

A ONE-PHASE PROBLEM FOR THE FRACTIONAL LAPLACIAN: REGULARITY OF FLAT FREE BOUNDARIES

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Dedicated to Neil Trudinger on the occasion of his 70th birthday

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Abstract

We consider a one-phase free boundary problem involving a fractional Laplacian $(-\Delta)^\alpha$, $0 < \alpha < 1$, and we prove that “flat free boundaries” are $C^{1,\gamma}$. We thus extend the known result for the case $\alpha = 1/2$.

1. Introduction

In the last decade, a large amount of work has been devoted to non linear equations involving non local operators with special attention for the so-called fractional laplacian $(-\Delta)^\alpha$, where $\alpha \in (0, 1)$. This is a Fourier multiplier in \mathbb{R}^n whose symbol is $|\xi|^{2\alpha}$. The main feature of this operator is its non locality, which can be seen from the alternative definition given by

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its integral representation (see [11])

$$(-\Delta)^\alpha u(x) = PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} dy$$

where PV denotes the Cauchy principal value (up to a renormalizing constant depending on n and α .)

This paper investigates the regularity properties of a free boundary problem involving the fractional Laplacian. More precisely, we are interested in a Bernoulli-type one-phase problem. The classical one is given by

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \cap \{u > 0\}, \\ |\nabla u| = 1, & \text{on } \Omega \cap \partial\{u > 0\}, \end{cases} \quad (1.1)$$

with Ω a domain in \mathbb{R}^n . A pioneering investigation of (1.1) was that of Alt and Caffarelli [1] (variational context), and then Caffarelli [2, 3, 4] (viscosity solutions context).

As a natural generalization of (1.1), we consider the following problem (see for instance the book [7])

$$\begin{cases} (-\Delta)^\alpha u = 0, & \text{in } \Omega \cap \{u > 0\}, \\ \lim_{t \rightarrow 0^+} \frac{u(x_0 + t\nu(x_0))}{t^\alpha} = \text{const.}, & \text{on } \Omega \cap \partial\{u > 0\}, \end{cases} \quad (1.2)$$

with u defined on the whole \mathbb{R}^n with prescribed values outside of Ω . This problem has been first investigated by Caffarelli, Roquejoffre and the third author in [5].

The non locality of the fractional Laplacian makes computations hard to handle directly on the equation. However by a result by Caffarelli and Silvestre [6], one can realize it as a boundary operator in one more dimension. More precisely, given $\alpha \in (0, 1)$ and a function $u \in H^\alpha(\mathbb{R}^n)$ we consider the minimizer g to

$$\min \left\{ \int_{\mathbb{R}_+^{n+1}} z^\beta |\nabla g|^2 dx dz : g|_{\partial\mathbb{R}_+^{n+1}} = u \right\} \quad (1.3)$$

with

$$\beta := 1 - 2\alpha \in (-1, 1).$$

The “extension” g solves the Dirichlet problem

$$\begin{cases} \operatorname{div}(z^\beta \nabla g) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ g = u & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases}$$

and $(-\Delta)^\alpha u$ is a Dirichlet to Neumann type operator for g . Precisely in [6] it is shown that

$$(-\Delta)^\alpha u = -d_\alpha \lim_{z \rightarrow 0^+} z^\beta \partial_z g,$$

where d_α is a positive constant depending only on n and α , and the equality holds in the distributional sense.

Due to the variational structure of the extension problem, one can consider the following functional, associated to (1.2),

$$J(g, B_1) = \int_{B_1} |z|^\beta |\nabla g|^2 dx dz + \mathcal{L}_{\mathbb{R}^n}(\{g > 0\} \cap \mathbb{R}^n \cap B_1).$$

The minimizers of J have been investigated in [5], where general properties (optimal regularity, nondegeneracy, classification of global solutions), corresponding to those proved in [1] for the classical Bernoulli problem (1.1), have been obtained. In [5], only a partial result concerning the regularity of the free boundary is obtained. The question of the regularity of the free boundary in the case $\alpha = 1/2$ was subsequently settled in a series of papers co-authored by the first and the second author of this note [13, 14, 15].

