \Delta^{(r)} + k^2 |df|^2 + k\left(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*\right), \tag{2.1}$$

where $\mathcal{L}_{\nabla f}$ denotes the Lie derivative of ∇f and $\mathcal{L}_{\nabla f}^*$ the formal adjoint of $\mathcal{L}_{\nabla f}$ with respect to the L^2-inner product $(\cdot|\cdot)$. Note that $\Delta_k^{(r)}$ is elliptic. Moreover, $\Delta_k^{(r)}$ has the local expression

$$\Delta_k^{(r)} = \Delta^{(r)} + k^2 |df|^2 + k \sum_{i,j} \mathrm{Hess}_f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\left[dx^j \wedge, \iota_{(dx^i)^\sharp}\right], \tag{2.2}$$

where $\mathrm{Hess}_f = \nabla^{TM} df$ is the Hessian form with respect to the Levi-Civita connection ∇^{TM} and $\left(dx^i\right)^\sharp$ is the dual element to dx^i with respect to the inner product $\langle\cdot|\cdot\rangle$.

We extend the Witten Laplaican to $\Delta_k^{(r)} : \mathrm{Dom}\Delta_k^{(r)} \to L_r^2(M)$, where

$$\mathrm{Dom}\Delta_k^{(r)} = \left\{\omega \in L_r^2(M) : \Delta_k^{(r)}\omega \in L_r^2(M)\right\}.$$

Note that $\Omega^r(M) \subset \mathrm{Dom}\Delta_k^{(r)}$ is dense in $L_r^2(M)$. Also, denote the adjoint of $\Delta_k^{(r)}$ with respect to the L^2-inner product by $\Delta_k^{(r)*} : \mathrm{Dom}\Delta_k^{(r)*} \to L_r^2(M)$, where $\mathrm{Dom}\Delta_k^{(r)*}$ consists of elements ω in $L_r^2(M)$ for which there exists a constant $C > 0$ such that

$$\left|\left(\omega|\Delta_k^{(r)}\eta\right)\right| \leq C\|\eta\|$$

for each $\eta \in \mathrm{Dom}\Delta_k^{(r)}$. In fact, we can see that $\Delta_k^{(r)}$ is self-adjoint and non-negative.

Let $\mathrm{Spec}\Delta_k^{(r)}$ be the spectrum of $\Delta_k^{(r)}$ and $E_{\lambda,k}^{(r)}(M)$ be the eigenspace of $\Delta_k^{(r)}$ corresponding to the eigenvalue λ. We have the following property related to the alternative sum of the dimensions of the eigenspaces.

Proposition 2.2. *For each k, for each r, and for each $\mu \in \mathrm{Spec}\Delta_k^{(r)} \setminus \{0\}$, we have*

$$\sum_{j=0}^{r} (-1)^{r-j} \dim E_{\mu,k}^{(j)}(M) = \dim d_k\left(E_{\mu,k}^{(r)}(M)\right),$$

where $d_k\left(E_{\mu,k}^{(r)}(M)\right) = \left\{d_k\omega : \omega \in E_{\mu,k}^{(r)}(M)\right\}$. Subsequently, we obtain for each r,

$$\sum_{j=0}^{r} (-1)^{r-j} \dim E_{\mu,k}^{(j)}(M) \geq 0,$$

and if $r = n$, the equality occurs; namely,

$$\sum_{j=0}^{n} (-1)^{n-j} \dim E_{\mu,k}^{(j)}(M) = 0.$$

Proof. It follows from the fact that the complex $d_k : E_{\mu,k}^{(r)}(M) \to E_{\mu,k}^{(r+1)}(M)$ is exact. $\square$

For each $t > 0$, define the heat operator $e^{-t\Delta_k^{(r)}} : L_r^2(M) \to \mathrm{Dom}\Delta_k^{(r)}$ such that $e^{-t\Delta_k^{(r)}}\omega \in \Omega^r(\mathbb{R}^+ \times M)$ and the operator satisfies

$$\begin{cases} \frac{\partial}{\partial t} e^{-t\Delta_k^{(r)}}\omega + \Delta_k^{(r)} e^{-t\Delta_k^{(r)}}\omega = 0 \\ \lim_{t\to 0^+} \left\| e^{-t\Delta_k^{(r)}}\omega - \omega \right\| = 0 \end{cases}.$$

The associated distribution kernel

$$e^{-t\Delta_k^{(r)}}(x,y) \in \mathcal{C}^\infty\left(\mathbb{R}^+ \times M \times M; \bigwedge^r (T^*M) \boxtimes \left(\bigwedge^r (T^*M)\right)^*\right)$$

is called the heat kernel with respect to $\Delta_k^{(r)}$. Here, we denote by $E \boxtimes F^*$ the vector bundle over $M \times M$ whose fiber at (x,y) is the space of linear transformations from F_y to E_x.

The heat kernel can be expressed in different ways. First, choose a local orthonormal frame $\{E^I\}_I$ for $\bigwedge^r (T^*M)$ and denote by $(E^I)^*$ the dual element to E^I, and we can write

$$e^{-t\Delta_k^{(r)}}(x,y) = \sum_{I,J}{}' e^{-t\Delta_k^{(r)}}{}_{I,J}(x,y)\, E^I(x) \otimes \left(E^J\right)^*(y),$$

where $E^I(x) \otimes \left(E^J\right)^*(y) \in \bigwedge^r T_x^*M \otimes \left(\bigwedge^r T_y^*M\right)^*$ satisfies

$$\left(E^I(x) \otimes \left(E^J\right)^*(y)\right)(E_K(y)) = \langle E^K(y) \,|E^J(y)\rangle \cdot E^I(x) = \delta_J^K E^I(x),$$

and the corresponding component function $e^{-t\Delta_k^{(r)}}{}_{I,J}(x,y) \in \mathcal{C}^\infty(\mathbb{R}^+ \times M \times M)$.

Set $d_\lambda = \dim E_{\lambda,k}^{(r)}(M)$ and let $\{\varphi_i^\lambda\}_{i=1,\dots,d_\lambda,\lambda\in\mathrm{Spec}\Delta_k^{(r)}}$ be a complete orthonormal basis for $L_r^2(M)$ such that $\Delta_k^{(r)}\varphi_i^\lambda = \lambda\varphi_i^\lambda$, and we can write the heat kernel as

$$e^{-t\Delta_k^{(r)}}(x,y) = \sum_{\lambda\in\mathrm{Spec}\Delta_k^{(r)}} \sum_{i=1}^{d_\lambda} e^{-t\lambda}\varphi_i^\lambda(x) \otimes \left(\varphi_i^\lambda\right)^*(y). \tag{2.3}$$

In fact, this series converges uniformly on compact subsets of $\mathbb{R}^+ \times M \times M$.

2.3. Analytic tools

In this subsection, we review some well-known analytic tools. First, we review the notion of Sobolev norms and adopt it for differential forms.

We begin by recalling that for each $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, the Fourier transform of f is defined by $\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\sqrt{-1}x\cdot\xi} f(x)\,dx$, where $x \cdot \xi =$

$\sum_{i=1}^{n} x_i\xi_i$ is the standard dot product on $\mathbb{R}^n$ and $dx = dx^1 \cdots dx^n$ is the standard volume element on $\mathbb{R}^n$. Recall that the L^2 space on $\mathbb{R}^n$ is given by $L^2(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R} \text{ measurable functions} : \int_{\mathbb{R}^n} |f|^2 \, dx \right\}$ together with the inner product $(f|g) := \int_{\mathbb{R}^n} f\overline{g} dx$, where $\overline{g}$ is the complex conjugate of g. It is well-known that we can extend the notion of Fourier transform to L^2 functions. Recall also that the Parseval's formula: $\left(\widehat{f}|\widehat{g}\right) = (f|g)$ for any two $f, g \in L^2(\mathbb{R}^n)$.

Let $m \in \mathbb{N} \cup \{0\}$. Define the Sobolev norm of order m on $L^2(\mathbb{R}^n)$ by

$$\|f\|_m := \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^m \left|\hat{f}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}}.$$

We put $W^m(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) : \|f\|_m < \infty \right\}$. By the Parseval's formula, we see that $\|f\|_0 = \|f\|_{L^2(\mathbb{R}^n)}$. Let $U \subset \mathbb{R}^n$ be an open subset and we set

$$W_c^m(U) := \{ f \in W^m(\mathbb{R}^n) : \operatorname{supp} f \subset U \text{ is compact} \}.$$

Let U be a subset of $\mathbb{R}^n$. For each $l \in \mathbb{N} \cup \{0\}$, define the $\mathcal{C}^l$-norm on $\mathcal{C}^l(U)$ by

$$\|f\|_{\mathcal{C}^l(U)} := \sum_{|\alpha| \le l} \sup_U |\partial_x^\alpha f|$$

for each $f \in \mathcal{C}^l(U)$. Let us state the Sobolev embedding theorem as follows:

Theorem 2.3. *If $f \in W^m(\mathbb{R}^n)$, $m \ge \frac{n}{2} + 1 + l$, then $f \in \mathcal{C}^l(\mathbb{R}^n)$ and*

$$\|f\|_{\mathcal{C}^l(\mathbb{R}^n)} \le C \|f\|_m.$$

Let $m \in \mathbb{N} \cup \{0\}$. We define the Sobolev norm of order $-m$ by duality: given an open subset $U \subset \mathbb{R}^n$, let $W_c^{-m}(U)$ be the set consisting of measurable functions f on U for which there exists $C > 0$ such that $\left|\int_U fg dx\right| \le C \|g\|_m$ for each $g \in W_c^m(U)$; then we define the Sobolev norm of order $-m$ on $W_c^m(U)$ by

$$\|f\|_{-m} := \sup_{\substack{g \in W_c^m(U) \\ g \neq 0}} \frac{\left|\int_U fg dx\right|}{\|g\|_m}.$$

If $m = 0$, then this duality defined norm coincides the usual L^2-norm $\|\cdot\|_{L^2(U)}$, so we included this case as we defined the negative norms.

It is clear to see $\mathcal{C}_c^\infty(U) \subset W_c^{-m}(U)$. In fact, the negative norm of $f \in \mathcal{C}_c^\infty(U)$ has an upper bound in terms of Fourier transform.

Proposition 2.4. *Let $U \subset \mathbb{R}^n$ be an open subset and let $f \in \mathcal{C}_c^\infty(U)$. Then for each $m \in \mathbb{N} \cup \{0\}$,*

$$\|f\|_{-m} \leq \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-m} \left|\widehat{f}(\xi)\right|^2 \, d\xi\right)^{\frac{1}{2}}. \tag{2.4}$$

Proof. By the Parseval's formula, we derive that

$$\begin{aligned}\left|\int_U fgd\xi\right| &= \left|\int_{\mathbb{R}^n} \hat{f}\hat{g}d\xi\right| = \left|\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{m}{2}} \hat{f} \cdot \left(1+|\xi|^2\right)^{\frac{m}{2}} \hat{g}d\xi\right| \\ &\leq \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-m} \left|\widehat{f}(\xi)\right|^2 \, d\xi\right)^{\frac{1}{2}} \|g\|_m,\end{aligned}$$

which implies (2.4). □

To extend the notions of Sobolev norms and $\mathcal{C}^l$-norms to the Riemannian vector bundle (E, M) over a Riemannian manifold M, we choose a pair $(\mathcal{V}, \mathcal{P}, \mathcal{E})$ as follows: let $\mathcal{V}$ be a set given by chosen coordinate charts of M such that their coordinate domains cover M, let $\mathcal{P}$ be a partition of unity $\mathcal{P}$ subordinate to the open cover from $\mathcal{V}$ and $\sum_{\psi\in\mathcal{P}} \psi^2 = 1$, and let $\mathcal{E}$ be the set collecting chosen local orthonormal frames $\{E_I\}_I$ of E over the coordinate domains from $\mathcal{V}$.

Let $m \in \mathbb{Z}$. For each smooth section $s \in \mathcal{C}_c^\infty(M, E)$, define the Sobolev norm of order m by

$$\|s\|_m := \Big(\sum_{\psi_i\in\mathcal{P}} \|\psi_i s\|_m^2\Big)^{\frac{1}{2}},$$

where

$$\|\psi_i s\|_m := \Big(\sum_I \left\|(\psi_i s_I)\circ\varphi_i^{-1}\right\|_m^2\Big)^{\frac{1}{2}},$$

and $(V_i, \varphi_i) \in \mathcal{V}$, $\psi_i s = \sum_I \psi_i s_I E_I$ in V_i, and $\left\|(\psi_i s_I)\circ\varphi_i^{-1}\right\|_m$ is then defined as above. Note that $\|\cdot\|_0 = \|\cdot\|_{L^2(M,E)}$.

Let $U \subset M$ be a subset. For each $l \in \mathbb{N} \cup \{0\}$, define the $\mathcal{C}^l$-norm of $s \in \mathcal{C}^\infty(M, E)$ by

$$\|s\|_{\mathcal{C}^l(U)} := \Big(\sum_{\psi_i \in \mathcal{P}} \|\psi_i s\|^2_{\mathcal{C}^l(U)} \Big)^{\frac{1}{2}},$$

where

$$\|\psi_i s\|_{\mathcal{C}^l(U)} := \sup_{U \cap V_i} \sum_{|\alpha| \leq l} \Big(\sum_I \left|\partial^\alpha \left[(\psi_i s_I) \circ \varphi_i^{-1} \right] \right|^2 \Big)^{\frac{1}{2}}$$

and $(V_i, \varphi_i) \in \mathcal{V}$, $\psi_i s = \sum_I \psi_i s_I E_I$ in V_i.

If both of the manifold M and the vector bundle E are global (for example, $M = \mathbb{R}^n$ and $E = \bigwedge^r T^*\mathbb{R}^n$), then choosing a partition of unity to define the Sobolev norms and $\mathcal{C}^l$-norms is redundant.

Finally, we review the spectral theorem in functional analysis.

Theorem 2.5. *Let X be a Hilbert space and let $A : \mathrm{Dom}A \subset X \to X$ be a self-adjoint operator with the spectrum $S = \mathrm{Spec}A$. Then there exists a finite measure μ on $S \times \mathbb{N}$ and a unitary operator $U : X \to L^2(S \times \mathbb{N})$ that is one-to-one, onto with the following properties:*

(a) *Let $\eta \in X$. Then $\eta \in \mathrm{Dom}A$ if and only if $s \cdot U(\eta) \in L^2(S \times \mathbb{N})$;*

(b) *Define the operator $\mathcal{S}$ by*

$$\begin{aligned} \mathcal{S} : \mathrm{Dom}\mathcal{S} \subset L^2(S \times \mathbb{N}) &\to L^2(S \times \mathbb{N}) \\ g(s,n) &\mapsto sg(s,n), \end{aligned}$$

where

$$\mathrm{Dom}\mathcal{S} = \left\{ g(s,n) \in L^2(S \times \mathbb{N}) : sg(s,n) \in L^2(S \times \mathbb{N}) \right\};$$

then $UAU^{-1} = \mathcal{S}$ on $U(\mathrm{Dom}A)$.

It follows from Theorem 2.5 that we can identify the element $\omega \in \mathrm{Dom}A$ with the element $g = U(\omega) \in L^2(S \times \mathbb{N})$, the operator A with the operator $\mathcal{S}$. Additionally, if A is non-negative, then the heat operator e^{-tA} can be identified with the operator defined by

$$\begin{aligned} P(t) : L^2(S \times \mathbb{N}) &\to L^2(S \times \mathbb{N}) \\ g(s,n) &\mapsto e^{-ts} g(s,n). \end{aligned}$$

3. Scaled Heat Kernel Asymptotics

In this section, we prove Theorem 1.1. To do so, we introduce the localization and scaling technique.

3.1. Localization

To capture the local geometric data attached to the critical points of a Morse function, we introduce the locally flat metric as follows:

Theorem 3.1 ([4], [15]). *Let M be a compact orientable smooth manifold of dimension n and f be a Morse function. Then M admits a metric g and coordinate charts (U_p, φ_p) around critical points $p \in \mathrm{Crit}\,(f)$ such that in each of the coordinate charts (U_p, φ_p), we have*

$$f\,(x) = -\sum_{i=1}^{l} \frac{1}{2}\left(x^i\right)^2 + \sum_{i=l+1}^{n} \frac{1}{2}\left(x^i\right)^2$$

with $l = \mathrm{Ind}_f p$ and

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta^i_j,$$

where δ^i_j is the Kronecker delta.

The construction of this metric is due to the Morse lemma and techniques in [4], but for the reader's convenience, we give a proof:

***Proof of Theorem* 3.1.** By the Morse lemma, we have for each $p \in \mathrm{Crit}\,(f)$, there exists a coordinate chart (V_p, φ_p) around p such that f is expressed as shown above.

