

CONSTRUCTING SOLITON SOLUTIONS OF GEOMETRIC FLOWS BY SEPARATION OF VARIABLES

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Abstract

This note surveys and compares results in [12] and [21, 22] on the separation of variables construction for soliton solutions of curvature equations including the Kähler-Ricci flow and the Lagrangian mean curvature flow. In the last section, we propose some new generalizations in the Lagrangian mean curvature flow case.

1. Introduction

Let us recall a simple example of separation of variables constructions of solutions of elliptic PDE's. Consider the Laplace equation $\Delta^{\mathbb{R}^n} f = 0$ on \mathbb{R}^n . In terms of polar coordinates, the equation becomes

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \Delta^{S^{n-1}} f = 0. \quad (1)$$

The separation of variables ansatz assumes that f takes the form $f(r, \theta) = f_1(r)f_2(\theta)$ for $r \in (0, \infty)$ and $\theta \in S^{n-1}$, and equation (1) can be written as

$$-\frac{\frac{1}{r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f_1}{\partial r} \right)}{f_1(r)} = \frac{\Delta^{S^{n-1}} f_2}{f_2} = \text{constant}.$$

Thus, the original equation reduces to an ODE for f_1 and a PDE for a function f_2 on the sphere.

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Suppose instead we would like to solve an inhomogeneous equation of the form $\Delta^{\mathbb{R}^n} f = g$. Assuming again that g takes the form $g(r, \theta) = g_1(r)g_2(\theta)$, for $g_1(r)$ a function on $(0, \infty)$ and $g_2(\theta)$ a function on S^{n-1} , the ansatz $f(r, \theta) = f_1(r)f_2(\theta)$ for suitable choices of $g_2(\theta)$ and $f_2(\theta)$ reduces the original equation to an ODE for $f_1(r)$ involving $g_1(r)$. Thus the method of separation of variables produces solutions of homogeneous equations and reduces an inhomogeneous equation to an ODE.

2. Application to Nonlinear Elliptic Curvature Equations

We discuss two types of geometric parabolic equations, the Lagrangian mean curvature flow and the Kähler-Ricci flow, and their soliton solutions. First we review the elliptic case.

Suppose a submanifold is given by an embedding X (or more generally an immersion) into the Euclidean space. The mean curvature equation $H(X) = 0$ corresponds to minimal submanifolds. We recall that $H(X) = \Delta X$ for the embedding X where Δ is the Laplace operator with respect to the induced metric. Thus this should be considered as the Laplace equation for submanifolds.

On the other hand, suppose a Riemannian metric g is given. The Ricci curvature equation $Ric(g) = 0$ corresponds to Ricci flat manifolds. We recall that $Ric(g) = -\frac{1}{2}\Delta g$ at a point with respect to a harmonic normal coordinate system, and we can view the Ricci equation as a Laplace equation for Riemannian metrics.

For both nonlinear elliptic equations, cone-like solutions appear naturally in the study of singularity formation through blow-up, or rescaling process. Minimal cone in \mathbb{R}^n is of the form $rX, r > 0$ where $X : \Sigma \rightarrow S^{n-1}$ is a minimal submanifold in S^{n-1} . On the other hand, a Ricci flat cone is of the form $dr^2 + r^2 ds_\Sigma^2$ for an Einstein manifold Σ . Both are solutions of homogeneous elliptic equations and provide ansatz for separation of variable. An elliptic geometric curvature problem is often accompanied by a parabolic problem, which is the gradient flow in case the elliptic problem is variational. Parabolic blow-up (rescaling) gives soliton solutions which are self-similar in space and time variables. To classify singularities of parabolic flows, it is important to study solitons. It turns out solitons of parabolic equations satisfy elliptic inhomogeneous equations.

In the case of mean curvature flows, suppose X_t is a family of embeddings that satisfies the mean curvature flow equation:

$$\frac{\partial}{\partial t} X_t = H(X_t).$$

Special solutions called solitons or self-similar solutions can be obtained in the following way. First one solves the elliptic equation for an embedding X such that

$$H(X) = (aX + \vec{b})^\perp$$

for a constant real number a and a constant vector \vec{b} . Then one defines a one-parameter family of embeddings X_t by

$$X_t = a(t)X + \vec{b}(t).$$

One can check that X_t forms a mean curvature flow that is moving by scaling and translation for suitable choices of $a(t)$ and $\vec{b}(t)$. The sign of a determines whether the flow is expanding, shrinking, or steady (translating).