In this paper, in view of the previous discussion, we consider the following thin one-phase problem associated to the extension

$$\begin{cases} \operatorname{div}(|z|^\beta \nabla g) = 0, & \text{in } B_1^+(g) := B_1 \setminus \{(x, 0) : g(x, 0) = 0\}, \\ \frac{\partial g}{\partial t^\alpha} = 1, & \text{on } F(g) := \partial_{\mathbb{R}^n} \{x \in \mathcal{B}_1 : g(x, 0) > 0\} \cap \mathcal{B}_1, \end{cases} \quad (1.4)$$

where $\beta = 1 - 2\alpha$,

$$\frac{\partial g}{\partial t^\alpha}(x_0) := \lim_{t \rightarrow 0^+} \frac{g(x_0 + t\nu(x_0))}{t^\alpha}, \quad x_0 \in F(g) \quad (1.5)$$

and $\mathcal{B}_r \subset \mathbb{R}^n$ is the n -dimensional ball of radius r (centered at 0).

A special class of viscosity solutions to (1.4) (with the constant 1 replaced by a precise constant A depending on n and α) is provided by mini-

mizers of the functional J above.

We explain below the free boundary condition (1.5). In Section 2 we show that in the case $n = 1$, a particular 2-dimensional solution $U(t, z)$ to our free boundary problem is given by

$$U = \left(r^{1/2} \cos \frac{\theta}{2} \right)^{2\alpha}, \quad (1.6)$$

with r, θ the polar coordinates in the (t, z) plane. This function is simply the “extension” of $(t^+)^{\alpha}$ to the upper half-plane, reflected evenly across $z = 0$. By boundary Harnack estimate (see Theorem 2.14), any solution g to

$$\operatorname{div}(|z|^{\beta} \nabla g) = 0, \quad \text{in } \mathbb{R}^2 \setminus \{(t, 0) | t \leq 0\}$$

that vanishes on the negative t axis satisfies the following expansion near the origin

$$g = U(a + o(1)),$$

for some constant a . Then $\frac{\partial g}{\partial t^{\alpha}}(0) = a$ and the constant a can be thought as a “normal” derivative of g at the origin.

The 2-dimensional solution U describes also the general behavior of g near the free boundary $F(g)$. Indeed, in the n -dimensional case, if $0 \in F(g)$ and $F(g)$ is C^2 then the same expansion as above holds in the 2-dimensional plane perpendicular to $F(g)$ at the origin. We often denote the limit in (1.5) as $\partial g / \partial U$ and it represents the first coefficient of U in the expansion of g as above.

We now state our main result about the regularity of $F(g)$ under appropriate flatness assumptions (for all the relevant definitions see Section 2).

Theorem 1.1. *There exists a small constant $\bar{\epsilon} > 0$ depending on n and α , such that if g is a viscosity solution to (1.4) satisfying*

$$\{x \in \mathcal{B}_1 : x_n \leq -\bar{\epsilon}\} \subset \{x \in \mathcal{B}_1 : g(x, 0) = 0\} \subset \{x \in \mathcal{B}_1 : x_n \leq \bar{\epsilon}\}, \quad (1.7)$$

then $F(g)$ is $C^{1,\gamma}$ in $\mathcal{B}_{1/2}$, with $\gamma > 0$ depending on n and α .

The previous theorem has the following corollary.

Corollary 1.2. *There exists a universal constant $\bar{\epsilon} > 0$, such that if u is a viscosity solution to (1.2) in \mathcal{B}_1 satisfying*

$$\{x \in \mathcal{B}_1 : x_n \leq -\bar{\epsilon}\} \subset \{x \in \mathcal{B}_1 : u(x, 0) = 0\} \subset \{x \in \mathcal{B}_1 : x_n \leq \bar{\epsilon}\},$$

then $F(u)$ is $C^{1,\gamma}$ in $\mathcal{B}_{1/2}$.