Now, assume $V_p \cap V_{p'} = \emptyset$ if $p \neq p'$. For each p, let $U_p \subset \overline{U_p} \subset V_p$ and put

$$W = M \setminus \bigsqcup_{p \in \mathrm{Crit}(f)} \overline{U}_p.$$

Note that W is open. Let $\{W_\alpha\}_\alpha$ be an open cover of W consisting of coordinate neighborhoods. Hence, $\{G_\beta\}_\beta = \{W_\alpha\}_\alpha \cup \{V_p\}_{p \in \mathrm{Crit}(f)}$ is an open cover of M. (G_β, φ_β) denotes the coordinate chart for each β. Define

$$g_\beta = \varphi_\beta^* \overline{g},$$

where $\overline{g}$ is the usual Euclidean metric on $\varphi_\beta (G_\beta)$ and $\varphi_\beta^* \overline{g}$ is the pullback of $\overline{g}$ by φ_β.

Finally, let $\{\psi_\beta\}_\beta$ be a partition of unity subordinate to the cover $\{G_\beta\}_\beta$ and put

$$g = \sum_\beta \psi_\beta g_\beta.$$

Then g is a Riemannian metric on M. Note that in each of the coordinate charts (U_p, φ_p), we have

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta^i_j,$$

since $\psi_p (x) = 1$ in U_p. Therefore, we have furnished the desired metric. $\square$

In the sequel, we adopt this metric throughout this paper together with the local charts around the critical points of a Morse function in question.

Under such a metric, the local representation (2.2) can be reduced to

$$\Delta_k^{(r)} \left(\omega dx^I\right) = \left[-\sum_{i=1}^n \frac{\partial^2}{\partial \left(x^i\right)^2} + k^2 \left(x^i\right)^2 + k \sum_{i=1}^n \varepsilon_i \varepsilon_i^I\right] \omega dx^I \tag{3.1}$$

under the coordinate chart (U_p, φ_p) around $p \in \mathrm{Crit} f$, where ε_i and ε_i^I are defined respectively by

$$\varepsilon_i = \begin{cases} -1 \ ; & i \leq \mathrm{Ind}_f p \\ 1 \ ; & \text{otherwise} \end{cases}$$

and by

$$\varepsilon_i^I = \begin{cases} 1 \ ; & i \text{ appears in } I \\ -1 \ ; & \text{otherwise} \end{cases},$$

for each smooth r-form $\omega = \omega_I dx^I$ acting on $\Delta_k^{(r)}$ with $I = (i_1, \ldots, i_r)$ in increasing order. As you can see, this local representation looks exactly like the harmonic oscillator with perturbation accordingly by the critical point p.

For each $p \in \operatorname{Crit}(f)$, define $\Delta_{f,p}^{(r)}$ to be the differential operator acting on smooth r-forms on an open subset in $\mathbb{R}^n$ given by

$$\Delta_{f,p}^{(r)}\left(\omega dx^I\right) = \Delta_1^{(r)}\left(\omega dx^I\right) = \left[-\sum_{i=1}^{n}\frac{\partial^2}{\partial\left(x^i\right)^2} + \left(x^i\right)^2 + \sum_{i=1}^{n}\varepsilon_i\varepsilon_i^I\right]\omega dx^I. \tag{3.2}$$

Note that $\Delta_{f,p}^{(r)}$ is the system of the harmonic oscillators attached to the critical point $p \in \operatorname{Crit}(f)$ as in Witten's paper [15].

3.2. Scaling technique

Let $p \in \operatorname{Crit}(f)$ and let (U_p, φ_p) be the coordinate chart around p such that $U_p = \varphi_p^{-1}\left(B_{\frac{3}{2}}(0)\right)$ with a fixed $\delta > 0$.

Given a sufficiently large $k > 0$, put

$$U_p^k := \varphi_p^{-1}\left(B_{k^{-\frac{1}{2}+\varepsilon}}(0)\right) \subset U_p$$

with $\varepsilon \in \left(0, \frac{1}{2}\right)$.

Let $\omega = \sum_I' \omega_I dx^I \in \Omega^r\left(B_{k^\varepsilon}(0)\right)$ and define

$$\omega_{[k]}(x) = \sum_I{}' \omega_I\left(\sqrt{k}x\right)dx^I \in \Omega^r\left(B_{k^{-\frac{1}{2}+\varepsilon}}(0)\right).$$

Through the coordinate map φ_p, we see that the pulled back form of $\omega_{[k]}$ is of $\Omega^r\left(U_p^k\right)$ and we still denote it by $\omega_{[k]}$.

We give the following formula to illustrate how these two operators $\Delta_{f,p}^{(r)}$ and $\Delta_k^{(r)}$ relate to one another.

Proposition 3.2. *For each $k > 0$, and for each $\omega \in \Omega^r\left(B_{k^\varepsilon}(0)\right)$,*

$$\Delta_{f,p}^{(r)}\omega = \frac{1}{k}\left(\Delta_k^{(r)}\omega_{[k]}\right)_{\left[\frac{1}{k}\right]}. \tag{3.3}$$

Proof. This formula follows from the local expression (3.1) and change of variables. □

Define the scaled heat kernel at $p \in \mathrm{Crit}\,(f)$ by

$$A^r_{(k),p}\,(t,x,y) := k^{-\frac{n}{2}} e^{-\frac{t}{k}\Delta^{(r)}_k}\left(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\right)$$
$$\in \mathcal{C}^\infty\left(\mathbb{R}^+ \times B_{k^\varepsilon}\,(0) \times B_{k^\varepsilon}\,(0)\,, \bigwedge^r (T^*\mathbb{R}^n) \boxtimes \left(\bigwedge^r (T^*\mathbb{R}^n)\right)^*\right),$$

where $\frac{x}{\sqrt{k}} \in \mathbb{R}^n$, for the sake of convenience, stands for $\varphi_p^{-1}\left(\frac{x}{\sqrt{k}}\right)$. We can write $A^r_{(k),p}\,(t,x,y)$ as

$$A^r_{(k),p}\,(t,x,y) = {\sum_{I,J}}' A^r_{(k),pI,J}\,(t,x,y)\,dx^I\,(x) \otimes \left(dx^J\right)^*(y)\,,$$

where the component functions

$$A^r_{(k),p}\,(t,x,y)\,_{I,J}\,(t,x,y) = k^{-\frac{n}{2}} e^{-\frac{t}{k}\Delta^{(r)}_k}\,_{I,J}\Big(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\Big).$$

For each $t>0$, define the scaled heat operator at p by

$$A^r_{(k),p}\,(t) : \Omega^r_c\,(B_{k^\varepsilon}\,(0)) \to \Omega^r\,(B_{k^\varepsilon}\,(0))$$
$$\omega \mapsto \int_{B_{k^\varepsilon}(p)} A^r_{(k),p}\,(t,x,y)\,\omega\,(y)\;dy.$$

Note that $A^r_{(k),p}\,(t)\,\omega \in \Omega^r\,(\mathbb{R}^+ \times B_{k^\varepsilon}\,(0))$ for each $\omega \in \Omega^r_c\,(B_{k^\varepsilon}\,(0))$.

Let us show how the scaled heat kernel/operator relates to the ordinary heat kernel/operator as follows:

Proposition 3.3. *For each $k>0$*

$$A^r_{(k),p}\,(t)\,\omega = \left(e^{-\frac{t}{k}\Delta^{(r)}_k}\omega_{[k]}\right)_{\left[\frac{1}{k}\right]} \tag{3.4}$$

for each $\omega \in \Omega^r_c\,(B_{k^\varepsilon}\,(0))$.

Proof. By change of variables, for each $\omega \in \Omega^r_c\,(B_{k^\varepsilon}\,(0))$, we deduce

$$\left(A^r_{(k),p}\,(t)\,\omega\right)(x) = k^{-\frac{n}{2}} \int_{B_{k^\varepsilon}(p)} e^{-\frac{t}{k}\Delta^{(r)}_k}\Big(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\Big)\omega\,(y)\,dy$$
$$= \int_{B_{k^{-\frac{1}{2}+\varepsilon}}(p)} e^{-\frac{t}{k}\Delta^{(r)}_k}\Big(\frac{x}{\sqrt{k}},y\Big)\omega_{[k]}\,(y)\;dy$$

$$= \int_M e^{-\frac{t}{k}\Delta_k^{(r)}} \Big(\frac{x}{\sqrt{k}}, y\Big) \omega_{[k]}(y)\, dV_M = \Big(e^{-\frac{t}{k}\Delta_k^{(r)}} \omega_{[k]}\Big)_{\left[\frac{1}{k}\right]}(x). \qquad \square$$

Now, we have the following important observation:

Lemma 3.4. *For each $k > 0$*

$$\begin{cases} \frac{\partial}{\partial t} A^r_{(k),p}(t)\,\omega + \Delta^{(r)}_{f,p} A^r_{(k),p}(t)\,\omega = 0 \\ \lim_{t\to 0^+} \left\| A^r_{(k),p}(t)\,\omega - \omega \right\|_{L^2(B_{k^\varepsilon}(0))} = 0 \end{cases}$$

for each $\omega \in \Omega^r_c(B_{k^\varepsilon}(0))$.

Proof. This lemma essentially follows from (3.3) and (3.4). Given $\omega \in \Omega^r_c(B_{k^\varepsilon}(0))$, we deduce

$$\begin{aligned} \frac{\partial}{\partial t} A^r_{(k),p}(t)\,\omega &= \frac{1}{k}\left(\frac{\partial}{\partial t} e^{-\frac{t}{k}\Delta_k^{(r)}} \omega_{[k]}\right)_{\left[\frac{1}{k}\right]} = -\frac{1}{k}\left(\Delta_k^{(r)} e^{-\frac{t}{k}\Delta_k^{(r)}} \omega_{[k]}\right)_{\left[\frac{1}{k}\right]} \\ &= -\Delta^{(r)}_{f,p}\left(e^{-\frac{t}{k}\Delta_k^{(r)}} \omega_{[k]}\right)_{\left[\frac{1}{k}\right]} = -\Delta^{(r)}_{f,p} A^r_{(k),p}(t)\,\omega. \end{aligned}$$

where the first equation holds by the chain rule. Hence, we obtain

$$\frac{\partial}{\partial t} A^r_{(k),p}(t)\,\omega + \Delta^{(r)}_{f,p} A^r_{(k),p}(t)\,\omega = 0$$

for each $\omega \in \Omega^r_c(B_{k^\varepsilon}(0))$.

Finally, note that

$$\begin{aligned} \lim_{t\to 0^+} \left\| A^r_{(k),p}(t)\,\omega - \omega \right\|_{L^2\left(B_{k^\varepsilon}(0)\right)} &= \lim_{t\to 0^+} \left\| e^{-\frac{t}{k}\Delta_k^{(r)}} \omega_{[k]} - \omega_{[k]} \right\|_{L^2\left(B_{k^{-\frac{1}{2}+\varepsilon}}(0)\right)} \\ &= \lim_{t\to 0^+} \left\| e^{-\frac{t}{k}\Delta_k^{(r)}} \omega_{[k]} - \omega_{[k]} \right\|_{L^2(M)} = 0. \qquad \square \end{aligned}$$

Lemma 3.4 motivates us to seek a local bound for scaled heat kernels stated as follows:

Theorem 3.5 (Local Boundedness of the Scaled Heat Kernels)**.** *Given $p \in \mathrm{Crit}(f)$, let T and K be compact subsets in $\mathbb{R}^+$ and in $\mathbb{R}^n$, respectively. Then for each $l \in \mathbb{N} \cup \{0\}$, the sequence $\left\{A^r_{(k),p}(t,x,y)\right\}_k$ is uniformly bounded in $T \times K \times K$ with respect to the $\mathcal{C}^l$-norm; namely, there exists a constant*

$C(T,K)>0$ *such that*

$$\left\|A^r_{(k),p}(t,x,y)\right\|_{\mathcal{C}^l(T\times K\times K)}\leq C(T,K)$$

for each k *sufficiently large.*

We will prove Theorem 3.5 in Section 3.3.

3.3. Locally uniform bound for scaled heat kernels

We find a locally uniform bound through investigating the scaled heat operators. The key of finding such a bound is that we manage to show the scaled heat operators are "locally uniformly bounded" by a constant independent of k. We illustrate it in what we call the mapping property as follows:

Theorem 3.6 (Mapping Property)**.** *Given a compact subset* $K\subset\mathbb{R}^n$*, choose a bounded open subset* U *such that* $K\subset U$ *and choose a cut-off function* $\chi\in\mathcal{C}^\infty_c(U)$ *such that* $\chi=1$ *in* K*. Then for each* $p\in\mathrm{Crit}(f)$ *and for each cut-off function* $\tilde{\chi}\in\mathcal{C}^\infty_c(U)$*, there exists a constant* $C(\chi,\tilde{\chi},t)>0$ *such that*

$$\left\|\tilde{\chi}A^r_{(k),p}(t)\,\chi\omega\right\|_{2m}\leq C(\chi,\tilde{\chi},t)\,\|\omega\|_{-2m},$$

where $C(\chi,\tilde{\chi},t)$ *depends on the choices of* χ *and* $\tilde{\chi}$ *and smoothly on* t *only, for each* $\omega\in\Omega^r_c(U)$*, if* k *is large enough.*

Proof. Before we start, we point out that the following reasoning works for k large enough for us to have $U\subset B_{k^\varepsilon}(0)$, $\varepsilon\in\left(0,\frac{1}{2}\right)$.

Firstly, by the Gårding's inequality, we obtain

$$\begin{aligned}\left\|\tilde{\chi}A^r_{(k),p}(t)\,\chi\omega\right\|_{2m}\leq\,&C_1(\tilde{\chi})\left\|\chi_1\left(\Delta^{(r)}_{f,p}\right)^m A^r_{(k),p}(t)\,\chi\omega\right\|_0\\&+C_2(\tilde{\chi})\left\|\chi_2A^r_{(k),p}(t)\,\chi\omega\right\|_0,\end{aligned}$$

where $\chi_1,\chi_2\in\mathcal{C}^\infty_c(U)$ are the cut-off functions chosen to satisfy $\chi_1=1=\chi_2$ in $\operatorname{supp}\tilde{\chi}$. Therefore, we reduce to the L^2-norm estimates for the two term on the right.