On the other hand, the Ricci flow is a parabolic equation for a family of metrics g_t that satisfies:

$$\frac{\partial}{\partial t} g_t = -2Ric(g_t).$$

Self-similar solutions can be obtained by solving the elliptic Ricci soliton equation: $Ric(g) = ag + \mathcal{L}_V g$ for a metric g , where a is a constant real number and V is a smooth vector field on the underlying manifold.

The family of metrics g_t

$$g_t = a(t)\gamma_t^* g,$$

where γ_t is the one-parameter group of diffeomorphism generated by V , satisfies the Ricci flow equation and the corresponding flow is a combination of scalings and reparametrizations. Likewise, the sign of a determines whether the flow is expanding, shrinking, or steady.

3. The Kähler-Ricci Flow

Let us first look at the Kähler-Ricci flow. Suppose the Kähler form is $\omega = ig_{i\bar{j}}dz^i \wedge d\bar{z}^j$, the Ricci form is then given by $\rho(\omega) = -i\partial\bar{\partial} \log \det g_{i\bar{j}}$. Given a family of Kähler metrics, in terms of the associated Kähler forms, the Kähler-Ricci flow is the parabolic equation

$$\frac{d}{dt}\omega = -\frac{1}{2}\rho(\omega).$$

Correspondingly, the Kähler-Ricci soliton equation is

$$-\frac{1}{2}\rho(\omega) = \lambda\omega + \mathcal{L}_V\omega,$$

where $\lambda = 1, 0, -1$ corresponds to expanding, steady and shrinking Ricci solitons.

This is a gradient soliton if

$$\mathcal{L}_V\omega = i\partial\bar{\partial}Q$$

for a function Q . Therefore, the Kähler-Ricci gradient soliton equation for Kähler form ω and potential Q is:

$$-\frac{1}{2}\rho(\omega) = \lambda\omega + i\partial\bar{\partial}Q.$$

The Ansatz to apply the separation of variables method in this case involves a Sasakian manifold. Recall a $2m + 1$ dimensional Riemannian manifold (S, g) is Sasakian if $\bar{g} = dr^2 + r^2g$ is a Kähler metric on the cone $C(S)$ over S .

The restriction to S of $\xi = Jr\frac{\partial}{\partial r}$ is the Reeb vector field of the contact form that satisfies $i(\xi)\eta = 1$ and $i(\xi)d\eta = 0$. We have the following relation:

$$\eta = \frac{1}{r^2}\bar{g}(\xi, \cdot) = i(\bar{\partial} - \partial) \log r.$$

The vector field ξ defines a transverse Kähler structure on the local orbit space. ξ is killing on $(C(S), \bar{g})$ and its complexification $\xi - iJ\xi$ is holomorphic. In particular, the transverse metric satisfies $g^T + \eta \otimes \eta = g$

and the transverse Kähler form is

$$\omega^T = \frac{1}{2}d\eta = i\partial\bar{\partial}\log r.$$

The Kähler form on $C(S)$ is

$$\omega = \frac{i}{2}\partial\bar{\partial}r^2.$$

For example, we can take $S = S^{2m+1}$, $C(S) = \mathbb{C}^{m+1} - \{0\}$ and the orbit space is $\mathbb{C}\mathbb{P}^m$.

One can check that (S, g) is a Sasaki-Einstein manifold if and only if $(C(S), \bar{g})$ is a Calabi-Yau cone manifold. Also, S is η -Einstein ($Ric_g = \alpha g + \beta\eta \otimes \eta$ with $\alpha + \beta = 2m$) if and only if the transverse Kähler metric is Einstein ($\rho(\omega^T) = (\alpha + 2)\omega^T$).

Suppose we start with an η -Einstein Sasaki manifold. The ansatz for the Kähler form of a Kähler-Ricci gradient soliton is

$$\omega^T + i\partial\bar{\partial}F(s)$$

with $s = \log r$. This is similar to the Calabi Ansatz [4] for Calabi-Yau metrics which was applied to the total space of a Hermitian line bundles over a Kähler manifold.