The Theorem above extends the results in [13] to any power $0 < \alpha < 1$. We follow the strategy developed in [13]. Most of the proofs remain valid in this context as well, since they rely on basic facts such as Harnack Inequality, Boundary Harnack inequality, Comparison Principle and elementary properties of U .

The paper is organized as follows. In section 2 we introduce notation, definitions and preliminary results. In Section 3 we recall the notion of ϵ - domain variations and the corresponding linearized problem. Section 4 is devoted to Harnack inequality while Section 5 contains the proof of the main improvement of flatness theorem. In Section 6 the regularity of the linearized problem is investigated.

2. Preliminaries

In this Section we introduce notation, definitions, and preliminary results.

2.1. Notation

A point $X \in \mathbb{R}^{n+1}$ will be denoted by $X = (x, z) \in \mathbb{R}^n \times \mathbb{R}$. We will also use the notation $x = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1})$. A ball in \mathbb{R}^{n+1} with radius r and center X is denoted by $B_r(X)$ and for simplicity $B_r = B_r(0)$. Also we use \mathcal{B}_r to denote the n -dimensional ball $B_r \cap \{z = 0\}$.

Let $v(X)$ be a continuous non-negative function in B_1 . We associate to v the following sets:

$$B_1^+(v) := B_1 \setminus \{(x, 0) : v(x, 0) = 0\} \subset \mathbb{R}^{n+1};$$

$$\mathcal{B}_1^+(v) := B_1^+(v) \cap \mathcal{B}_1 \subset \mathbb{R}^n;$$

$$F(v) := \partial_{\mathbb{R}^n} \mathcal{B}_1^+(v) \cap \mathcal{B}_1 \subset \mathbb{R}^n.$$

Often subsets of \mathbb{R}^n are embedded in \mathbb{R}^{n+1} , as it will be clear from the context. $F(v)$ is called the free boundary of v .

We consider the free boundary problem,

$$\begin{cases} \operatorname{div}(|z|^\beta \nabla g) = 0, & \text{in } B_1^+(g), \\ \frac{\partial g}{\partial U} = 1, & \text{on } F(g), \end{cases} \quad (2.1)$$

where $\beta = 1 - 2\alpha$, $0 < \alpha < 1$

$$\frac{\partial g}{\partial U}(x_0) := \lim_{t \rightarrow 0^+} \frac{g(x_0 + t\nu(x_0), 0)}{t^\alpha}, \quad X_0 = (x_0, 0) \in F(g).$$

Here $\nu(x_0)$ denotes the unit normal to $F(g)$ at x_0 pointing toward $\mathcal{B}_1^+(g)$ and U is the function defined in (1.6).

2.2. The solution U

Recall that

$$U(t, z) = h^{2\alpha}, \quad h := r^{1/2} \cos \frac{\theta}{2}.$$

The function h is harmonic and it is easy to check that it satisfies

$$h_t = \frac{h}{2r}, \quad |\nabla h| = \frac{1}{2}r^{-1/2}, \quad \frac{h_z}{z} = \frac{1}{4rh}.$$

We obtain

$$\Delta U + \beta \frac{U_z}{z} = 2\alpha(2\alpha - 1)h^{2\alpha-2}(|\nabla h|^2 - h \frac{h_z}{z}) = 0,$$

and since U is C^2 in its positive set, it is a viscosity solution.

Clearly the $(n+1)$ dimensional function $U(X) := U(x_n, z)$ is a solution with the free boundary $F(U) = \{x_n = 0\}$. Notice that

$$\frac{U_n}{U} = \frac{U_t}{U} = \frac{\alpha}{r}.$$

2.3. Viscosity solutions

We now introduce the notion of viscosity solutions to (2.1). First we

need the following standard notion.