Next, to estimate the L^2-norm of the first term, observe that

$$\left\|\chi_1\left(\Delta_{f,p}^{(r)}\right)^m A_{(k),p}^r(t)\,\chi\omega\right\|_0 := \sup_{\eta\in\Omega_c^r(U),\eta\neq 0}\frac{\left|\left(\chi_1(\Delta_{f,p}^{(r)})^m A_{(k),p}^r(t)\,\chi\omega\middle|\eta\right)\right|}{\|\eta\|_0},$$

so we consider the inner product on the right hand side. By Theorem 2.5, (3.3) and (3.4), we see that

$$\begin{aligned}\left|\left(\chi_1(\Delta_{f,p}^{(r)})^m A_{(k),p}^r(t)\,\chi\omega\middle|\eta\right)\right| &= k^{-m}\left|\left((\chi_1)_{[k]}\left(\Delta_k^{(r)}\right)^m e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_{[k]}\omega_{[k]}\middle|\eta_{[k]}\right)\right|\\ &= k^{-m}\left|\left(\omega_{[k]}\middle|\chi_{[k]}e^{-\frac{t}{k}\Delta_k^{(r)}}\left(\Delta_k^{(r)}\right)^m(\chi_1)_{[k]}\,\eta_{[k]}\right)\right|\\ &= \left|\left(\omega\middle|\chi A_{(k),p}^r(t)\left(\Delta_{f,p}^{(r)}\right)^m\chi_1\eta\right)\right|\end{aligned}$$

for each $\eta\in\Omega_c^r(U)$ with $\eta\neq 0$. Moreover, by definition of Sobolev norms, we see that

$$\left|\left(\omega\middle|\chi A_{(k),p}^r(t)\,(\Delta_{f,p}^{(r)})^m\chi_1\eta\right)\right|\leq\|\omega\|_{-2m}\left\|\chi A_{(k),p}^r(t)\left(\Delta_{f,p}^{(r)}\right)^m\chi_1\eta\right\|_{2m}.$$

Again, by the Gårding's inequality, we obtain

$$\begin{aligned}\left\|\chi A_{(k),p}^r(t)\left(\Delta_{f,p}^{(r)}\right)^m\chi_1\eta\right\|_{2m}\leq{}& C_3(\chi)\left\|\chi_3\left(\Delta_{f,p}^{(r)}\right)^m A_{(k),p}^r(t)\left(\Delta_{f,p}^{(r)}\right)^m\chi_1\eta\right\|_0\\ &+C_4(\chi)\left\|\chi_4 A_{(k),p}^r(t)\left(\Delta_{f,p}^{(r)}\right)^m\chi_1\eta\right\|_0,\end{aligned}$$

where $\chi_3,\chi_4\in\mathcal{C}_c^\infty(U)$ are the cut-off functions chosen to satisfy $\chi_3=1=\chi_4$ in $\operatorname{supp}\chi$. Moreover, by Theorem 2.5, (3.3) and (3.4), we derive

$$\begin{aligned}&\left\|\chi_3\left(\Delta_{f,p}^{(r)}\right)^m A_{(k),p}^r(t)\left(\Delta_{f,p}^{(r)}\right)^m\chi_1\eta\right\|_0^2\\ &\leq k^{\frac{n}{2}-4m}\left\|\left(\Delta_k^{(r)}\right)^m e^{-\frac{t}{k}\Delta_k^{(r)}}\left(\Delta_k^{(r)}\right)^m(\chi_1\eta)_{[k]}\right\|_{L^2(M)}^2\\ &\leq k^{\frac{n}{2}}\int_{\mathcal{S}\times\mathbb{N}}\left(\frac{s}{k}\right)^{4m}e^{-2t\frac{s}{k}}|g(s,n)|^2\,d\mu\\ &\leq k^{\frac{n}{2}}C_5(t)\int_{\mathcal{S}\times\mathbb{N}}|g(s,n)|^2\,d\mu\\ &= C_5(t)\,\|\chi_1\eta\|_0^2\leq C_5(t)\,\|\eta\|_0^2,\end{aligned}$$

where $C_5(t) = \left(\frac{2m}{t}\right)^{4m} e^{-4m} > 0$ depends smoothly on t. Hence, we conclude

$$\left\|\chi_3 \left(\Delta_{f,p}^{(r)}\right)^m A_{(k),p}^r(t) \left(\Delta_{f,p}^{(r)}\right)^m \chi_1 \eta\right\|_0 \leq C_6(t) \|\eta\|_0, \tag{3.5}$$

where $C_6(t) = \sqrt{C_5(t)}$. Similarly, we can obtain

$$\left\|\chi_4 A_{(k),p}^r(t) \left(\Delta_{f,p}^{(r)}\right)^m \chi_1 \eta\right\|_0 \leq C_7(t) \|\eta\|_0, \tag{3.6}$$

where $C_7(t)$ depends smoothly on t. By (3.5) and (3.6), we conclude

$$\left\|\chi A_{(k),p}^r(t) \left(\Delta_{f,p}^{(r)}\right)^m \chi_1 \eta\right\|_{2m} \leq C_8(\chi, t) \|\eta\|_0,$$

for each $\eta \in \Omega_c^r(U)$ with $\eta \neq 0$, which in turn, implies

$$\left\|\chi_1 \left(\Delta_{f,p}^{(r)}\right)^m A_{(k),p}^r(t) \chi \omega\right\|_0 \leq C_9(\chi, t) \|\omega\|_{-2m}, \tag{3.7}$$

where $C_9(\chi, t)$ depends on χ and smoothly on t.

Using the similar argument, we can obtain

$$\left\|\chi_2 A_{(k),p}^r(t) \chi \omega\right\|_0 \leq C_{10}(\chi, t) \|\omega\|_{-2m}, \tag{3.8}$$

where $C_9(\chi, t)$ depends on χ and smoothly on t.

Finally, by (3.7) and (3.8), we have established

$$\left\|\tilde{\chi} A_{(k),p}^r(t) \chi \omega\right\|_{2m} \leq C(\chi, \tilde{\chi}, t) \|\omega\|_{-2m},$$

where $C(\chi, \tilde{\chi}, t)$ depends on $\chi, \tilde{\chi}$ and smoothly on t, for each $\omega \in \Omega_c^r(U)$ with $\operatorname{supp} \omega \subset K$. $\square$

Now, to show Theorem 3.5, let us recall the definition of approximate identities. Let $B_1(0)$ be the unit Euclidean ball centered at 0 and choose a cut-off function $\chi \in \mathcal{C}_c^\infty(B_1(0))$ such that

$$\int_{\mathbb{R}^n} \chi(x)\ dx = 1.$$

For each $x_0 \in \mathbb{R}^n$ and for each $\delta > 0$, define

$$\chi_{x_0,\delta}(x) := \delta^{-n}\chi\Big(\frac{x-x_0}{\delta}\Big) \in \mathcal{C}_c^\infty(B_\delta(x_0)). \tag{3.9}$$

We call the family $\{\chi_{x_0,\delta}\}_\delta$ an approximate identity with respect to the point x_0. Note that a change of variable gives

$$\int_{\mathbb{R}^n} \chi_{x_0,\delta}(x)\ dx = \int_{\mathbb{R}^n} \chi(x)\ dx = 1.$$

An important feature of an approximate identity is illustrated as below.

Lemma 3.7. *Given a multi-index α, there exists a constant $C > 0$ such that for each $x_0 \in \mathbb{R}^n$, for each α and for each I, set*

$$\omega_{\alpha,x_0,I,\delta} = \partial^\alpha \chi_{x_0,\delta}\ dx^I \in \Omega_c^r(B_\delta(x_0)) \tag{3.10}$$

and we obtain

$$\limsup_{\delta\to 0} \|\omega_{\alpha,x_0,I,\delta}\|_{-m} \leq C,$$

if m is sufficiently large.

Proof. By Proposition 2.4, it suffices to show there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-m} \left|\widehat{\omega_{x_0,\alpha,I,\delta}}(\xi)\right|^2\ d\xi \leq C$$

for each δ and for each x_0, if m is large enough.

First, observe that

$$\left|\widehat{\omega_{x_0,\alpha,I,\delta}}(\xi)\right|^2 = \left|\int_{\mathbb{R}^n} e^{-\sqrt{-1}x\cdot\xi}\partial^\alpha\chi_{x_0,\delta}(x)\ dx\right|^2 \leq |\xi|^{2|\alpha|}\int_{\mathbb{R}^n}\chi\ dx = |\xi|^{2|\alpha|}.$$

Now, if $m \geq \frac{n}{2} + |\alpha| + 1$, then

$$\begin{aligned}\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-m} \left|\widehat{\omega_{x_0,\alpha,I,\delta}}(\xi)\right|^2 d\xi &\leq \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-m} |\xi|^{2|\alpha|}\, d\xi \\ &\leq C_1 \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-m+|\alpha|} d\xi \leq C_2,\end{aligned}$$

where C_1, C_2 are independent of δ and the choice of x_0. Hence, we have established Lemma 3.7. $\square$

***Proof of Theorem* 3.5.** Let T be a compact interval in $\mathbb{R}^+$ and K be a compact subset in $\mathbb{R}^n$. We choose $M_K \in \mathbb{N}$ such that

$$K \subset B_{k^\varepsilon}(0)$$

for all $k \geq M_K$ and, without loss of generosity, discard the rest k's for which $K \not\subset B_{k^\varepsilon}(0)$.

Write

$$A^r_{(k),p}(t,x,y) = {\sum_{I,J}}' A^r_{(k),p}(t,x,y)\, dx^I(x) \otimes \left(dx^J\right)^*(y).$$

To establish Theorem 3.5, it suffices to show its component functions are locally uniformly bounded; in other words, we show for any two multi-indices I', J', there exists a constant $C(T,K)$ such that

$$\left\| A^r_{(k)I',J'}(t,x,y) \right\|_{\mathcal{C}^l(T\times K\times K)} \leq C(T,K).$$

First of all, we show that for each $t_0 \in T$, for each $y_0 \in K$, for each α, and for each J',

$$\lim_{\delta\to 0} \left\| A^r_{(k),p}(t_0)\, \omega_{\alpha,y_0,J',\delta} - {\sum_I}' \partial_y^\alpha A_{(k),pI,J'}(t_0,x,y_0)\, dx^I \right\|_{\mathcal{C}^l(K)} = 0, \quad (3.11)$$

where $\omega_{\alpha,y_0,J',\delta}$ is as defined in (3.10). To see it, for each $x \in K$, note that

$$\sum_{|\alpha|\leq l} \left({\sum_I}' \left| \partial_x^\gamma \left(\int_{\mathbb{R}^n} A^r_{(k),pI,J'}(t_0,x,y)\, \partial_y^\alpha \chi_{y_0,\delta} dy - \partial_y^\alpha A^r_{(k),pI,J'}(t_0,x,y_0) \right) \right|^2 \right)^{\frac{1}{2}}$$

$$= \sum_{|\alpha|\leq l} \Bigg({\sum_I}' \Bigg| \Big(\int_{\mathbb{R}^n} \partial_x^\gamma \big(\partial_y^\alpha A^r_{(k),pI,J'}(t_0,x,y)$$

$$- \partial_y^\alpha A^r_{(k),pI,J'}(t_0,x,y_0) \big) \chi_{y_0,\delta} dy \Big) \Bigg|^2 \Bigg)^{\frac{1}{2}}$$

$$= \sum_{|\alpha|\leq l} \Bigg({\sum_I}' \Bigg| \Big(\int_{\mathbb{R}^n} \big(\partial_x^\gamma \partial_y^\alpha A^r_{(k),pI,J'}(t_0,x,\delta y + y_0)$$

$$- \partial_x^\gamma \partial_y^\alpha A^r_{(k),pI,J'}(t_0,x,y_0)\,)\chi\, dy\Big)\Big|^2\Big)^{\frac{1}{2}}.$$

Now, since the function $\partial_x^\gamma \partial_y^\alpha A^r_{(k),pI,J'}(t_0,x,y)$ is uniformly continuous in $K \times K$, we obtain (3.11) and thus

$$\Big\| \sum_I{}' \partial_y^\alpha A^r_{(k),pI,J'}(t_0,x,y_0)\, dx^I \Big\|_{\mathcal{C}^l(K)} = \lim_{\delta\to 0} \left\| A_{(k),p}(t_0)\,\omega_{y_0,\alpha,J',\delta} \right\|_{\mathcal{C}^l(K)}. \tag{3.12}$$

Next, we deduce to find a local bound. Let W be a chosen open subset such that $K \subset W \Subset B_{k^\varepsilon}(p)$ for each $k \geq M_K$ and let $\chi, \tilde{\chi} \in \mathcal{C}_c^\infty(W)$ be two cut-off functions such that $\chi = 1$ in K and $\tilde{\chi} = 1$ in $\operatorname{supp}\chi$. By Theorem 2.3, there exists $C_1 > 0$ independent of k and y_0, such that

$$\left\| A_{(k),p}(t_0)\,\omega_{y_0,\alpha,J',\delta} \right\|_{\mathcal{C}^l(K)} \leq C_1 \left\| \tilde{\chi} A^r_{(k),p}(t_0)\,\chi\omega_{y_0,\alpha,J',\delta} \right\|_{2m}, \tag{3.13}$$

for some $m \in \mathbb{N}$. Moreover, by Theorem 3.6, we obtain

$$\left\| \tilde{\chi} A^r_{(k),p}(t_0)\,\chi\omega_{y_0,\alpha,J',\delta} \right\|_{2m} \leq C_2(\chi,\tilde{\chi},t)\, \|\omega_{y_0,\alpha,J,\delta}\|_{-2m} \leq C_3(\chi,\tilde{\chi},t_0), \tag{3.14}$$

where C_2, C_3 depend on $\chi, \tilde{\chi}, t_0$, but on neither k nor y_0. Hence, by (3.12), (3.13) and (3.14), we conclude

$$\begin{aligned} \left\| A^r_{(k),pI',J'}(t_0,x,y) \right\|_{\mathcal{C}^l(K\times K)} &\leq \Big\| \sum_I{}' A^r_{(k),pI,J'}(t_0,x,y)\, dx^I \Big\|_{\mathcal{C}^l(K\times K)} \\ &\leq C_3(\chi,\tilde{\chi},t_0) \end{aligned} \tag{3.15}$$

for any two I', J'.

Finally, to establish Theorem 3.5, it remains to deal with t-derivatives. By nature of scaled heat kernels, we see

$$\partial_t^\beta A^r_{(k),p}(t)\,\omega = -\left(\Delta^{(r)}_{f,p}\right)^\beta A^r_{(k),p}(t)\,\omega$$

for each $\omega \in \Omega^r_c(W)$ and the constant C_3 depends smoothly on $t_0 \in T$ (by Theorem 3.6), so by (3.15), we conclude that for each $l \in \mathbb{N} \cup \{0\}$ and for each I', J'

$$\left\| A^r_{(k),pI',J'}(t,x,y) \right\|_{\mathcal{C}^l(T\times K\times K)} \leq C(T,K),$$

where C depends on T and K. Therefore, Theorem 3.5 has been shown. $\square$

3.4. Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1.

Let $\{T_i\}_{i=1}^{\infty} \subset \mathbb{R}^+$ be a collection of compact subsets satisfying $T_i \subset T_{i+1}^{\circ}$ and $\bigcup_{i=1}^{\infty} T_i = \mathbb{R}^+$ and let $\{K_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$ be a collection of compact subsets satisfying $K_i \subset K_{i+1}^{\circ}$ and $\bigcup_{i=1}^{\infty} K_i = \mathbb{R}^n$.

Owing to Theorem 3.5 and the Arzela-Ascoli theorem, there exists a strictly increasing sequence $\{n_{1,k}\}_k \subset \mathbb{N}$ such that the subsequence $\left\{A^r_{(n_{1,k}),p}(t,x,y)\right\}_k$ that converges in $\mathcal{C}^1$-norm in $T_1 \times K_1 \times K_1$. Again, according to Theorem 3.5 and the Arzela-Ascoli theorem, there exists a strictly increasing sequence $\{n_{2,k}\}_k \subset \{n_{1,k}\}_k$ such that the subsequence $\left\{A^r_{(n_{2,k}),p}(t,x,y)\right\}_k$ that converges in $\mathcal{C}^2$-norm in $T_2 \times K_2 \times K_2$. We proceed in the same manner to obtain a subsequence $\left\{A^r_{(n_{i,k}),p}(t,x,y)\right\}_k$ for each i that converges in $\mathcal{C}^i$-norm in $T_i \times K_i \times K_i$. Finally, the diagonal argument enables us to find a subsequence $\left\{A^r_{(n_k),p}(t,x,y)\right\}_{n_k}$ such that

$$\lim_{k\to\infty} A^r_{(n_k),p}(t,x,y) = B^r_p(t,x,y)$$

in $\mathcal{C}^\infty$-topology in each compact subset of $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$. Note that

$$B^r_p(t,x,y) \in \mathcal{C}^\infty\Big(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n; \bigwedge^r (T^*\mathbb{R}^n) \boxtimes \Big(\bigwedge^r (T^*\mathbb{R}^n)\Big)^*\Big).$$

In addition, for each $t > 0$, define the operator $B^r_p(t) : \Omega^r_c(\mathbb{R}^n) \to \Omega^r(\mathbb{R}^n)$ by

$$\left(B^r_p(t)\,\omega\right)(x) = \int_{\mathbb{R}^n} B^r_p(t,x,y)\,\omega(y)\;dy.$$

Note that $B^r_p(t)\,\omega \in \Omega^r(\mathbb{R}^+ \times \mathbb{R}^n)$.

Now, to see $B^r_p(t,x,y) = e^{-t\Delta^{(r)}_{f,p}}(x,y)$, we need to show that for each $t > 0$,

$$B^r_p(t) : \Omega^r_c(\mathbb{R}^n) \to \mathrm{Dom}\Delta^{(r)}_{f,p} := \left\{\omega \in L^2_r(\mathbb{R}^n) : \Delta^{(r)}_{f,p}\omega \in L^2_r(\mathbb{R}^n)\right\} \quad (3.16)$$

and that

$$\begin{cases} \frac{\partial}{\partial t} B_p^r(t)\,\omega + \Delta_{f,p}^{(r)} B_p^r(t)\,\omega = 0 \\ \lim_{t\to 0^+} \left\| B_p^r(t)\,\omega - \omega \right\|_{L^2(\mathbb{R}^n)} = 0 \end{cases} \tag{3.17}$$

for each $\omega \in \Omega_c^r(\mathbb{R}^n)$.