The equation for $F(s)$ can be derived as follows. Set $\sigma = 1 + F'(s)$ and $\phi(\sigma) = F''(s)$. We have

$$\omega_\phi = \sigma\omega^T + \phi(\sigma)i\partial s \wedge \bar{\partial} s$$

where $i\partial s \wedge \bar{\partial} s$ is the Kähler form of the cylindrical metric on $\mathbb{C} - \{0\}$.

As a comparison, the metric on the Calabi-Yau cone ($Ric = \rho(\omega) = 0$) is given by $F(s) = \frac{r^2}{2} - \log r = \frac{1}{2}e^{2s} - s$ and $\phi(\sigma) = 2\sigma$. The Ricci form of ω_ϕ is

$$\rho(\omega_\phi) = \kappa\omega^T - i\partial\bar{\partial}\log(\sigma^m\phi(\sigma))$$

with $\kappa = \alpha + 2$.

For a gradient soliton, if the potential Q is a function of s , Q must be of the form $Q = \mu\sigma + c$ for constants μ and c . Therefore, the gradient soliton

equation becomes

$$\phi'(\sigma) + \left(\frac{m}{\sigma} - \mu\right)\phi(\sigma) - (\kappa + 2\lambda\sigma) = 0$$

for m, μ, κ, λ constants.

Studying of the ODE gives the following theorem:

Theorem 1 ([12]). *Let S be a compact Sasaki manifold such that the transverse Kähler metric g^T satisfies Einstein equation*

$$\text{Ric}^T = \kappa g^T$$

for some $\kappa < 0$ (or S is η -Einstein). Then there exists a complete expanding soliton on the Kähler cone $C(S)$.

On total space of line bundles over Fano manifolds, we also obtain shrinking and expanding solitons that can be glued together to form eternal solution on $(-\infty, \infty)$.

Theorem 2 ([12]). *Let M be a Fano manifold of dimension m , and $L \rightarrow M$ be a positive line bundle with $K_M = L^{-p}$, $p \in \mathbb{Z}^+$. For $0 < k < p$, let S be the $U(1)$ -bundle associated with L^{-k} , which is a regular Sasaki manifold. Let Z be the zero section of L^{-k} . Suppose that S admits a Sasaki-Einstein metric (or M admits a K-E metric). Then there exist shrinking and expanding solitons on $L^{-k} - Z$, and they can be pasted together to form an eternal solution of the Kähler-Ricci flow on $(L^{-k} - Z) \times (-\infty, \infty)$. The shrinking soliton for $t \in (-\infty, 0)$ extends smoothly to the zero section Z .*

Our construction generalizes the work of H.-D. Cao [5], Feldman-Ilmanen-Knopf [11] which are rotationally symmetric. B. Yang [27] and Dancer-Wang [10] constructed cohomogeneity-one examples. All these examples have very large symmetry group. In contrast, our examples do not carry any continuous symmetry in general.

4. The Lagrangian mean curvature flow

An n -dimensional submanifold $L \subset \mathbb{C}^n$ is Lagrangian if $\omega|_L = 0$ where ω is the standard Kähler form of \mathbb{C}^n . An $n - 1$ dimensional submanifold $\Sigma \subset S^{2n-1}$ is Legendrian if the cone L over Σ is Lagrangian. L is minimal

in \mathbb{C}^n if and only if Σ is minimal in S^{2n-1} . Being Lagrangian is a closed condition preserved by the mean curvature flow [25] and thus it makes sense to consider the Lagrangian mean curvature flow. This is similar to being Kählerian is a closed condition that is preserved by the Ricci flow.

Consider a minimal Legendrian $X : \Sigma^{n-1} \rightarrow S^{2n-1} \subset \mathbb{C}^n$ and a curve $\gamma(s) : I \rightarrow \mathbb{C}^*$. An Ansatz for Lagrangian submanifold can be taken to be $\gamma(s) \cdot X : I \times \Sigma^{n-1} \rightarrow \mathbb{C}^n$ where \cdot denotes complex multiplication. The soliton equation for the Lagrangian mean curvature flow reduces to an ODE for $\gamma(s)$ (Angenent, Ilmanen, Anciaux [1],[2]).