Definition 2.1. Given g, v continuous, we say that v touches g by below (resp. above) at $X_0 \in B_1$ if $g(X_0) = v(X_0)$, and

$$g(X) \geq v(X) \quad (\text{resp. } g(X) \leq v(X)) \quad \text{in a neighborhood } O \text{ of } X_0.$$

If this inequality is strict in $O \setminus \{X_0\}$, we say that v touches g strictly by below (resp. above).

Definition 2.2. We say that $v \in C(B_1)$ is a (strict) comparison subsolution to (2.1) if v is a non-negative function in B_1 which is even with respect to $\{z = 0\}$, v is C^2 in the set where it is positive and it satisfies

- (i) $\operatorname{div}(|z|^\beta \nabla v) \geq 0$ in $B_1 \setminus \{z = 0\}$;
- (ii) $F(v)$ is C^2 and if $x_0 \in F(v)$ we have

$$v(x, z) = aU((x - x_0) \cdot \nu(x_0), z) + o(|(x - x_0, z)|^\alpha), \quad \text{as } (x, z) \rightarrow (x_0, 0),$$

with

$$a \geq 1,$$

where $\nu(x_0)$ denotes the unit normal at x_0 to $F(v)$ pointing toward $\mathcal{B}_1^+(v)$;

- (iii) Either v satisfies (i) strictly or $a > 1$.

Similarly one can define a (strict) comparison supersolution.

Definition 2.3. We say that g is a viscosity solution to (2.1) if g is a continuous non-negative function in B_1 which satisfies

- (i) g is locally $C^{1,1}$ in $B_1^+(g)$, even with respect to $\{z = 0\}$ and solves (in the viscosity sense)

$$\operatorname{div}(|z|^\beta \nabla g) = 0 \quad \text{in } B_1 \setminus \{z = 0\};$$

- (ii) Any (strict) comparison subsolution (resp. supersolution) cannot touch g by below (resp. by above) at a point $X_0 = (x_0, 0) \in F(g)$.

Remark 2.4. Observe that the equation in (i) can be written in the following non-divergence form

$$\Delta g + \beta \frac{gz}{z} = 0.$$

This fact will be used throughout the paper.

Remark 2.5. We notice that in view of Lemma 2.1 in [16], g satisfies part (i) in Definition 2.3 if and only if g solves

$$\operatorname{div}(|z|^\beta \nabla g) = 0 \quad \text{in } B_1^+(g),$$

in the distributional sense. Equivalently, g is a local minimizer in $B_1^+(g)$ to the energy functional

$$\int |z|^\beta |\nabla g|^2 dX.$$

In view of this remark, we can apply the standard maximum/comparison principle to functions that satisfy part (i) of Definition 2.3.

Remark 2.6. We remark that if g is a viscosity solution to (2.1) in B_ρ , then

$$g_\rho(X) = \rho^{-\alpha} g(\rho X), \quad X \in B_1 \tag{2.2}$$

is a viscosity solution to (2.1) in B_1 .

We also introduce the notion of viscosity solutions for the fractional Laplace free boundary problem (1.2) in the Introduction.

Definition 2.7. We say that u is a viscosity solution to (1.2) if u is a non-negative continuous function in Ω and it satisfies

- (i) $(-\Delta)^\alpha u = 0$ in Ω ;
- (ii) at any point $x_0 \in F(u) \cap \Omega$ that admits a tangent ball from either the positive set $\{u > 0\}$ or from the zero set $\{u = 0\}$ we have

$$u(x) = ((x - x_0)^\alpha \cdot \nu(x_0))^+ + o(|(x - x_0)|^\alpha),$$

where $\nu(x_0)$ denotes the unit normal at x_0 to $F(u)$ pointing toward $B_1^+(u)$.

2.4. Expansion at regular points

In order to explain better the free boundary conditions in the definitions above we recall Lemma 7.5 from [14] about the expansion of solutions g to the equation

$$\operatorname{div}(|z|^\beta \nabla g) = 0 \quad \text{in } B_1^+(g), \quad (2.3)$$

near points on $F(g)$ that have a tangent ball either from the positive side of g or from the zero-side. The proof in [14] is for the case $\alpha = 1/2$, however it uses only boundary Harnack inequality (see Theorem 2.14) and it works identically for any $\alpha \in (0, 1)$.