To see (3.16), it suffices to show for each $m \in \mathbb{N} \cup \{0\}$,

$$\left\| \left(\Delta_{f,p}^{(r)} \right)^m B_p^r(t)\,\omega \right\|_{L^2(\mathbb{R}^n)} < \infty$$

for each $\omega \in \Omega_c^r(\mathbb{R}^n)$. Observe that, by Fatou's lemma, we obtain

$$\left\| \left(\Delta_{f,p}^{(r)} \right)^m B_p^r(t)\,\omega \right\|_{L^2(\mathbb{R}^n)} \leq \liminf_{R\to\infty} \left\| \left(\Delta_{f,p}^{(r)} \right)^m B_p^r(t)\,\omega \right\|_{L^2\left(\overline{B_R(0)}\right)}.$$

Now, choose two cut-off function $\chi, \tilde{\chi} \in \mathcal{C}_c^\infty(B_1(0))$ such that $\tilde{\chi} = 1$ in $\operatorname{supp}\chi$, and for each k, put

$$\chi_k(x) := \chi\left(\frac{x}{k^\varepsilon}\right) \in \mathcal{C}_c^\infty\left(B_{k^\varepsilon}(0)\right), \tilde{\chi}_k := \tilde{\chi}\left(\frac{x}{k^\varepsilon}\right) \in \mathcal{C}_c^\infty\left(B_{k^\varepsilon}(0)\right).$$

Then by Theorem 2.5, we derive

$$\begin{aligned} \left\| \left(\Delta_{f,p}^{(r)} \right)^m B_p^r(t)\,\omega \right\|_{L^2\left(\overline{B_R(0)}\right)} &\leq \lim_{k\to\infty} \left\| \tilde{\chi}_{n_k} \left(\Delta_{f,p}^{(r)} \right)^m A_{(n_k),p}^r(t)\,\chi_{n_k}\omega \right\|_{L^2(\mathbb{R}^n)} \\ &\leq C(t)\,\|\omega\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where $C(t)$ is a constant that depends smoothly on t but is independent of R, k. Hence, we conclude

$$\left\| \left(\Delta_{f,p}^{(r)} \right)^m B_p^r(t)\,\omega \right\|_{L^2(\mathbb{R}^n)} \leq C(t)\,\|\omega\|_{L^2(\mathbb{R}^n)} < \infty$$

and (3.16) is now established.

To see (3.17), let $\omega \in \Omega_c^r(\mathbb{R}^n)$. For each $\omega \in \Omega_c^r(\mathbb{R}^n)$, by Theorem 3.5, we see

$$\begin{aligned}\left(\frac{\partial}{\partial t}B_p^r(t)\right)\omega &= \int_{\mathbb{R}^n}\frac{\partial}{\partial t}B_p^r(t,x,y)\,\omega(y)\,dy = \lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{\partial}{\partial t}A_{(n_k),p}^r(t,x,y)\,\omega(y)dy\\ &= -\lim_{k\to\infty}\int_{\mathbb{R}^n}\Delta_{f,p}^{(r)}A_{(n_k),p}^r(t,x,y)\,\omega(y)\,dy\\ &= -\int_{\mathbb{R}^n}\Delta_{f,p}^{(r)}B_p^r(t,x,y)\,\omega(y)\,dy = -\Delta_{f,p}^{(r)}B_p^r(t)\,\omega\end{aligned}$$

for each $(t,x)\in K$, where K is a compact subset in each compact subset of $\mathbb{R}^+\times\mathbb{R}^n$. This shows $\frac{\partial}{\partial t}B_p^r(t)\,\omega+\Delta_{f,p}^{(r)}B_p^r(t)\,\omega=0$.

To see $\lim_{t\to 0^+}\left\|B_p^r(t)\,\omega-\omega\right\|_{L^2(\mathbb{R}^n)}=0$, by Fatou's lemma, we obtain for each $t>0$,

$$\left\|B_p^r(t)\,\omega-\omega\right\|_{L^2(\mathbb{R}^n)}\le\liminf_{R\to\infty}\left\|B_p^r(t)\,\omega-\omega\right\|_{L^2\left(\overline{B_R(0)}\right)}.$$

Again, let $\chi_k,\tilde{\chi}_k\in\mathcal{C}_c^\infty(B_{k^\varepsilon}(0))$ as previously given. By Theorem 2.5 and the mean value theorem, we derive

$$\begin{aligned}\left\|B_p^r(t)\,\omega-\omega\right\|^2_{L^2\left(\overline{B_R(0)}\right)} &\le\lim_{k\to\infty}\left\|\tilde{\chi}_{n_k}\left(A_{(n_k),p}^r(t)\,\chi_{n_k}\omega-\chi_{n_k}\omega\right)\right\|^2_{L^2(\mathbb{R}^n)}\\ &\le\lim_{k\to\infty}(n_k)^{\frac{n}{2}}\left\|e^{-\frac{t}{n_k}\Delta_{n_k}^{(r)}}(\chi_{n_k}\omega)_{[n_k]}-(\chi_{n_k}\omega)_{[n_k]}\right\|^2_{L^2(M)}\\ &=\lim_{k\to\infty}(n_k)^{\frac{n}{2}}\int_{\mathcal{S}\times\mathbb{N}}\left(e^{-\frac{t}{n_k}s}-1\right)^2|g|^2\,d\mu\\ &=\lim_{k\to\infty}(n_k)^{\frac{n}{2}}\left(\frac{t}{n_k}\right)^2\int_{\mathcal{S}\times\mathbb{N}}e^{-\frac{t_k}{n_k}s}\,|sg|^2\,d\mu\\ &\le\lim_{k\to\infty}(n_k)^{\frac{n}{2}}\,t^2\left\|\frac{1}{n_k}\Delta_{n_k}^{(r)}(\chi_{n_k}\omega)_{[n_k]}\right\|^2_{L^2(M)}\\ &=t^2\left\|\Delta_{f,p}^{(r)}\omega\right\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

where $t_k\in(0,t)$ and $g\in L^2(S\times\mathbb{N})$ is identified with $(\chi_{n_k}\omega)_{[n_k]}$ according to Theorem 2.5.

This implies for each $\omega\in\Omega_c^r(\mathbb{R}^n)$,

$$\left\|B_p^r(t)\,\omega-\omega\right\|_{L^2(\mathbb{R}^n)}\le t\left\|\Delta_{f,p}^{(r)}\omega\right\|_{L^2(\mathbb{R}^n)}\to 0$$

as $t \to 0^+$.

Finally, to see

$$B_p^r(t,x,y) = e^{-t\Delta_{f,p}^{(r)}}(x,y)$$

in $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$, it suffices to show

$$B_p^r(t) = e^{-t\Delta_{f,p}^{(r)}}$$

in $\Omega_c^r(\mathbb{R}^n)$. To see it, we can first observe that

$$\lim_{h\to 0}\left\|\left(\frac{B^r(t+h)-B^r(t)}{h} - \frac{\partial}{\partial t}B^r(t)\right)\omega\right\|_{L^2(\mathbb{R}^n)} = 0$$

from the following estimate obtained by Theorem 2.5:

$$\left\|\left(\frac{B_p^r(t+h)-B_p^r(t)}{h} - \frac{\partial}{\partial t}B^r(t)\right)\omega\right\|_{L^2(\mathbb{R}^n)} \leq h\left\|\left(\Delta_{f,p}^{(r)}\right)^2\omega\right\|_{L^2(\mathbb{R}^n)}$$

for each $\omega \in \Omega_c^r(\mathbb{R}^n)$. This shows

$$\frac{d}{dt}\left(B^r(t)\,\omega\middle|\eta\right) = \left(\frac{\partial}{\partial t}B^r(t)\,\omega\middle|\eta\right)$$

for each $\omega \in \Omega_c^r(\mathbb{R}^n)$ and for each $\eta \in L_r^2(\mathbb{R}^n)$. Hence, by the fundamental theorem of Calculus and the fact that $B^r(t)\,\omega \in \mathrm{Dom}\Delta_{f,p}^{(r)}$ for each $\omega \in \Omega_c^r(\mathbb{R}^n)$, we derive

$$\begin{aligned}
&\left(B_p^r(t)\,\omega\middle|\eta\right) - \left(\omega\middle|e^{-t\Delta_{f,p}^{(r)}}\eta\right) = \lim_{q\to 0^+}\int_q^t \frac{d}{ds}\left(B_p^r(s)\,\omega\middle|e^{-(t+q-s)\Delta_{f,p}^{(r)}}\eta\right)ds \\
&= \lim_{q\to 0^+}\int_q^t\left(\left(\left(\frac{\partial}{\partial s}B_p^r(s)\right)\omega\middle|e^{-(t+q-s)\Delta_{f,p}^{(r)}}\eta\right) - \left(B_p^r(s)\,\omega\middle|\left(\frac{\partial}{\partial s}e^{-(t+q-s)\Delta_{f,p}^{(r)}}\right)\eta\right)\right)ds \\
&= \lim_{q\to 0^+}\int_q^t\left(\left(-\Delta_{f,p}^{(r)}B_p^r(s)\,\omega\middle|e^{-(t+q-s)\Delta_{f,p}^{(r)}}\eta\right) + \left(B_p^r(s)\,\omega\middle|\Delta_{f,p}^{(r)}e^{-(t+q-s)\Delta_{f,p}^{(r)}}\eta\right)\right)ds \\
&= \lim_{q\to 0^+}\int_q^t\left(\left(-\Delta_{f,p}^{(r)}B_p^r(s)\,\omega\middle|e^{-(t+q-s)\Delta_{f,p}^{(r)}}\eta\right) + \left(\Delta_{f,p}^{(r)}B_p^r(s)\,\omega\middle|e^{-(t+q-s)\Delta_{f,p}^{(r)}}\eta\right)\right)ds \\
&= 0
\end{aligned}$$

for any two $\omega, \eta \in \Omega_c^r(\mathbb{R}^n)$. This shows

$$\left(B_p^r(t)\,\omega\middle|\eta\right) = \left(\omega\middle|e^{-t\Delta_{f,p}^{(r)}}\eta\right) = \left(e^{-t\Delta_{f,p}^{(r)}}\omega\middle|\eta\right)$$

for any two $\omega, \eta \in \Omega_c^r(\mathbb{R}^n)$, implying $B_p^r(t) = e^{-t\Delta_{f,p}^{(r)}}$ in $\Omega_c^r(\mathbb{R}^n)$ and thus $B_p^r(t,x,y) = e^{-t\Delta_{f,p}^{(r)}}(x,y)$ in $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$.

Note that the previous argument implies that every convergent subseqeunce of the sequence of the scaled heat kernels $\left\{A_{(k),p}^{(r)}(t,x,y)\right\}_k$ converges to the same limit, so together with Theorem 3.5, we can further conclude this sequence in effect converges to that limit. Therefore, we have established Theorem 1.1.

4. Heat Kernel Asymptotics away from Critical Points

In this section, we give proofs of Theorem 1.2 and Theorem 1.3.

The key point of obtaining the two theorems is based on the remark that we can retrieve pointwise bound for the heat kernels in question by establishing their corresponding mapping properties (see Theorem 4.4 and Theorem 4.7). These mapping properties are established based on a Bochner-type estimate related to differential forms with support away from the critical points (see Lemma 4.1 and Lemma 4.5).

Throughout this section, for each $p \in \mathrm{Crit}(f)$, we additionally identify the coordinate neighborhood U_p of p with the Euclidean ball $B_{\frac{3}{2}}(0)$ and p with 0, under the coordinate chart φ_p. For each large $k > 0$, put

$$\mathcal{U}^k = \bigcup_{p\in\mathrm{Crit}(f)} U_p^k,$$

where U_p^k is identified with the Euclidean ball $B_{k^{-\frac{1}{2}+\varepsilon}}(0)$, $\varepsilon \in \left(0, \frac{1}{2}\right)$ under the coordinate chart φ_p, as in Subsection 3.2.

4.1. Proof of Theorem 1.2

Choose $D > 1$ and let k large enough so that $2D < k^{\varepsilon}$. For each

$p \in \mathrm{Crit}(f)$, let $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$ and put

$$A_x^k := \varphi_p^{-1}\Big(B_{k^{-\frac{1}{2}+\varepsilon}}(0) \setminus B_{\frac{|x|}{2}k^{-\frac{1}{2}}}(0)\Big) \subset U_p^k$$

The choice of D plays a role in the following Bochner type estimate:

Lemma 4.1. *If D is large enough, let $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$ and we have*

$$\left(\Delta_k^{(r)}\omega|\omega\right) \geq Ck\,|x|^2\,\|\omega\|^2 \tag{4.1}$$

for each $\omega \in \Omega^r(M)$ with $\operatorname{supp}\omega \subset A_x^k$ and for large k, where C is independent of D, x, k.

Proof. Recall that we can write the Witten Laplacian $\Delta_k^{(r)}$ as

$$\Delta_k^{(r)} = \Delta^{(r)} + k^2\,|df|^2 + k\left(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*\right).$$

Put $A = \mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*$.

Note that

$$|df|^2 \geq \frac{1}{4k}\,|x|^2$$

in A_x^k (with respect to (U_p, φ_p)). Moreover, by the local expression (2.2) and partition of unity, there exists $m \in \mathbb{R}$ such that $(A\eta|\eta) \geq m\,\|\eta\|^2$ for each $\eta \in \Omega^r(M)$. For this m, let D large enough such that

$$\left|\frac{m}{D^2}\right| < \frac{1}{8}.$$

Now, for each $\omega \in \Omega^r(M)$ with $\operatorname{supp}\omega \subset A_x^k$, we deduce

$$\left(\Delta_k^{(r)}\omega|\omega\right) \geq k\,|x|^2\left(\frac{1}{4} + \frac{m}{|x|^2}\right)\|\omega\|^2 \geq Ck\,|x|^2\,\|\omega\|^2$$

with $C = \frac{1}{4}$. ∎

Let $D > 1$ be large enough such that Lemma 4.1 holds and let $k > 0$ large enough such that $2D < k^\varepsilon$. Given $p \in \mathrm{Crit}(f)$, for each $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$, let $\chi^{[1]} \in \mathcal{C}_c^\infty(U_p)$ be a cut-off function such that $\chi^{[1]} = 1$ in $\varphi_p^{-1}\left(B_{\frac{1}{2}}(0)\right)$

and $\chi^{[1]} = 0$ in $M \setminus \varphi_p^{-1}\left(B_{\frac{3}{4}}(0)\right)$. Put $\chi^{[1]}_{x,k} \in \mathcal{C}_c^\infty\left(U_p^k\right)$ such that

$$\chi^{[1]}_{x,k} \circ \varphi_p^{-1}(q) = \chi^{[1]}\left(|x|^{-1} k^{\frac{1}{2}} q\right)$$

for each $q \in \varphi_p(U_p) \subset \mathbb{R}^n$. Similarly, let $\chi^{[2]} \in \mathcal{C}_c^\infty(U_p)$ be a cut-off function such that $\chi^{[2]} = 1$ in $\varphi_p^{-1}\left(B_{\frac{5}{4}}(0)\right)$ and $\chi^{[2]} = 0$ in $\varphi_p^{-1}\left(B_{\frac{3}{2}}(0)\right)$. Put $\chi^{[2]}_{x,k} \in \mathcal{C}_c^\infty\left(U_p^k\right)$ such that

$$\chi^{[2]}_{x,k} \circ \varphi_p^{-1}(q) = \chi^{[2]}\left(|x|^{-1} k^{\frac{1}{2}} q\right)$$

for each $q \in \varphi_p(U_p) \subset \mathbb{R}^n$. Finally, set $\chi_{x,k} \in \mathcal{C}_c^\infty\left(A_x^k\right)$ by

$$\chi_{x,k} = \chi^{[2]}_{x,k} - \chi^{[1]}_{x,k}. \tag{4.2}$$

In particular, $\chi_{x,k} = 1$ in $\varphi_p^{-1}\left(B_{\frac{5|x|}{4}k^{-\frac{1}{2}}}(0) \setminus B_{\frac{3|x|}{4}k^{-\frac{1}{2}}}(0)\right)$. Note that

$$\sup_{\mathbb{R}^n}\left|D^\alpha \chi^{[1]}_{x,k}\right| \leq C\left(\chi^{[1]}, \chi^{[2]}\right)|x|^{-|\alpha|} k^{\frac{|\alpha|}{2}}, \tag{4.3}$$

where $C\left(\chi^{[1]}, \chi^{[2]}\right)$ depends on $\chi^{[1]}$ and $\chi^{[2]}$, with respect to the coordinates given by (U_p, φ_p). The construction of $\chi_{x,k}$ can be given from the same cut-off functions $\chi^{[1]}$ and $\chi^{[2]}$ as long as $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$ and $k > 0$, in which circumstance, the constant $C\left(\chi^{[1]}, \chi^{[2]}\right)$ in (4.3) does not rely on x and k.