An Ansatz which differs from the minimal Legendrian one but still fits into the separation of variable method is the quadric Ansatz. Here is an example of a soliton of the Lagrangian mean curvature flow which was constructed in [21, 22]. Consider a quadric in \mathbb{R}^n given by

$$\Sigma = \{(x_1, \dots, x_n) \mid \lambda_1(x_1)^2 + \dots + \lambda_n(x_n)^2 = 1\}$$

where λ_i non-zero and $\sum_{i=1}^n \lambda_i > 0$. Then

$$L = \{(x_1 e^{i\lambda_1 s}, \dots, x_n e^{i\lambda_n s} \mid (x_1, \dots, x_n) \in \Sigma, s \in I\}$$

is a soliton solution.

In fact, the trace of the corresponding flow in \mathbb{R}^n is given by the family of hypersurfaces defined by

$$\sum_{i=1}^n \lambda_i x_i^2 = (-2t) \sum_{i=1}^n \lambda_i.$$

We note that the flow is shrinking for $t < 0$ and expanding for $t > 0$. The solution for $t \in (-\infty, \infty)$ forms a weak eternal solution of the mean curvature flow and the singularity at $t = 0$ corresponds to a neck pinching with topological change.

The quadric ansatz was used earlier by Joyce [18, 19] to construct special Lagrangian submanifolds. Joyce-Lee-Tsui in [20] constructed translating solitons of the Lagrangian mean curvature flow of the form

$$L = \{(x_1 w_1(s), \dots, x_n w_n(s) \mid (x_1, \dots, x_n) \in \Sigma, s \in I\},$$

where $w_1(s), \dots, w_n(s)$ satisfy an ODE system. This is a Lagrangian version of “Grim Reaper” for curve shortening flows. Further generalizations were obtained in [24]. There are also uniqueness results for minimal Lagrangians and Lagrangian solitons, see [23], [17], [7, 8].

5. Generalizations

In this section, we discuss a generalization of both the minimal Legendrian and the quadric Ansatz. Take an $(n - 1)$ dimensional submanifold Σ of \mathbb{C}^n that is isotropic and consider the image of $\mathfrak{A}(s)\Sigma$ where $\mathfrak{A}(s)$ is a curve in the complex affine group which is the semi-direct product of $GL(n, \mathbb{C})$ and \mathbb{C}^n .

One can ask for what kind of Σ there will exist a solution $\mathfrak{A}(s)$ such that $\mathfrak{A}(s)\Sigma$ is a Lagrangian submanifold. The answer is that Σ is an *initial data* in the following sense:

A k -dimensional isotropic submanifold Σ of \mathbb{C}^n is said to be an *initial data* with respect to (B, b) with $B \in GL(n, \mathbb{C})$ and $b \in \mathbb{C}^n$ if at any point $p \in \Sigma$,

$$\langle V, BX_p + b \rangle = 0 \tag{2}$$

for any $V \in T_p\Sigma$, where $\langle \cdot, \cdot \rangle$ is the Hermitian product on \mathbb{C}^n .

For example, let Λ be an $n \times n$ Hermitian matrix and let \mathfrak{H} be the real hypersurface in \mathbb{C}^n defined by $\{X \mid \text{Re}\langle \Lambda X, X \rangle = c\}$ for a constant c . Then

$$\gamma(\cdot) = \omega(\Lambda X, \cdot)$$

is a contact form. Let Σ be a Legendrian submanifold of \mathfrak{H} with respect to γ , then Σ is an initial data.

Theorem 3. *Given any initial data (Σ, B, b) and any smooth complex curve $\alpha : I \rightarrow \mathbb{R}$, suppose $(A(s), a(s))$ satisfies*

$$\begin{aligned} A^*(s)\dot{A}(s) &= \alpha(s)B \\ A^*(s)\dot{a}(s) &= \alpha(s)b \text{ with } A(0) = I \text{ and } a(0) = 0 \end{aligned} \tag{3}$$

then $A(s)\Sigma + a(s)$ is an isotropic submanifold.