Proposition 2.8. *Let $g \in C^\alpha(B_1)$, $g \geq 0$, satisfy (2.3). If*

$$0 \in F(g), \quad \mathcal{B}_{1/2}(1/2e_n) \subset B_1^+(g),$$

then

$$g = aU + o(|X|^\alpha), \quad \text{for some } a > 0.$$

The same conclusion holds for some $a \geq 0$ if

$$\mathcal{B}_{1/2}(-1/2e_n) \subset \{g = 0\}.$$

Since viscosity solutions have the optimal C^α regularity (see [5], [14]), a consequence of the proposition above is the following

Corollary 2.9. *The function u is a viscosity solution to (1.2) if and only if its extension to \mathbb{R}^{n+1} (reflected evenly across $z = 0$) is a viscosity solution to (2.1).*

2.5. Flatness assumption

Theorem 1.1 is stated under the flatness assumption of the free boundary $F(g)$. As in Lemma 7.9 in [14] this implies closeness between the function g and the one-dimensional solution U . Precisely we have

Lemma 2.10. *Assume g solves (2.1). Given any $\epsilon > 0$ there exist $\bar{\epsilon} > 0$ and $\delta > 0$ depending on ϵ such that if*

$$\{x \in \mathcal{B}_1 : x_n \leq -\bar{\epsilon}\} \subset \{x \in \mathcal{B}_1 : g(x, 0) = 0\} \subset \{x \in \mathcal{B}_1 : x_n \leq \bar{\epsilon}\},$$

then the rescaling g_δ (see (2.2)) satisfies

$$U(X - \epsilon e_n) \leq g_\delta(X) \leq U(X + \epsilon e_n) \quad \text{in } B_1.$$

In view of Lemma (2.10) we may assume from now on that

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in } B_1,$$

for some $\epsilon > 0$.

2.6. Comparison principle

We state the comparison principle for problem (2.1), which in view of Remark 2.5 holds in this setting as well. Its proof is standard and can be found in [13]. As an immediate consequence one obtains Corollary 2.12 which is the formulation of the Comparison Principle used in this paper.

Lemma 2.11 (Comparison Principle). *Let $g, v_t \in C(\overline{B}_1)$ be respectively a solution and a family of subsolutions to (2.1), $t \in [0, 1]$. Assume that*

- i. $v_0 \leq g$, in \overline{B}_1 ;*
- ii. $v_t \leq g$ on ∂B_1 for all $t \in [0, 1]$;*
- iii. $v_t < g$ on $\mathcal{F}(v_t)$ which is the boundary in ∂B_1 of the set $\partial \mathcal{B}_1^+(v_t) \cap \partial B_1$, for all $t \in [0, 1]$;*
- iv. $v_t(x)$ is continuous in $(x, t) \in \overline{B}_1 \times [0, 1]$ and $\overline{\mathcal{B}_1^+(v_t)}$ is continuous in the Hausdorff metric.*

Then

$$v_t \leq g \quad \text{in } \overline{B}_1, \text{ for all } t \in [0, 1].$$

Corollary 2.12. *Let g be a solution to (2.1) and let v be a subsolution to (2.1) in B_2 which is strictly monotone increasing in the e_n -direction in $B_2^+(v)$. Call*

$$v_t(X) := v(X + te_n), \quad X \in B_1.$$

Assume that for $-1 \leq t_0 < t_1 \leq 1$

$$v_{t_0} \leq g, \quad \text{in } \overline{B}_1,$$

and

$$v_{t_1} \leq g \quad \text{on } \partial B_1, \quad v_{t_1} < g \quad \text{on } \mathcal{F}(v_{t_1}).$$

Then

$$v_{t_1} \leq g \quad \text{in } \overline{B_1}.$$

2.7. Harnack inequalities for A_2 weights

The weight involved in our problem, i.e. $w(z) = |z|^\beta$ where $\beta = 1 - 2\alpha$ with $\alpha \in (0, 1)$ belongs to the well-known class of A_2 functions as defined by Muchenhaupt [12]. Equations in divergence form involving such weights have been studied in a series of papers by Fabes et al in [8, 9, 10]. In the following, we review the results needed for our purposes.