With Lemma 4.1 at hand, we can obtain the following L^2-estimate.

Theorem 4.2. *Let $D > 1$ be large enough such that Lemma 4.1 holds, and for large $k > 0$ and for each $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$, let $\chi_{x,k}$ be the cut-off function given by the cut-off function $\chi^{[1]}, \chi^{[2]}$ as in (4.2). Then for each $N \in \mathbb{N}$,*

$$\left\|\chi_{x,k} e^{-\frac{t}{k}\Delta_k^{(r)}} \omega\right\| \leq C\left(\chi^{[1]}, \chi^{[2]}, t, N\right)|x|^{-N}\|\omega\|, \tag{4.4}$$

for each $\omega \in \Omega^r(M)$, where $C\left(\chi^{[1]}, \chi^{[2]}, t, N\right)$ depends on $\chi^{[1]}, \chi^{[2]}$, N and smoothly on t but is independent of x and k.

Proof. First, we show (4.4) holds for $N = 1$. For higher positive integers N, we can achieve similarly.

Since $\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega$ has its support in A_x^k, by Lemma 4.1, we obtain

$$\begin{aligned}&\left\|\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2\\&\leq C_1k^{-1}\left|x\right|^{-2}\left(\Delta_k^{(r)}\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\middle|\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\\&=C_1k^{-1}\left|x\right|^{-2}\left(\left\|d_k\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2+\left\|d_k^*\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\right).\end{aligned}\tag{4.5}$$

Hence, the L^2-norm estimate of $\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega$ is determined by the two L^2-norms in the right hand side (4.5). Before we proceed, note that by direct computation, we see the following analogous Leibniz rules:

$$d_k\left(\chi_{x,k}\eta\right)=d\chi_{x,k}\wedge\eta+\chi_{x,k}d_k\eta\tag{4.6}$$

and that

$$d_k^*\left(\chi_{x,k}\eta\right)=\left(-1\right)^{n(r+1)+1}*\left(d\chi_{x,k}\wedge*\eta\right)+\chi_{x,k}d_k^*\eta\tag{4.7}$$

for each $\eta\in\Omega^r\left(M\right)$.

To deal with the first term on the right hand side of (4.5), by (4.6) with $\eta=e^{-\frac{t}{k}\Delta_k^{(r)}}\omega$, we see that

$$\left\|d_k\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\leq4\left(\left\|d\chi_{x,k}\wedge e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2+\left\|\chi_{x,k}d_ke^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2\right).\tag{4.8}$$

Note that $\chi_{x,k}\in\mathcal{C}_c^\infty\left(U_p^k\right)$ and has the local expression $d\chi_{x,k}=\sum_{i=1}^n\frac{\partial\chi_{x,k}}{\partial y^i}dy^i$ in $\left(U_p^k,\varphi_p\right)$. Hence, by the local nature of $\chi_{x,k}$ (4.3) and Theorem 2.5, we obtain

$$\begin{aligned}\left\|d\chi_{x,k}\wedge e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2&\leq C_2\left(\chi^{[1]},\chi^{[2]}\right)k\left|x\right|^{-2}\left\|e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2\\&\leq C_3\left(\chi^{[1]},\chi^{[2]},t\right)k\left|x\right|^{-2}\left\|\omega\right\|^2,\end{aligned}\tag{4.9}$$

where $C_3\left(\chi^{[1]},\chi^{[2]},t\right)$ depends on $\chi^{[1]},\chi^{[2]}$ and smoothly on t but independent of D,x,k. Moreover, by Theorem 2.5, we obtain

$$\begin{aligned}\left\|\chi_{x,k}d_ke^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2\leq\left\|d_ke^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2&\leq\left(\Delta_k^{(r)}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\middle|e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\\&\leq C_4\left(t\right)k\left\|\omega\right\|^2.\end{aligned}\tag{4.10}$$

Hence, using (4.8), (4.9) along with the fact that $|x|^{-2} < 1$, and (4.10), we conclude

$$\left\|d_k\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2 \leq C_5\left(\chi^{[1]},\chi^{[2]},t\right)k\left\|\omega\right\|^2, \tag{4.11}$$

where $C_5\left(\chi^{[1]},\chi^{[2]},t\right)$ depends on $\chi^{[1]},\chi^{[2]}$ and smoothly on t but independent of D,x,k.

We can achieve an upper bound for the second term in the similar fashion and let us briefly go through the deduction. By (4.7) with $\eta = e^{-\frac{t}{k}\Delta_k^{(r)}}\omega$, we see

$$\begin{aligned}&\left\|d_k^*\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\\ &\quad\leq 4\left(\left\|*\left(d\chi_{x,k}\wedge *e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2+\left\|\chi_{x,k}d_k^*e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2\right);\end{aligned} \tag{4.12}$$

by doing so, we break down this L^2-norm term into two terms, one is with $\chi_{x,k}$ differentiated and the other without. By local nature of $\chi_{x,k}$ along with Theorem 2.5, we can obtain

$$\left\|*\left(d\chi_{x,k}\wedge *e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2 \leq C_6\left(\chi^{[1]},\chi^{[2]},t\right)k\left|x\right|^{-2}\left\|\omega\right\|^2. \tag{4.13}$$

By Theorem 2.5, we can see

$$\left\|\chi_{x,k}d_k^*e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2 \leq C_7\left(\chi^{[1]},\chi^{[2]},t\right)k\left\|\omega\right\|^2. \tag{4.14}$$

Finally, by (4.12), (4.13), and (4.14), we can conclude

$$\left\|d_k^*\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2 \leq C_8\left(\chi^{[1]},\chi^{[2]},t\right)k\left\|\omega\right\|^2, \tag{4.15}$$

where $C_8\left(\chi^{[1]},\chi^{[2]},t\right)$ depends on $\chi^{[1]},\chi^{[2]}$ and smoothly on t but independent of D,x,k.

Subsequently, by (4.5), (4.11), and (4.15), we conclude

$$\left\|\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2 \leq C_9\left(\chi^{[1]},\chi^{[2]},t\right)\left|x\right|^{-2}\left\|\omega\right\|^2,$$

where $C_9\left(\chi^{[1]},\chi^{[2]},t\right)$ depends on $\chi^{[1]},\chi^{[2]}$ and smoothly on t but independent of D,x,k. Namely, we have established (4.4) for $N=1$.

Similarly, we can obtain (4.4) holds for each positive integer $N > 1$. To explain, we divide it into two cases: N is even and N is odd. If N is even, then we can apply Lemma 4.1 repeatedly until we obtain

$$\begin{aligned}\left\|\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2 \leq & C_{10}k^{-N}\left|x\right|^{-2N}\left(\left\|(d_kd_k^*)^{\frac{N}{2}}\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\right. \\ & \left.+\left\|(d_k^*d_k)^{\frac{N}{2}}\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\right);\end{aligned}$$

By the Leibniz rules (4.6), (4.7), local nature of $\chi_{x,k}$, and Theorem 2.5, we can deduce

$$\begin{aligned}\left\|\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2 &\leq C_{11}\left(\chi^{[1]},\chi^{[2]},t,N\right)k^{-N}\left|x\right|^{-2N}\cdot k^N\left\|\omega\right\|^2 \\ &= C_{11}\left(\chi^{[1]},\chi^{[2]},t,N\right)\left|x\right|^{-2N}\left\|\omega\right\|^2,\end{aligned}$$

where $C_{11}\left(\chi^{[1]},\chi^{[2]},t,N\right)$ depends on N due to (4.6) and (4.7) but is independent of D, x, k.

If N is odd, we can obtain

$$\begin{aligned}\left\|\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2 \leq & C_{12}k^{-N}\left|x\right|^{-2N}\left(\left\|(d_kd_k^*)^{\frac{N-1}{2}}d_k\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\right. \\ & \left.+\left\|(d_k^*d_k)^{\frac{N-1}{2}}d_k^*\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\right)\end{aligned}$$

and so we can proceed to obtain

$$\left\|\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right\|^2 \leq C_{13}\left(\chi^{[1]},\chi^{[2]},t,N\right)\left|x\right|^{-2N}\left\|\omega\right\|^2.$$

Hence, we have established Theorem 4.2. □

More generally, using similar iterative argument, we can in effect obtain the following L^2-estimate involving the Witten Laplacian $\Delta_k^{(r)}$:

Corollary 4.3. *Let $D > 1$ be large enough such that Lemma 4.1 holds, and for large $k > 0$ and for each $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$, let $\chi_{x,k}$ be the cut-off function given by the cut-off function $\chi^{[1]},\chi^{[2]}$ as in (4.2). Then for each $m \in \mathbb{N}\cup\{0\}$ and for each $N \in \mathbb{N}$,*

$$\left\|\left(\Delta_k^{(r)}\right)^m\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\| \leq C\left(\chi^{[1]},\chi^{[2]},t,m,N\right)k^m\left|x\right|^{-N}\left\|\omega\right\|, \quad (4.16)$$

for each $\omega \in \Omega^r(M)$*, where* $C\left(\chi^{[1]}, \chi^{[2]}, t, m, N\right)$ *depends on* $\chi^{[1]}, \chi^{[2]}, m, N$ *and smoothly on* t *but is independent of* x *and* k.

Proof. Corollary 4.3 can be obtained similarly to Theorem 4.2.

Since $d_k^2 = 0 = (d_k^*)^2$, we see that

$$\begin{aligned}&\left\|\left(\Delta_k^{(r)}\right)^m\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\\ &\leq 4\left(\left\|(d_kd_k^*)^m\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2+\left\|(d_k^*d_k)^m\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\right).\end{aligned}$$

As in the proof of Theorem 4.2, repeated use of Lemma 4.1 gives

$$\begin{aligned}\left\|\left(\Delta_k^{(r)}\right)^m\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2 \leq C_1k^{-N}|x|^{-2N}\Bigg(&\left\|(d_kd_k^*)^{\frac{N}{2}+m}\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\\ &+\left\|(d_k^*d_k)^{\frac{N}{2}+m}\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\Bigg);\end{aligned}$$

for each even positive integer N, and

$$\begin{aligned}&\left\|\left(\Delta_k^{(r)}\right)^m\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\\ &\leq C_1k^{-N}|x|^{-2N}\left(\left\|(d_kd_k^*)^{\frac{N-1}{2}+m}d_k\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\right.\\ &\qquad\left.+\left\|(d_k^*d_k)^{\frac{N-1}{2}+m}d_k^*\left(\chi_{x,k}e^{-\frac{t}{k}\Delta_k^{(r)}}\omega\right)\right\|^2\right);\end{aligned}$$

for each odd positive integer N. Hence, the Leibniz rules (4.6), (4.7), the local nature of $\chi_{x,k}$ (see (4.3)) and Theorem 2.5 allow us to establish Corollary 4.3. Note that the constant $C\left(\chi^{[1]}, \chi^{[2]}, t, m, N\right)$ in (4.16) depends on m due to not only the Leibniz rules (4.6) and (4.7) but also Theorem 2.5, and on N due to again (4.6) and (4.7). $\square$

Finally, we can show the following mapping property:

Theorem 4.4. *Let* $D > 1$ *be large enough such that Lemma* 4.1 *holds, and for large* $k > 0$ *and for each* $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$*, let* $\chi_{x,k}$ *be the cut-off function given by the cut-off function* $\chi^{[1]}, \chi^{[2]}$ *as in* (4.2)*. Then for each* $m \in \mathbb{N} \cup \{0\}$ *and for each* $N \in \mathbb{N}$*, there exists* $C\left(\chi^{[1]}, \chi^{[2]}, t, m, N\right) > 0$ *such* *that*

$$\left\|(\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k),p}(t)\,(\chi_{x,k})_{\left[\frac{1}{k}\right]}\,\omega\right\|_{2m} \leq C\left(\chi^{[1]},\chi^{[2]},t,m,N\right)|x|^{-N}\,\|\omega\|_{-2m}\,, \tag{4.17}$$

for each $\omega\in\Omega^r_c\left(B_{k^\varepsilon}(0)\right)$, *where* $C\left(\chi^{[1]},\chi^{[2]},t,m,N\right)$ *depends on* $\chi^{[1]},\chi^{[2]},m,$ N *and smoothly on* t *and is independent of* D, k, x.

Proof. Thanks to Gårding's inequality and (3.3), (3.4), we obtain

$$\begin{aligned}
&\left\|(\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k),p}(t)\,(\chi_{x,k})_{\left[\frac{1}{k}\right]}\,\omega\right\|_{2m}\\
&\leq C_1\left\|\left(\Delta^{(r)}_{f,p}\right)^m(\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k),p}(t)\,(\chi_{x,k})_{\left[\frac{1}{k}\right]}\,\omega\right\|_0\\
&\quad+C_2\left\|(\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k),p}(t)\,(\chi_{x,k})_{\left[\frac{1}{k}\right]}\,\omega\right\|_0\\
&\leq C_3\big(\chi^{[1]},\chi^{[2]}\big)\left\|\tilde\chi_1\left(\Delta^{(r)}_{f,p}\right)^m A^r_{(k),p}(t)\,(\chi_{x,k})_{\left[\frac{1}{k}\right]}\,\omega\right\|_0\\
&\quad+C_4\left(\chi^{[1]},\chi^{[2]}\right)\left\|\tilde\chi_2 A^r_{(k),p}(t)\,(\chi_{x,k})_{\left[\frac{1}{k}\right]}\,\omega\right\|_0,
\end{aligned}$$

where $C_3\left(\chi^{[1]},\chi^{[2]}\right), C_4\left(\chi^{[1]},\chi^{[2]}\right)$ depends on $\chi^{[1]}$ and $\chi^{[2]}$, but is independent of x,k since $(\chi_{x,k})_{\left[\frac{1}{k}\right]}(\cdot)=\chi^{[1]}\big(|x|^{-1}\cdot\big)-\chi^{[2]}\big(|x|^{-1}\cdot\big)$ with respect to $\left(U^k_p,\varphi_p\right)$ and $|x|>D>1$, and $\tilde\chi_1,\tilde\chi_2\in\mathcal{C}^\infty_c\big(\sqrt{k}A^k_x\big)$ are cut-off functions such that $\tilde\chi_1=1=\tilde\chi_2$ in $\sqrt{k}\,\mathrm{supp}\,\chi_{x,k}$.

Next, for each $\eta\in\Omega^r_c\left(B_{k^\varepsilon}(0)\right)$ with $\eta\neq 0$, note that

$$\begin{aligned}
&\frac{\left(\tilde\chi_1\big(\Delta^{(r)}_{f,p}\big)^m A^r_{(k),p}(t)\,(\chi_{x,k})_{\left[\frac{1}{k}\right]}\,\omega\middle|\eta\right)}{\|\eta\|_0}\\
&\qquad\leq\frac{\|\omega\|_{-2m}\left\|(\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k),p}(t)\left(\Delta^{(r)}_{f,p}\right)^m\tilde\chi_1\eta\right\|_{2m}}{\|\eta\|_0}.
\end{aligned}$$

By Gårding's inequality, Corollary 4.3 and Theorem 2.5,

$$\begin{aligned}
&\left\|(\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k)}(t)\left(\Delta^{(r)}_{f,p}\right)^m\tilde\chi_1\eta\right\|_{2m}\\
&\leq C_5\left\|\left(\Delta^{(r)}_{f,p}\right)^m(\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k)}(t)\left(\Delta^{(r)}_{f,p}\right)^m\tilde\chi_1\eta\right\|_0\\
&\quad+C_6\left\|(\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k)}(t)\left(\Delta^{(r)}_{f,p}\right)^m\tilde\chi_1\eta\right\|_0
\end{aligned}$$

$$
\begin{aligned}
&= C_5 k^{\frac{n}{2}-2m} \left\| \left(\Delta_k^{(r)}\right)^m \chi_{x,k} e^{-\frac{t}{k}\Delta_k^{(r)}} \left(\Delta_k^{(r)}\right)^m (\tilde{\chi}_1 \eta)_{[k]} \right\|_{L^2(M)} \\
&\quad + C_6 k^{\frac{n}{2}-m} \left\| \chi_{x,k} e^{-\frac{t}{k}\Delta_k^{(r)}} \left(\Delta_k^{(r)}\right)^m (\tilde{\chi}_1 \eta)_{[k]} \right\|_{L^2(M)} \\
&\leq C_7 \Big(\chi^{[1]}, \chi^{[2]}, \frac{t}{2}, m, N\Big) |x|^{-N} k^{\frac{n}{2}-m} \left\| e^{\frac{t}{2k}\Delta_k^{(r)}} \left(\Delta_k^{(r)}\right)^m (\tilde{\chi}_1 \eta)_{[k]} \right\|_{L^2(M)} \\
&\leq C_8 \Big(\chi^{[1]}, \chi^{[2]}, \frac{t}{2}, m, N\Big) |x|^{-N} \|\eta\|_0 ,
\end{aligned}
$$

where $C_8\left(\chi^{[1]}, \chi^{[2]}, \frac{t}{2}\right)$ depends on $\chi^{[1]}, \chi^{[2]}, m, N$ and smoothly on t but is independent of D, x, k. Thus, we deduce

$$
\left\| \tilde{\chi}_1 \left(\Delta_{f,p}^{(r)}\right)^m A^r_{(k),p}(t) \left(\chi_{x,k}\right)_{\left[\frac{1}{k}\right]} \omega \right\|_0 \leq C_8 \Big(\chi^{[1]}, \chi^{[2]}, \frac{t}{2}, m, N\Big) |x|^{-N} \|\omega\|_{-2m} .
$$

Similarly, we can obtain

$$
\left\| \tilde{\chi}_2 A^r_{(k),p}(t) \left(\chi_{x,k}\right)_{\left[\frac{1}{k}\right]} \omega \right\|_0 \leq C_9 \Big(\chi^{[1]}, \chi^{[2]}, \frac{t}{2}, m, N\Big) |x|^{-N} \|\omega\|_{-2m} ,
$$

where $C_9\left(\chi^{[1]}, \chi^{[2]}, \frac{t}{2}, m, N\right)$ depends on $\chi^{[1]}, \chi^{[2]}, m, N$ and smoothly on t but is independent of D, x, k.