In order to reduce the equation to an ODE, it is necessary that the Lagrangian angle of $\mathfrak{A}(s)\Sigma$ depends only on the parameter s . Suppose (Σ, B, b) is an initial data of dimension $n - 1$ in \mathbb{C}^n . For any complex curve $\alpha(s)$, the Lagrangian angle of $\mathfrak{A}(s)\Sigma = A(s)\Sigma + a(s)$ at (s, p) is given by

$$\theta(s, p) = \arg \det A(s) + \arg \alpha(s) + \theta(0, p). \tag{4}$$

Therefore, the Lagrangian angle of $\mathfrak{A}(s)\Sigma$ depends only on the parameter s if Σ is an *initial data with constant Lagrangian angle* in the following sense: We say (Σ, B, b) is an *initial data with constant Lagrangian angle* if the Lagrangian angle of $T_p\Sigma \oplus \mathbb{R}(BX_p + b)$ is a constant in $p \in \Sigma$.

Example 4. Suppose Σ is an $(n - 1)$ dimensional Legendrian submanifold of $\mathfrak{H} = \{X \mid \langle \Lambda X, X \rangle = c\} \subset \mathbb{C}^n$. The Σ is an initial data with constant Lagrangian angle if

$$\omega\left(-H + \frac{\Lambda^2 X}{|\Lambda X|^2}, \cdot\right) = 0 \tag{5}$$

on Σ where H is the mean curvature vector of Σ in \mathfrak{H} .

Both minimal Legendrians of S^{2n-1} ($\Lambda = I$) and real quadrics in \mathbb{R}^n ($\mathfrak{H} \cap \mathbb{R}^n$ for Λ diagonal) satisfy this condition.

For an initial data with constant Lagrangian angle Σ , a mean curvature equation (minimal Lagrangian equation or Lagrangian soliton equation) on the Lagrangian submanifold $A(s)\Sigma$ reduces to an ODE system on $A(s)$ that can be solved.

Theorem 5. *Suppose (Σ, B, b) is an initial data with constant Lagrangian angle. Let $(A(s), a(s))$ be a solution of*

$$\begin{aligned} A^*(s)\dot{A}(s) &= \overline{\det A(s)}B \\ A^*(s)\dot{a}(s) &= \overline{\det A(s)}b \text{ with } A(0) = I \text{ and } a(0) = 0 \end{aligned} \tag{6}$$

then $A(s)\Sigma + a(s)$ is a special Lagrangian submanifold.

We also consider the multiple developing case, i.e.

$$A(s_1, \dots, s_k) = A_1(s_1) \cdots A_k(s_k)$$

on an $(n - k)$ dimensional initial data and the twisted product of two initial data.

Theorem 6. *For $p > 1$ and $q > 1$. Suppose Σ_1 is a $p - 1$ dimensional isotropic submanifold of \mathbb{C}^p that is an initial data of constant Lagrangian angle with respect to B_1 and Σ_2 is a $q - 1$ dimensional isotropic submanifold of \mathbb{C}^q that is an initial data of constant Lagrangian angle with respect to B_2 , then $\Sigma = \Sigma_1 \times \Sigma_2 \subset \mathbb{C}^{p+q}$ is an initial data of constant Lagrangian angle in \mathbb{C}^{p+q} with respect to any $c_{11}\tilde{B}_1 + c_{12}\tilde{B}_2, c_{21}\tilde{B}_1 + c_{22}\tilde{B}_2$ where $\tilde{B}_1 = \begin{bmatrix} B_1 & 0_{p \times q} \\ 0_{q \times p} & 0_{q \times q} \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} 0_{p \times p} & 0_{p \times q} \\ 0_{q \times p} & B_2 \end{bmatrix}$, and $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \in GL(2, \mathbb{R})$.*

The twisted product of Legendrians by Castro-Li-Urbano [9] (see also Haskins-Kapouleas [15]) is such an example. New examples can be constructed by taking Σ_1 to be a real quadric in \mathbb{C}^p and Σ_2 to be a minimal Legendrian in $S^{2q-1} \subset \mathbb{C}^q$.

Further questions to be studied include:

1. The construction in the last section only addresses the local solvability of the soliton equation. It would be interesting to see whether these solutions are complete.

2. Are there more general initial data as “minimal Legendrians” of general real codimensional 1 hyperquadrics in \mathbb{C}^n ?

3. What is the relation to complex affine geometry? There should be a rigidity theorem for those solitons whose level sets of Lagrangian angles are real hyperquadrics in a Lagrangian subspace? Note that the level sets are congruent under actions of the complex affine group.

4. Is there a similar construction of “initial data” in the Kähler-Ricci flow case? Does it lead to a further generalization of η -Sasaki-Einstein manifolds?

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