Theorem 2.13 (Harnack inequality [8]). *Let $u \geq 0$ be a solution of*

$$\operatorname{div}(|z|^\beta \nabla u) = 0 \quad \text{in } B_1 \subset \mathbb{R}^n.$$

Then,

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u$$

for some constant C depending only on n and β .

Theorem 2.14 (Boundary Harnack principle [9]). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $0 \in \partial\Omega$. Let $u > 0$ and v be solutions of*

$$\operatorname{div}(|z|^\beta \nabla u) = \operatorname{div}(|z|^\beta \nabla v) = 0 \quad \text{in } B_1 \setminus (\Omega \times \{0\}),$$

that vanish continuously on $B_1 \cap (\Omega \times \{0\})$. Then,

$$\left[\frac{v}{u} \right]_{C^\gamma(B_{1/2})} \leq C$$

for some constants C, γ depending on n and the Lipschitz constant of $\partial\Omega$.

3. The Function \tilde{g} and the Linearized Problem

In this section we recall the notion of ϵ -domain variations of a viscosity solution to (2.1). We also introduce the linearized problem associated to (2.1).

3.1. The function \tilde{g}

Let $\epsilon > 0$ and let g be a continuous non-negative function in \overline{B}_ρ . Let

$$P := \{X \in \mathbb{R}^{n+1} : x_n \leq 0, z = 0\}, \quad L := \{X \in \mathbb{R}^{n+1} : x_n = 0, z = 0\}.$$

To each $X \in \mathbb{R}^{n+1} \setminus P$ we associate $\tilde{g}_\epsilon(X) \subset \mathbb{R}$ such that

$$U(X) = g(X - \epsilon w e_n), \quad \forall w \in \tilde{g}_\epsilon(X). \quad (3.1)$$

We call \tilde{g}_ϵ the ϵ -domain variation associated to g . By abuse of notation, from now on we write $\tilde{g}_\epsilon(X)$ to denote any of the values in this set. As noted in [13], if g satisfies

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in } B_\rho, \quad (3.2)$$

for all $\epsilon > 0$ we can associate to g a possibly multi-valued function \tilde{g}_ϵ defined at least on $B_{\rho-\epsilon} \setminus P$ and taking values in $[-1, 1]$ which satisfies

$$U(X) = g(X - \epsilon \tilde{g}_\epsilon(X) e_n). \quad (3.3)$$

Moreover if g is strictly monotone in the e_n -direction in $B_\rho^+(g)$, then \tilde{g}_ϵ is single-valued.

The following comparison principle is proved in [13] in the case $\alpha = 1/2$. The proof remains still valid as it only involves Corollary 2.12 and some elementary considerations following from the definition of \tilde{g} .

Lemma 3.1. *Let g, v be respectively a solution and a subsolution to (2.1) in B_2 , with v strictly increasing in the e_n -direction in $B_2^+(v)$. Assume that g satisfies the flatness assumption (3.2) in B_2 for $\epsilon > 0$ small and that \tilde{v}_ϵ is defined in $B_{2-\epsilon} \setminus P$ and satisfies*

$$|\tilde{v}_\epsilon| \leq C.$$

If,

$$\tilde{v}_\epsilon + c \leq \tilde{g}_\epsilon \quad \text{in } (B_{3/2} \setminus \overline{B_{1/2}}) \setminus P, \quad (3.4)$$

then

$$\tilde{v}_\epsilon + c \leq \tilde{g}_\epsilon \quad \text{in } B_{3/2} \setminus P. \quad (3.5)$$