Finally, we conclude

$$
\left\| \left(\chi_{x,k}\right)_{\left[\frac{1}{k}\right]} A^r_{(k),p}(t) \left(\chi_{x,k}\right)_{\left[\frac{1}{k}\right]} \omega \right\|_{2m} \leq C_{10} \left(\chi^{[1]}, \chi^{[2]}, t, m, N\right) |x|^{-N} \|\omega\|_{-2m} ,
$$

where $C_{10}\left(\chi^{[1]}, \chi^{[2]}, t, m, N\right)$ depends on $\chi^{[1]}, \chi^{[2]}, m, N$ and smoothly on t but is independent of D, x, k. $\square$

As a consequence of Theorem 4.4, we can prove Theorem 1.2:

***Proof of Theorem* 1.2.** The main key of proving this theorem is similar to Theorem 3.5.

Write

$$
\begin{aligned}
A^r_{(k),p}(t,x,x) &= \sum_{I,J}{}' A^r_{(k),pI,J}(t,x,x)\, dx^I \otimes \left(dx^J\right)^* \\
&\in \mathcal{C}^\infty\Big(\mathbb{R}^+ \times B_{k^\varepsilon}(0) \times B_{k^\varepsilon}(0), \bigwedge^r T^*\mathbb{R}^n \boxtimes \Big(\bigwedge^r T^*\mathbb{R}^n\Big)^*\Big).
\end{aligned}
$$

Recall that for each $S \in \bigwedge^r T^*_x\mathbb{R}^n \otimes \left(\bigwedge^r T^*_x\mathbb{R}^n\right)^*$, the norm of S is defined to be $|S|_x := \sup_{\omega_x \in \bigwedge^r T_x^* \mathbb{R}^n, \omega_x \neq 0} \frac{|S\omega_x|}{|\omega_x|}$, and note that

$$\left| A^r_{(k),p}(t,x,x) \right|_x \leq \left(\sum_{I,J} \left| A^r_{(k),pI,J}(t,x,x) \right|^2 \right)^{\frac{1}{2}},$$

so it suffices to show for any two I, J and for each $N \in \mathbb{N}$,

$$\left| A^r_{(k),pI,J}(t,x,x) \right| \leq C(t,N) |x|^{-N},$$

where $C(t,N)$ depends on N and smoothly on t and is independent of D, x, k.

Let $D > 1$ be large enough such that Lemma 4.1 holds. Let $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$, and put $\omega_{x,I,\delta} = \chi_{x,I,\delta} dx^I$ and $\omega_{x,J,\delta'} = \chi_{x,J,\delta'} dx^J$ as in (3.10). By integration by part, we see that

$$A^r_{(k),pI,J}(t,x,x) = \lim_{\delta \to 0} \lim_{\delta' \to 0} \left(A^r_{(k),p}(t) \omega_{x,J,\delta'} \middle| \omega_{x,I,\delta} \right).$$

Now, let $\chi_{x,k}$ be the cut-off function as in (4.2) given by $\chi^{[1]}, \chi^{[2]}$. Using Theorem 4.4 and Lemma 3.7 with fixed large m, we obtain, if δ, δ' are small enough,

$$\begin{aligned} \left| \left(A^r_{(k),p}(t) \omega_{x,J,\delta'} \middle| \omega_{x,I,\delta} \right) \right| &\leq \left\| (\chi_{x,k})_{\left[\frac{1}{k}\right]} A^r_{(k),p}(t) (\chi_{x,k})_{\left[\frac{1}{k}\right]} \omega_{x,J,\delta'} \right\|_{2m} \|\omega_{x,I,\delta}\|_{-2m} \\ &\leq C\left(\chi^{[1]}, \chi^{[2]}, t, m, N\right) |x|^{-N}, \end{aligned}$$

where $C\left(\chi^{[1]}, \chi^{[2]}, t, m, N\right)$ depends on $\chi^{[1]}, \chi^{[2]}, m, N$ and smoothly on t but independent of D, x, k, δ, δ', for large k.

For each $x \in B_{k^\varepsilon}(0) \setminus B_{2D}(0)$, observe that the cut-off function $\chi_{x,k}$ is constructed from the same cut-off functions $\chi^{[1]}, \chi^{[2]}$, leading to $A^r_{(k),pI,J}(t,x,x)$ enjoying the same upper bound. Hence, we conclude

$$\left| A^r_{(k),pI,J}(t,x,x) \right| \leq C(t,N) |x|^{-N},$$

where $C(t,N)$ depends on N and smoothly on t and independent of D, x, k and N is an arbitrary positive integer. (The contribution of m to the constant is insignificant and thus is not marked in the parenthesis). $\square$

4.2. Proof of Theorem 1.3

The main idea of this proof is similar to Theorem 1.2. Recall that for each $p \in \operatorname{Crit}(f)$, we additionally identify the coordinate neighborhood U_p of p with the Euclidean ball $B_{\frac{3}{2}}(0)$ and p with 0, and for each $k > 0$, put

$$\mathcal{U}^k = \bigcup_{p \in \operatorname{Crit}(f)} U_p^k,$$

where U_p^k is identified with the Euclidean ball $B_{k^{-\frac{1}{2}+\varepsilon}}(0)$, $\varepsilon \in \left(0, \frac{1}{2}\right)$.

First, we need the following estimate of Bochner type.

Lemma 4.5. *If k is sufficiently large,*

$$\left(\Delta_k^{(r)}\omega|\omega\right) \geq Ck^{1+2\varepsilon} \|\omega\|^2 \tag{4.18}$$

for each $\omega \in \Omega^r(M)$ with $\operatorname{supp}\omega \subset M \setminus \mathcal{U}^k$, where C is independent of k.

Proof. The key point of this proof is virtually the same as in Lemma 4.1.

Since M is compact and $df = 0$ at $p \in \operatorname{Crit}(f)$, we can see

$$|df|^2 \geq k^{-1+2\varepsilon}$$

in $M \setminus \mathcal{U}^k$ if k is sufficiently large; furthermore, we can let k large enough such that

$$\left|\frac{m}{k^{2\varepsilon}}\right| < \frac{1}{2},$$

where $m \in \mathbb{R}$ is given from the fact that $(A\omega|\omega) \geq m\|\omega\|^2$ for each $\omega \in \Omega^r(M)$ in which $A = \mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*$. Hence, for each $\omega \in \Omega^r(M)$ with $\operatorname{supp}\omega \subset M \setminus \mathcal{U}^k$, we deduce

$$\left(\Delta_k^{(r)}\omega|\omega\right) \geq k^{-1+2\varepsilon}\left(1 + mk^{-2\varepsilon}\right)\|\omega\|^2 \geq \frac{1}{2}k^{-1+2\varepsilon}\|\omega\|^2. \qquad \square$$

Put $\mathcal{U} = \bigcup_{p \in \operatorname{Crit}(f)} U_p$ (with U_p identified as $B_{\frac{3}{2}}(0)$ under the coordinate chart φ_p for each $p \in \operatorname{Crit}(f)$), and let $\tau \in \mathcal{C}_c^\infty(\mathcal{U})$ be a cut-off function such that $\tau = 1$ in $\bigcup_{p \in \operatorname{Crit}(f)} \varphi_p^{-1}\left(B_{\frac{1}{2}}(0)\right)$ and $\tau = 0$ in

$M \setminus \bigcup_{p \in \mathrm{Crit}(f)} \varphi_p^{-1}(B_1(0))$. For (large) k, put $\tau_k \in \mathcal{C}_c^\infty(\mathcal{U}^k)$ such that

$$\tau_k \circ \varphi_p^{-1}(q) = \tau\left(k^{\frac{1}{2}-\varepsilon} q\right)$$

for each $q \in \varphi_p(U_p) \subset \mathbb{R}^n$ and for each $p \in \mathrm{Crit}(f)$. Finally, we set

$$\chi_k = 1 - \tau_k \in \mathcal{C}_c^\infty\left(M \setminus \bigcup_{p \in \mathrm{Crit}(f)} \varphi_p^{-1}\left(B_{\frac{1}{2}k^{-\frac{1}{2}+\varepsilon}}(0)\right)\right) \tag{4.19}$$

Note that $\chi_k = 1$ in $M \setminus \mathcal{U}^k$. Moreover, we can see $D^\alpha \chi_k \in \mathcal{C}_c^\infty(\mathcal{U}^k)$ and

$$\sup_{\mathbb{R}^n} |D^\alpha \chi_k| \leq C(\tau)\, k^{-\varepsilon|\alpha|} k^{\frac{|\alpha|}{2}}$$

with respect to each of the coordinate charts (U_p, φ_p), for each multi-index $\alpha \neq 0$.

Again, we have the following L^2-estimate via Lemma 4.5.

Theorem 4.6. *Let χ_k be the cut-off function given by the cut-off function τ as in (4.19). If k is large, then for each $N \in \mathbb{N}$ and for each fixed $m \in \mathbb{N}$,*

$$\left\| \left(\Delta_k^{(r)}\right)^m \left(\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}} \omega\right) \right\| \leq C(\tau, t, m, N)\, k^{m-N} \|\omega\| \tag{4.20}$$

for each $\omega \in \Omega^r(M)$, where $C(\tau, t, m, N)$ depends on τ, m, N and smoothly on t but is independent of k.

Proof. Theorem 4.6 can be established by the arguments in Theorem 4.2 and Corollary 4.3 by replacing $|x|$ with k^ε, and $\chi^{[1]}, \chi^{[2]}$ with τ. ☐

Theorem 4.7. *For each $p \in \mathrm{Crit}(f)$, let χ_k be the cut-off function given by the cut-off function τ as in (4.19). If k is sufficiently large, then for each $N \in \mathbb{N}$, for each $m \in \mathbb{N} \cup \{0\}$,*

$$\left\| \chi_k e^{-\frac{t}{k}\Delta_k^{(r)}} \chi_k \omega \right\|_{2m} \leq C(\tau, t, m, N)\, k^{-N} \|\omega\|_{-2m} \tag{4.21}$$

for each $\omega \in \Omega_c^r(V)$, where $C(\tau, t, m, N) > 0$ depends on τ, m, N and smoothly on t but is independent of k.

Proof. The main idea of proving Theorem 4.7 is very similar to Theorem 3.6. Choose a pair $(\mathcal{V}, \mathcal{P}, \mathcal{E})$ so that the Sobolev norms are defined on M.

By definition,

$$\left\|\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right\|_{2m}^2 = \sum_{\psi\in\mathcal{P}} \left\|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right\|_{2m}^2. \tag{4.22}$$

By Gårding's inequality, we see that

$$\begin{aligned}\left\|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right\|_{2m} \le\, & C_1(\psi)\, k^m \left\|\tilde{\psi}_1 \left(\Delta_k^{(r)}\right)^m \left(\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right)\right\|_0 \\ & + C_2(\psi)\, k^m \left\|\tilde{\psi}_2\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right\|_0,\end{aligned}$$

where $\tilde{\psi}_1, \tilde{\psi}_2$ are cut-off functions with compact support in the coordinate domain in which $\operatorname{supp}\psi$ lies and $\tilde{\psi}_1 = 1 = \tilde{\psi}_2$ in $\operatorname{supp}\psi$. Moreover, by Theorem 4.7, we obtain for each $N_1 \in \mathbb{N}$,

$$\begin{aligned}&\left\|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right\|_{2m} \\ \le&\left(C_3\left(\psi,\tau,\frac{t}{2},m,N\right)k^{2m-N_1} + C_4\left(\psi,\tau,\frac{t}{2},m,N\right)k^{m-N_1}\right)\left\|e^{-\frac{t}{2k}\Delta_k^{(r)}}\chi_k\omega\right\|_{L^2(M)} \\ \le& C_5(\psi,\tau,t,m,N)\, k^{2m-N_1}\left\|e^{-\frac{t}{2k}\Delta_k^{(r)}}\chi_k\omega\right\|_{L^2(M)},\end{aligned} \tag{4.23}$$

where $C_5(\psi,\tau,t,m,N)$ depends on ψ,τ,m,N and smoothly on t but is independent of k.

Now, since $\sum_{\psi\in\mathcal{P}}\psi^2 = 1$, for each $\eta \in \Omega^r(M)$, $\eta \neq 0$, we deduce

$$\begin{aligned}\frac{\left|\left(e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\middle|\eta\right)\right|}{\|\eta\|_{L^2(M)}} = \frac{\left|\left(\omega\middle|\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\eta\right)\right|}{\|\eta\|_{L^2(M)}} &\le \sum_{\psi\in\mathcal{P}} \frac{\left|\left(\psi\omega\middle|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\eta\right)\right|}{\|\eta\|_{L^2(M)}} \\ &\le \sum_{\psi\in\mathcal{P}} \frac{\|\psi\omega\|_{-2m}\left\|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\eta\right\|_{2m}}{\|\eta\|_{L^2(M)}}.\end{aligned}$$

Using Gårding's inequality again, we obtain

$$\begin{aligned}&\left\|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\eta\right\|_{2m} \\ &\quad\le C_6(\psi)\, k^m \left\|\tilde{\psi}_3\left(\Delta_k^{(r)}\right)^m \chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\eta\right\|_0 + C_7(\psi)\, k^m\left\|\tilde{\psi}_4\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\eta\right\|_0,\end{aligned}$$

where $\tilde{\psi}_3, \tilde{\psi}_4$ are cut-off functions with compact support in the coordinate domain in which $\operatorname{supp}\psi$ lies and $\tilde{\psi}_3 = 1 = \tilde{\psi}_4$ in $\operatorname{supp}\psi$. By Theorem 4.7, we obtain for each $N_2 \in \mathbb{N}$,

$$\left\|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\eta\right\|_{2m} \leq C_8\left(\psi,\tau,t,m,N\right)k^{2m-N_2}\left\|\eta\right\|_{L^2(M)},$$

$C_8\left(\psi,\tau,t,m,N\right)$ depends on ψ,τ,m,N and smoothly on t but is independent of k. Hence, we conclude

$$\begin{aligned}\left\|e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right\|_{L^2(M)} &\leq C_9\left(\tau,t,m,N\right)k^{2m-N_2}\sum_{\psi\in\mathcal{P}}\left\|\psi\omega\right\|_{-2m}\\ &\leq C_{10}\left(\tau,t,m,N\right)k^{2m-N_2}\left(\sum_{\psi\in\mathcal{P}}\left\|\psi\omega\right\|_{-2m}^2\right)^{\frac{1}{2}}\\ &= C_{10}\left(\tau,t,m,N\right)k^{2m-N_2}\left\|\omega\right\|_{-2m}. \end{aligned} \tag{4.24}$$

where $C_{10}\left(\tau,t,m,N\right)$ depends on τ,m,N and smoothly on t but is independent of ψ,k.

Hence, by (4.22), (4.23), and (4.24), we deduce for each $N_1,N_2 \in \mathbb{N}$,

$$\left\|\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right\|_{2m} = C_{11}\left(\tau,t,m,N\right)k^{4m-N_1-N_2}\left\|\omega\right\|_{-2m},$$

where $C_{11}\left(\tau,t,m,N\right)$ depends on τ,m,N and smoothly on t but is independent of ψ,k. Finally, take N_1,N_2 large enough, we can conclude for each $N \in \mathbb{N}$,

$$\left\|\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega\right\|_{2m} = C_{12}\left(\tau,t,m,N\right)k^{-N}\left\|\omega\right\|_{-2m}$$

as desired. □

Now, we can start to prove Theorem 1.3.

***Proof of Theorem* 1.3.** The argument of Theorem 1.3 is very similar to Theorem 1.2.

Choose a pair $(\mathcal{V},\mathcal{P},\mathcal{E})$ such that the sup-norm $\|\cdot\|_{\mathcal{C}^0}$ and the Sobolev norms are defined.

For each large $k > 0$, let $x \in M \setminus \mathcal{U}^k$. Put $\omega_{x,I,\delta} = \chi_{x,I,\delta}E^I$ and $\omega_{x,J,\delta'} = \chi_{x,J,\delta'}E^J$ with $\chi_{x,I,\delta},\chi_{y,J,\delta'}$ as in (3.9) and $\left\{E^I\right\}_I \subset \mathcal{E}$ is a local orthonormal frame. By integration by part, we see that

$$\left|e^{-\frac{t}{k}\Delta_k^{(r)}}{}_{I,J}\left(t,x,x\right)\right| = \lim_{\delta\to 0}\lim_{\delta'\to 0}\left|\left(e^{-\frac{t}{k}\Delta_k^{(r)}}\omega_{x,J,\delta'}\middle|\omega_{x,I,\delta}\right)\right|.$$

Let χ_k be the cut-off function as in (4.19). Fix a large m, and we obtain

$$\begin{aligned}\left|\left(e^{-\frac{t}{k}\Delta_k^{(r)}}\omega_{x,J,\delta'}\middle|\omega_{x,I,\delta}\right)\right| &\le \sum_{\psi\in\mathcal{P}}\left|\left(\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega_{x,J,\delta'}\middle|\psi\omega_{x,I,\delta}\right)\right| \\ &\le \sum_{\psi\in\mathcal{P}}\left\|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega_{x,J,\delta'}\right\|_{2m}\left\|\psi\omega_{x,I,\delta}\right\|_{-2m}.\end{aligned}$$

Then in view of Lemma 3.7, we can deduce $\left\|\psi\omega_{x,I,\delta}\right\|_{-2m}\le C_1\left(\psi\right)$; moreover, by definition, $\left\|\psi\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega_{x,J,\delta'}\right\|_{2m}\le\left\|\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega_{x,J,\delta'}\right\|_{2m}$. Hence, by Theorem 4.7, we conclude for each $N\in\mathbb{N}$,

$$\left|\left(e^{-\frac{t}{k}\Delta_k^{(r)}}\omega_{x,J,\delta'}\middle|\omega_{x,I,\delta}\right)\right| \le C_2\left\|\chi_k e^{-\frac{t}{k}\Delta_k^{(r)}}\chi_k\omega_{x,J,\delta'}\right\|_{2m}\le C_3\left(\tau,t,N\right)k^{-N},$$

where $C_3\left(\tau,t,N\right)$ depends on τ,N and smoothly on t but is independent of x,k,δ,δ' (Dependence of m is again redundant). In addition, note that the cut-off function χ_k is constructed so that $\chi_k=1$ in $M\setminus\mathcal{U}^k$, which indicates $e^{-\frac{t}{k}\Delta_k^{(r)}}{}_{I,J}\left(t,x,x\right)$ shares the same upper bound for each $x\in M\setminus\mathcal{U}^k$. Therefore, we obtain for each $t\in T$, for each $x\in M\setminus\mathcal{U}^k$, and for each $N\in\mathbb{N}$,

$$\left|e^{-\frac{t}{k}\Delta_k^{(r)}}{}_{I,J}\left(t,x,x\right)\right|\le C_4\left(t,N\right)k^{-N},$$

where $C_4\left(t,N\right)$ depends on N and smoothly on t and is independent of x,k.

Finally, for each $t>0$ and for each $N\in\mathbb{N}$, we deduce

$$\begin{aligned}&\left\|e^{-\frac{t}{k}\Delta_k^{(r)}}\left(t,x,x\right)\right\|^2_{\mathcal{C}^0\left(M\setminus\mathcal{U}^k\right)} \\ &\quad=\sum_{\psi\in\mathcal{P}}\left\|\psi e^{-\frac{t}{k}\Delta_k^{(r)}}\left(t,x,x\right)\right\|^2_{\mathcal{C}^0\left(M\setminus\mathcal{U}^k\right)} \\ &\quad=\sum_{\psi\in\mathcal{P}}\left(\sup_{\operatorname{supp}\psi\cap M\setminus\mathcal{U}^k}\left(\sum_{I,J}\left|\psi e^{-\frac{t}{k}\Delta_k^{(r)}}{}_{I,J}\left(t,x,x\right)\right|^2\right)^{\frac{1}{2}}\right)^2\le C_5\left(t,N\right)k^{-N},\end{aligned}$$

where $C_5\left(t,N\right)$ depends on N and smoothly on t. This furnishes Theorem 1.3. $\square$

5. Morse Inequalities

In this section, we give a new analytic proof of the Morse inequalities as an application of our heat kernel results: Theorems 1.1, 1.2, and 1.3.

First, we review the Morse inequalities:

Theorem 5.1 (The Morse Inequalities). *Let M be a compact orientable smooth manifold of dimension n and let f be a Morse function. Then*

(a) *for each $0 \le r \le n$,*

$$\dim H^r_{dR}(M) \le m_r; \tag{5.1}$$

(b) *for each $0 \le r \le n$,*

$$\sum_{j=0}^{r} (-1)^{r-j} \dim H^j_{dR}(M) \le \sum_{j=0}^{r} (-1)^{r-j} m_j \tag{5.2}$$

and the equality holds if $r = n$; namely,

$$\sum_{j=0}^{n} (-1)^{n-j} \dim H^j_{dR}(M) = \sum_{j=0}^{n} (-1)^{n-j} m_j. \tag{5.3}$$

To prove Theorem 5.1, let us recall the local index theory. Let X be an inner product space and let $\{E_I\}_I$ be an orthonormal basis for X. Recall that a trace of a linear transformation $A : X \to X$ is given by

$$\operatorname{tr} A = \sum_I \langle AE_I | E_I \rangle.$$

Then we can derive the following McKean-Singer type trace integral formula:

Lemma 5.2 (McKean-Singer Type Trace Integral Formula, cf. [14]). *Let M be a compact orientable Riemannian manifold of dimension n. Then for each $t > 0$ and for each $k > 0$, we have*

(a) *for each r,*

$$\dim H^r_{dR}(M) \le \int_M \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}}(x,x)\ dV;$$

(b) *for each r,*

$$\sum_{j=0}^{r}(-1)^{r-j}\dim H_{dR}^{j}(M)\leq\sum_{j=0}^{r}(-1)^{r-j}\int_{M}\operatorname{tr}e^{-\frac{t}{k}\Delta_{k}^{(j)}}(x,x)\ dV,$$

and the equality holds if $r=n$; namely,

$$\sum_{j=0}^{n}(-1)^{n-j}\dim H_{dR}^{j}(M)=\sum_{j=0}^{n}(-1)^{n-j}\int_{M}\operatorname{tr}e^{-\frac{t}{k}\Delta_{k}^{(j)}}(x,x)\ dV.$$

Proof. We begin by noting that, from (2.3),

$$\begin{aligned}\operatorname{tr}e^{-\frac{t}{k}\Delta_{k}^{(r)}}(x,x)&=\sum_{\lambda\in\operatorname{Spec}\Delta_{k}^{(r)}}\sum_{i=1}^{d_\lambda}e^{-\frac{t}{k}\lambda}\operatorname{tr}\varphi_{i}^{\lambda}(x)\otimes\left(\varphi_{i}^{\lambda}\right)^{*}(x)\\&=\sum_{\lambda\in\operatorname{Spec}\Delta_{k}^{(r)}}\sum_{i=1}^{d_\lambda}e^{-\frac{t}{k}\lambda}\left|\varphi_{i}^{\lambda}(x)\right|^{2},\end{aligned}$$

where $d_\lambda=\dim E_{\lambda,k}^{(r)}(M)$.

For each r, observe that

$$Z^{r}=\int_{M}\operatorname{tr}e^{-\frac{t}{k}\Delta_{k}^{(r)}}(x,x)\,dV=\dim H_{k}^{r}(M)+\sum_{\lambda\in\operatorname{Spec}\Delta_{k}^{(r)}\setminus\{0\}}e^{-\frac{t}{k}\lambda}\dim E_{\lambda,k}^{(r)}(M),$$

which follows from the fact that the order of integral and infinite sum can interchange since the series representation (2.3) converges uniformly on compact subsets. Hence, we conclude

$$\dim H_{k}^{r}(M)\leq Z^{r}.$$

This proves (a).

To see (b), notice that

$$\begin{aligned}&\sum_{j=0}^{r}(-1)^{r-j}Z^{j}\\&=\sum_{j=0}^{r}(-1)^{r-j}\dim H_{k}^{j}(M)+\sum_{j=0}^{r}(-1)^{r-j}\sum_{\lambda^{(j)}\in\operatorname{Spec}\Delta_{k}^{(j)}\setminus\{0\}}e^{-\frac{t}{k}\lambda^{(j)}}\dim E_{\lambda^{(j)},k}^{(j)}(M)\end{aligned}$$

$$= \sum_{j=0}^{r} (-1)^{r-j} \dim H_k^j(M) + \sum_{\lambda \in \mathbb{R}^+} e^{-\frac{t}{k}\lambda} \sum_{j=0}^{r} (-1)^{r-j} \dim E_{\lambda,k}^{(j)}(M),$$

where we interpret $E_{\lambda,k}^{(j)}(M) = \{0\}$ if λ is not an eigenvalue of $\Delta_k^{(j)}$. Finally, by Proposition 2.2 and Proposition 2.1, we have established (b). $\square$

Thanks to Lemma 5.2, proving Theorem 5.1 boils down to investigating the trace integral of the heat kernel $e^{-\frac{t}{k}\Delta_k^{(r)}}(x,y)$, from which our main results (Theorems 1.1, 1.2, 1.3) come in handy.

5.1. Model kernels

To deal with the trace integral in question, it is important to know of the heat kernel $e^{-t\Delta_{f,p}^{(r)}}(x,y)$ with respect to $\Delta_{f,p}^{(r)}$ that we call the model kernel in this paper. In this subsection, we give the explicit expression for the trace of this model kernel by the Mehler's formula (see Theorem 5.3). With that in mind, we will be able to see that the trace integral of $e^{-t\Delta_{f,p}^{(r)}}(x,y)$ can be considered as an indicator of critical points of index r (see Theorem 5.5).

First, we review some facts about the usual harmonic operators. Let L be the harmonic oscillator given by

$$L = -\frac{d^2}{dx^2} + x^2$$

on $\mathrm{Dom}L := \left\{ f \in L^2(\mathbb{R}) : Lf \in L^2(\mathbb{R}) \right\}$. It is well-known that the eigenfunctions of L are given by for each $N \in \mathbb{N} \cup \{0\}$,

$$\Phi_N(x) = \frac{H_N(x)\, e^{-\frac{x^2}{2}}}{\pi^{\frac{1}{4}}\sqrt{2^N N!}},$$

where

$$H_N(x) = (-1)^N e^{x^2} \frac{d^N}{dx^N} e^{-x^2},$$

and with respect to which the eigenvalue is $2N+1$.

Also, we have the following well-known Mehler's formula:

Theorem 5.3 (Mehler's formula)**.** *For each $\rho \in [0,1)$ and for $x, y \in \mathbb{R}$, we* *have*

$$\sum_{n\geq 0}\frac{\rho^n}{2^n n!}H_n\left(x\right)H_n\left(y\right)e^{-\frac{x^2+y^2}{2}}=\frac{1}{\sqrt{1-\rho^2}}\exp\left(\frac{4xy\rho-\left(1+\rho^2\right)\left(x^2+y^2\right)}{2\left(1-\rho^2\right)}\right). \tag{5.4}$$

Similarly, put

$$L^{\pm}=-\frac{d^2}{dx^2}+x^2\pm 1$$

on $\mathrm{Dom}L^{\pm}:=\left\{f\in L^2\left(\mathbb{R}\right):L^{\pm}f\in L^2\left(\mathbb{R}\right)\right\}$. Note that the eigenfuctions of both operators $L^{\pm}$ are again given by $\{\Phi_N\}_{N\in\mathbb{N}\cup\{0\}}$, but the eigenvalue corresponding to Φ_N is $2N+2$ for L^+ while is $2N$ for L^-. Thus, by the Mehler's formula (5.4) with $\rho=e^{-2t}$, we obtain

$$\begin{aligned}
e^{-tL^{\pm}}\left(x,y\right)&=e^{(-1\mp 1)t}\sum_{N\in\mathbb{N}\cup\{0\}}e^{-2Nt}\Phi_N\left(x\right)\Phi_N\left(y\right)\\
&=e^{(-1\mp 1)t}\frac{1}{\pi^{\frac{1}{2}}}\sum_{N\in\mathbb{N}\cup\{0\}}e^{-2Nt}\frac{1}{2^N N!}H_N\left(x\right)H_N\left(y\right)e^{-\frac{x^2+y^2}{2}}\\
&=e^{(-1\mp 1)t}\frac{1}{\pi^{\frac{1}{2}}}\frac{1}{\sqrt{1-e^{-2t}}}\exp\left(\frac{4xye^{-2t}-\left(1+e^{-4t}\right)\left(x^2+y^2\right)}{2\left(1-e^{-4t}\right)}\right).
\end{aligned} \tag{5.5}$$

To write down the heat kernel explicitly, recall that for each $p\in\mathrm{Crit}\left(f\right)$ and for each multi-index I,

$$\Delta_{f,p}^{(r)}\left(gdx^I\right)=\left[-\sum_{i=1}^{n}\frac{\partial^2}{\partial\left(x^i\right)^2}+\left(x^i\right)^2+\sum_{i=1}^{n}\varepsilon_i\varepsilon_i^I\right]gdx^I,$$

where $\varepsilon_i,\varepsilon_I$ as indicated in (3.1). For each strictly increasing multi-index I with $|I|=r$, define $\Delta_{f,p}^I:\mathrm{Dom}\Delta_{f,p}^I\to\mathcal{C}^\infty\left(\mathbb{R}^n\right)$ by

$$\Delta_{f,p}^I g=-\sum_{i=1}^{n}\frac{\partial^2 g}{\partial\left(x^i\right)^2}+\left(x^i\right)^2 g+\sum_{i=1}^{n}\varepsilon_i\varepsilon_i^I g,$$

where $\mathrm{Dom}\Delta_{f,p}^I=\left\{g\in L^2\left(\mathbb{R}^n\right):\Delta_{f,p}^I g\in L^2\left(\mathbb{R}^n\right)\right\}$. As we can see, $\Delta_{f,p}^I g$ is given by taking the I-coefficient from $\Delta_{f,p}^{(r)}\left(gdx^I\right)$ (Note that $\left\{dx^I\right\}_I$ is a global orthonormal frame for $\bigwedge^r T^*\mathbb{R}^n$); namely,

$$\Delta^I_{f,p} g = \left(dx^I\right)^* \left(\Delta^{(r)}_{f,p}\left(g dx^I\right)\right)$$

for each $g \in \mathrm{Dom}\Delta^I_{f,p}$. Moreover, $\Delta^I_{f,p}$ is self-adjoint and non-negative, so we can speak of its heat kernel $e^{-t\Delta^I_{f,p}}(x,y) \in \mathcal{C}^\infty\left(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n\right)$.