Finally, we recall the following useful fact. Given $\epsilon > 0$ small and a Lipschitz function $\tilde{\varphi}$ defined on $B_\rho(\bar{X})$, with values in $[-1, 1]$, there exists a unique function φ_ϵ defined at least on $B_{\rho-\epsilon}(\bar{X})$ such that

$$U(X) = \varphi_\epsilon(X - \epsilon\tilde{\varphi}(X)e_n), \quad X \in B_\rho(\bar{X}). \quad (3.6)$$

Moreover such function φ_ϵ is increasing in the e_n -direction. If g satisfies the flatness assumption (3.2) in B_1 and $\tilde{\varphi}$ is as above then (say $\rho, \epsilon < 1/4$, $\bar{X} \in B_{1/2}$),

$$\tilde{\varphi} \leq \tilde{g}_\epsilon \quad \text{in } B_\rho(\bar{X}) \setminus P \Rightarrow \varphi_\epsilon \leq g \quad \text{in } B_{\rho-\epsilon}(\bar{X}). \quad (3.7)$$

3.2. The linearized problem

We introduce here the linearized problem associated to (2.1). Here and later U_n denotes the x_n -derivative of the function U defined in (1.6).

Given $w \in C(B_1)$ and $X_0 = (x'_0, 0, 0) \in B_1 \cap L$, we call

$$|\nabla_r w|(X_0) := \lim_{(x_n, z) \rightarrow (0, 0)} \frac{w(x'_0, x_n, z) - w(x'_0, 0, 0)}{r}, \quad r^2 = x_n^2 + z^2.$$

Once the change of unknowns (3.1) has been done, the linearized problem associated to (2.1) is

$$\begin{cases} \operatorname{div}(|z|^\beta \nabla(U_n w)) = 0, & \text{in } B_1 \setminus P, \\ |\nabla_r w| = 0, & \text{on } B_1 \cap L. \end{cases} \quad (3.8)$$

Our notion of viscosity solution for this problem is below.

Definition 3.2. We say that w is a solution to (3.8) if $w \in C_{loc}^{1,1}(B_1 \setminus P)$, w is even with respect to $\{z = 0\}$ and it satisfies (in the viscosity sense)

i. $\operatorname{div}(|z|^\beta \nabla(U_n w)) = 0$ in $B_1 \setminus \{z = 0\}$;

ii. Let ϕ be continuous around $X_0 = (x'_0, 0, 0) \in B_1 \cap L$ and satisfy

$$\phi(X) = \phi(X_0) + a(X_0) \cdot (x' - x'_0) + b(X_0)r + O(|x' - x'_0|^2 + r^{1+\gamma}),$$

for some $\gamma > 0$ and

$$b(X_0) \neq 0.$$

If $b(X_0) > 0$ then ϕ cannot touch w by below at X_0 , and if $b(X_0) < 0$ then ϕ cannot touch w by above at X_0 .

In Section 8, we will investigate the regularity of solutions to (3.8) and obtain the following corollary, which we use in the proof of the improvement of flatness.

Corollary 3.3. *There exists a universal constant $\rho > 0$ such that if w solves (3.8) and $|w| \leq 1$ in B_1 , $w(0) = 0$ then*

$$a_0 \cdot x' - \frac{1}{8}\rho \leq w(X) \leq a_0 \cdot x' + \frac{1}{8}\rho \quad \text{in } B_{2\rho}$$

for some vector $a_0 \in \mathbb{R}^{n-1}$.

4. Harnack Inequality

This section is devoted to a Harnack type inequality for solutions to our free boundary problem (2.1).