Set

$$L_i^{\pm} - \frac{\partial^2}{\partial\left(x^i\right)^2} + \left(x^i\right)^2 \pm 1.$$

and, we can write

$$\Delta^I_{f,p} = \sum_{\{i\in I: i\le \mathrm{Ind}_f p\}} L_i^- + \sum_{\{i\in I: i> \mathrm{Ind}_f p\}} L_i^+ + \sum_{\{i\notin I: i\le \mathrm{Ind}_f p\}} L_i^- + \sum_{\{i\notin I: i> \mathrm{Ind}_f p\}} L_i^+. \tag{5.6}$$

From the previous presented facts, we can see

$$\Phi_{N_1,\ldots,N_n}\left(x^1,\ldots,x^n\right) = \Phi_{N_1}\left(x^1\right)\cdots\Phi_{N_n}\left(x^n\right)$$

constitute the set of orthonormal eigenfunctions for $\Delta^I_{f,p}$, with respect to which the eigenvalue is

$$\begin{aligned}
&\lambda_{N_1,\ldots,N_n} \\
&= \sum_{\{i\in I: i\le \mathrm{Ind}_f p\}} 2N_i + \sum_{\{i\in I: i> \mathrm{Ind}_f p\}} \left(2N_i+2\right) + \sum_{\{i\notin I: i\le \mathrm{Ind}_f p\}} 2N_i + \sum_{\{i\notin I: i> \mathrm{Ind}_f p\}} \left(2N_i+2\right).
\end{aligned}$$

Hence, put $x = \left(x^1,\ldots,x^n\right), y = \left(y^1,\ldots,y^n\right)$ and the heat kernel $e^{-t\Delta^I_{f,p}}$ can be given by

$$\begin{aligned}
e^{-t\Delta^I_{f,p}}(x,y) &= \sum_{N_1,\ldots,N_n\in\mathbb{N}} e^{-t\lambda_{N_1,\ldots,N_n}} \Phi_{N_1,\ldots N_n}(x)\,\Phi_{N_1,\ldots,N_n}(y) \\
&= \prod_{\{i\in I: i\le \mathrm{Ind}_f p\}} e^{-tL_i^-}\left(x^i,y^i\right) \prod_{\{i\in I: i> \mathrm{Ind}_f p\}} e^{-tL_i^+}\left(x^i,y^i\right) \\
&\quad\times \prod_{\{i\notin I: i\le \mathrm{Ind}_f p\}} e^{-tL_i^+}\left(x^i,y^i\right) \prod_{\{i\notin I: i> \mathrm{Ind}_f p\}} e^{-tL_i^-}\left(x^i,y^i\right),
\end{aligned} \tag{5.7}$$

and each of the heat kernels $e^{-tL_i^{\pm}}\left(x^i,y^i\right)$ can be written explicitly via (5.5).

Proposition 5.4. *For any two $x, y \in \mathbb{R}^n$, write*

$$e^{-t\Delta_{f,p}^{(r)}}(x,y) = \sum_{I,J}{}' e^{-t\Delta_{f,p}^{(r)}}{}_{I,J}(x,y)\left(dx^I\right)(x) \otimes \left(dx^J\right)^*(y).$$

Then

$$e^{-t\Delta_{f,p}^{(r)}}{}_{I,I}(x,y) = e^{-t\Delta_{f,p}^{I}}(x,y). \tag{5.8}$$

Proof. Put $A_{I,I}(t) : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ by

$$\left(A_{I,I}(t)\, g\right)(x) = \int_{\mathbb{R}^n} e^{-t\Delta_{f,p}^{(r)}}{}_{I,I}(x,y)\, g(y)\ dy.$$

To prove (5.8), it suffices to show $A_{I,I}(t)\, g \in \mathrm{Dom}\Delta_{f,p}^I$ and

$$\begin{cases} \frac{\partial}{\partial t} A_{I,I}(t)\, g + \Delta_{f,p}^I A_{I,I}(t)\, g = 0 \\ \lim\limits_{t \to 0^+} \left\| A_{I,I}(t)\, g - g \right\|_{L^2(\mathbb{R}^n)} = 0 \end{cases}$$

for each $g \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

Note that

$$\begin{aligned} \left\|\Delta_{f,p}^I A_{I,I}(t)\, g\right\|_{L^2(\mathbb{R}^n)} &= \left\|\left(dx^I\right)^*\left(\Delta_{f,p}^{(r)}\left(A_{I,I}(t)\, g\right) dx^I\right)\right\|_{L^2(\mathbb{R}^n)} \\ &\le \left\|\Delta_{f,p}^{(r)} e^{-t\Delta_{f,p}^{(r)}}\left(g dx^I\right)\right\|_{L^2(\mathbb{R}^n)} < \infty \end{aligned}$$

and

$$\begin{aligned} \Delta_{f,p}^I A_{I,I}(t)\, g &= \left(dx^I\right)^*\left[\Delta_{f,p}^{(r)}\left(\int_{\mathbb{R}^n} e^{-t\Delta_{f,p}^{(r)}}(x,y)\left(g(y)\ dx^I\right) dy\right)\right] \\ &= \left(dx^I\right)^*\left[-\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} e^{-t\Delta_{f,p}^{(r)}}(x,y)\left(g(y)\ dx^I\right) dy\right)\right] \\ &= -\frac{\partial}{\partial t} A_{I,I}(t)\, g \end{aligned}$$

for each $g \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Since $\left\{dx^I\right\}_I$ is an orthonormal frame, we see that for each $g \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\left\|e^{-t\Delta_{f,p}^{(r)}}\left(g dx^I\right) - g dx^I\right\|_{L^2(\mathbb{R}^n)}^2 \sum_{K \neq I}{}' \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} e^{-t\Delta_{f,p}^{r}}{}_{K,I}(x,y)\, g(y)\, dy\right|^2 dx$$

$$+\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} e^{-t\Delta^r_{f,p}}{}_{I,I}\left(x,y\right)g\left(y\right)dy-g\left(x\right)\right|^2 dx\to 0$$

as $t\to 0^+$, implying

$$\left\|A_{I,I}\left(t\right)g-g\right\|^2_{L^2(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} e^{-t\Delta^r_{f,p}}{}_{I,I}\left(x,y\right)g\left(y\right)dy-g\left(x\right)\right|^2 dx\to 0$$

as $t\to 0^+$. Finally, by using the fundamental theorem of Calculus, together with $\Delta^I_{f,p}A_{I,I}\left(t\right)g\in L^2\left(\mathbb{R}^n\right)$ as in the last part of proof of Theorem 1.1, we obtain $A_{I,I}\left(t\right)=e^{-t\Delta^I_{f,p}}$ in $\mathcal{C}^\infty_c\left(\mathbb{R}^n\right)$. Hence, we have furnished (5.8). □

Proposition 5.4 shows that, under the global orthonormal frame $\left\{dx^I\right\}_I$, the diagonal entries of the model kernel $e^{-t\Delta^{(r)}_{f,p}}\left(x,y\right)$ as an linear transformation from $\bigwedge^r T^*_x\mathbb{R}^n$ to $\bigwedge^r T^*_y\mathbb{R}^n$ are nothing but $e^{-t\Delta^I_{f,p}}\left(x,y\right)$. That being said, we see that the trace of $e^{-t\Delta^{(r)}_{f,p}}\left(x,y\right)$ is given by

$$\operatorname{tr}e^{-t\Delta^{(r)}_{f,p}}\left(x,y\right)=\sum_I{}' e^{-t\Delta^{(r)}_{f,p}}{}_{I,I}\left(x,y\right)=\sum_I{}' e^{-t\Delta^I_{f,p}}\left(x,y\right) \tag{5.9}$$

and $e^{-t\Delta^I_{f,p}}\left(x,y\right)$ can be written as in (5.7).

Theorem 5.5.

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}\operatorname{tr}e^{-t\Delta^{(r)}_{f,p}}\left(x,x\right)\,dV_x=\begin{cases}1 & ,\ \textit{if } r=\operatorname{Ind}_f p\\ 0 & ,\ \textit{otherwise}\end{cases}. \tag{5.10}$$

Proof. By (5.9), (5.7), and the Fubini theorem, the trace integral in question is determined by the (trace) integrals of $e^{-tL^\pm_i}\left(x,y\right)$:

$$\lim_{t\to\infty}\int_{\mathbb{R}} e^{-tL^\pm_i}\left(x^i,x^i\right)\,dx^i.$$

By (5.5), we can write

$$e^{-tL^\pm_i}\left(x^i,x^i\right)=e^{(-1\mp1)t}\frac{1}{\pi^{\frac{1}{2}}}\frac{1}{\sqrt{1-e^{-2t}}}\exp\left(\frac{4\left(x^i\right)^2e^{-2t}-2\left(1+e^{-4t}\right)\left(x^i\right)^2}{2\left(1-e^{-4t}\right)}\right).$$

Since $e^{-t}\to 0$ as $t\to\infty$, $e^{-tL^\pm_i}\left(x^i,x^i\right)$ are both bounded by an integrable function. Also, we can see

$$\lim_{t\to\infty} e^{-tL_i^+}\left(x^i,x^i\right)=0$$

and

$$\lim_{t\to\infty} e^{-tL_i^-}\left(x^i,x^i\right)=\frac{1}{\pi^{\frac{1}{2}}}e^{-\left(x^i\right)^2}.$$

Thus, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t\to\infty}\int_{\mathbb{R}} e^{-tL_i^+}\left(x^i,x^i\right)\ dx^i=0$$

and

$$\lim_{t\to\infty}\int_{\mathbb{R}} e^{-tL_i^-}\left(x^i,x^i\right)\ dx^i=1.$$

Now, to see (5.10), if $r=\mathrm{Ind}_f p$, choose $I_0=(1,\ldots,r)$ and we obtain

$$e^{-t\Delta_{f,p}^{I_0}}(x,x)=\prod_{i=1}^{n} e^{-tL_i^-}\left(x^i,x^i\right),$$

in which case, we deduce

$$\lim_{t\to\infty}\int_{\mathbb{R}^n} e^{-t\Delta_{f,p}^{I_0}}(x,x)\,dx=1.$$

For each of the other strictly increasing indices $I\neq I_0$, there must exists $s\in I$ such that $s>r=\mathrm{Ind}_f p$, implying, from (5.7), $e^{-t\Delta_{f,p}^{I}}(x,x)$ contains the kernel $e^{-tL_s^+}(x^s,x^s)$. This leads to

$$\lim_{t\to\infty}\int_{\mathbb{R}^n} e^{-t\Delta_{f,p}^{I}}(x,x)\,dx=0.$$

Hence, we conclude $\lim_{t\to\infty}\int_{\mathbb{R}^n}\mathrm{tr}\, e^{-t\Delta_{f,p}^{(r)}}(x,x)\,dx=1$ if $r=\mathrm{Ind}_f p$.

If $r\neq\mathrm{Ind}_f p$, we discuss in the two cases for strictly increasing indices I: I whose elements are lower or equal to $\mathrm{Ind}_f p$ and the others. For each strictly increasing index I whose elements are lower or equal to $\mathrm{Ind}_f p$, there must exist $s\notin I$ such that $s\leq\mathrm{Ind}_f p$. This implies $e^{-t\Delta_{f,p}^{I}}(x,x)$ contains $e^{-tL_s^+}(x^s,x^s)$, resulting in $\lim_{t\to\infty}\int_{\mathbb{R}^n} e^{-t\Delta_{f,p}^{I}}(x,x)\,dx=0$. For each of the other strictly increasing indices I, I must contains at least one element s such that $s>\mathrm{Ind}_f p$. This implies again $e^{-t\Delta_{f,p}^{I}}(x,x)$ contains

$e^{-tL_s^+}(x^s,x^s)$ and so $\lim_{t\to\infty}\int_{\mathbb{R}^n} e^{-t\Delta^I_{f,p}}(x,x)\,dx = 0$. Therefore, we conclude $\lim_{t\to\infty}\int_{\mathbb{R}^n}\operatorname{tr} e^{-t\Delta^{(r)}_{f,p}}(x,x)\,dx = 0$ if $r \neq \mathrm{Ind}_f p$. Hence, we have furnished Theorem 5.5. □

We will see in a moment that Theorem 5.5 plays a central role in tackling the trace integral of $e^{-\frac{t}{k}\Delta_k^{(r)}}(x,y)$.

5.2. Heat kernel proof

In this subsection, we give our heat kernel proof of the Morse inequalities. The essence of our proof is that our main results (Theorems 1.1, 1.2, and 1.3) allow us to see as $k\to\infty$, the trace integral of $e^{-\frac{t}{k}\Delta_k^{(r)}}(x,y)$ is approximately close to the sum of the trace integral of the model kernels, which further converges to the Morse number m_r by Theorem 5.5 as $t\to\infty$. In other words, we can deduce the following result regarding the trace integral in question based upon our main results:

Theorem 5.6.

$$\lim_{t\to\infty}\lim_{k\to\infty}\int_M \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}}(x,x)\ dV = m_r \tag{5.11}$$

for each r.

Proof. Let D be the positive number given in Theorem 1.2. For each $p\in \mathrm{Crit}(f)$, put $D_p^k = \varphi_p^{-1}\left(B_{2Dk^{-\frac{1}{2}}}(0)\right)$ and $\mathcal{D}^k = \bigcup_{p\in\mathrm{Crit}(f)} D_p^k$. Moreover, let $\mathcal{U}^k$ be as given in Theorem 1.3. Note that

$$\begin{aligned}
&\int_M \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}}(x,x)\,dV \\
&= \underbrace{\int_{\mathcal{D}^k} \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}}(x,x)\,dV}_{(A)} + \underbrace{\int_{\mathcal{U}^k\setminus\mathcal{D}^k} \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}}(x,x)\,dV}_{(B)} \\
&\quad + \underbrace{\int_{M\setminus\mathcal{U}^k} \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}}(x,x)\,dV}_{(C)}.
\end{aligned} \tag{5.12}$$

By a change of variable, we see that

$$(A) = \sum_{p \in \mathrm{Crit}(f)} \int_{B_{2D}(0)} \operatorname{tr} A^r_{(k),p}(t,x,x)\, dx.$$

By Theorem 1.1, we obtain

$$\lim_{k\to\infty}(A) = \sum_{p \in \mathrm{Crit}(f)} \int_{B_{2D}(0)} \operatorname{tr} e^{-t\Delta^{(r)}_{f,p}}(x,x)\, dx. \tag{5.13}$$

Similarly, we obtain

$$(B) = \sum_{p \in \mathrm{Crit}(f)} \int_{\mathbb{R}^n} \mathbf{1}_{B_{k^\varepsilon}(0)\setminus B_{2D}(0)} \operatorname{tr} A^r_{(k),p}(t,x,x)\, dx,$$

where $\mathbf{1}_{B_{k^\varepsilon}(0)\setminus B_{2D}(0)}$ is the characteristic function of $B_{k^\varepsilon}(0) \setminus B_{2D}(0)$. Now, Theorem 1.2 indicates that the integrand is bounded above by an integrable function $|x|^{-N}$ with large N; namely,

$$\left|\mathbf{1}_{B_{k^\varepsilon}(0)\setminus B_{2D}(0)} \operatorname{tr} A^r_{(k),p}(t,x,x)\right| \leq \frac{C_1}{|x|^N},$$

where C_1 is independent of D, x, k, for each $x \in \mathbb{R}^n$ and for large k. Hence, by Theorem 1.1 and Lebesgue dominated convergence theorem, we retrieve

$$\lim_{k\to\infty}(B) = \sum_{p \in \mathrm{Crit}(f)} \int_{\mathbb{R}^n\setminus B_{2D}(0)} e^{-t\Delta^{(r)}_{f,p}}(x,x)\, dx. \tag{5.14}$$

To deal with (C), choose a pair $(\mathcal{V}, \mathcal{P}, \mathcal{E})$ so that the $\mathcal{C}^0$-norm is defined on the compact manifold M. Using the partition of unity, we can rewrite (C) as the finite sum:

$$(C) = \sum_{\psi\in\mathcal{P}} \int_M \mathbf{1}_{\mathrm{supp}\,\psi\cap M\setminus\mathcal{U}^k} \cdot \psi \operatorname{tr} e^{-\frac{t}{k}\Delta^{(r)}_k}(x,x)\, dV.$$

By Theorem 1.3, choose a positive integer $N \in \mathbb{N}$ and we see that

$$\left|\mathbf{1}_{\mathrm{supp}\,\psi\cap M\setminus\mathcal{U}^k} \cdot \psi \operatorname{tr} e^{-\frac{t}{k}\Delta^{(r)}_k}(x,x)\right| \leq \frac{C_2(t)}{k^N} \to 0, \text{as } k \to \infty$$

where $C_2(t)$ depends on t but is independent of k and $\psi \in \mathcal{P}$, for each $x \in M$. Hence, by Lebesgue dominated convergence theorem, we conclude

$$\lim_{k\to\infty} (C) = 0. \tag{5.15}$$

Therefore, by (5.12), (5.13), (5.14), and (5.15), we obtain

$$\lim_{k\to\infty} \int_M \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}} (x,x)\, dV = \int_{\mathbb{R}^n} \operatorname{tr} e^{-t\Delta_{f,p}^{(r)}} (x,x)\, dx.$$

Finally, Theorem 5.5 gives

$$\lim_{t\to\infty} \lim_{k\to\infty} \int_M \operatorname{tr} e^{-\frac{t}{k}\Delta_k^{(r)}} (x,x)\, dV = m_r.$$

Hence, we have established Theorem 5.6. □

Morse inequalities now follows from by Lemma 5.2 and Theorem 5.6.

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