Theorem 4.1 (Harnack inequality). *There exists $\bar{\epsilon} > 0$ such that if g solves (2.1) and it satisfies*

$$U(X + \epsilon a_0 e_n) \leq g(X) \leq U(X + \epsilon b_0 e_n) \quad \text{in } B_\rho(X^*), \quad (4.1)$$

with

$$\epsilon(b_0 - a_0) \leq \bar{\epsilon}\rho,$$

then

$$U(X + \epsilon a_1 e_n) \leq g(X) \leq U(X + \epsilon b_1 e_n) \quad \text{in } B_{\eta\rho}(X^*), \quad (4.2)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad (b_1 - a_1) \leq (1 - \eta)(b_0 - a_0),$$

for a small universal constant η .

Let g be a solution to (2.1) which satisfies

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in } B_1.$$

Let A_ϵ be the following set

$$A_\epsilon := \{(X, \tilde{g}_\epsilon(X)) : X \in B_{1-\epsilon} \setminus P\} \subset \mathbb{R}^{n+1} \times [a_0, b_0]. \quad (4.3)$$

Since \tilde{g}_ϵ may be multivalued, we mean that given X all pairs $(X, \tilde{g}_\epsilon(X))$ belong to A_ϵ for all possible values of $\tilde{g}_\epsilon(X)$. An iterative argument (see [13]) gives the following corollary of Theorem 4.1.

Corollary 4.2. *If*

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in } B_1,$$

with $\epsilon \leq \bar{\epsilon}/2$, given $m_0 > 0$ such that

$$2\epsilon(1 - \eta)^{m_0} \eta^{-m_0} \leq \bar{\epsilon},$$

then the set $A_\epsilon \cap (B_{1/2} \times [-1, 1])$ is above the graph of a function $y = a_\epsilon(X)$ and it is below the graph of a function $y = b_\epsilon(X)$ with

$$b_\epsilon - a_\epsilon \leq 2(1 - \eta)^{m_0 - 1},$$

and a_ϵ, b_ϵ having a modulus of continuity bounded by the Hölder function αt^β for α, β depending only on η .

The proof of Harnack inequality follows as in the case $\alpha = 1/2$. The key ingredient is the lemma below.

Lemma 4.3. *There exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon \leq \bar{\epsilon}$ if g is a solution to (2.1) in B_1 such that*

$$g(X) \geq U(X) \quad \text{in } B_{1/2}, \quad (4.4)$$

and at $\bar{X} \in B_{1/8}(\frac{1}{4}e_n)$

$$g(\bar{X}) \geq U(\bar{X} + \epsilon e_n), \quad (4.5)$$

then

$$g(X) \geq U(X + \tau \epsilon e_n) \quad \text{in } B_\delta, \quad (4.6)$$

for universal constants τ, δ . Similarly, if

$$g(X) \leq U(X) \quad \text{in } B_{1/2},$$

and

$$g(\bar{X}) \leq U(\bar{X} - \epsilon e_n),$$

then

$$g(X) \leq U(X - \tau \epsilon e_n) \quad \text{in } B_\delta.$$

A preliminary basic result is the following.

Lemma 4.4. *Let $g \geq 0$ be $C_{loc}^{1,1}$ in $B_2^+(g)$ and solve (2.3) in $B_2 \setminus \{z = 0\}$ and let $\bar{X} = \frac{3}{2}e_n$. Assume that*

$$g \geq U \quad \text{in } B_2, \quad g(\bar{X}) - U(\bar{X}) \geq \delta_0$$

for some $\delta_0 > 0$, then

$$g \geq (1 + c\delta_0)U \quad \text{in } B_1 \quad (4.7)$$

for a small universal constant c .

In particular, for any $0 < \epsilon < 2$

$$U(X + \epsilon e_n) \geq (1 + c\epsilon)U(X) \quad \text{in } B_1, \quad (4.8)$$

with c small universal.

Its proof can be found in [13] (Lemma 5.1.) It remains valid since Maximum principle, Harnack Inequality, Boundary Harnack Inequality, and monotonicity of U in the e_n -direction, which are all the ingredients of the proof, are still valid. Harmonic functions in that proof are replaced by solutions to

$$\operatorname{div}(|z|^\beta \nabla g) = 0. \quad (4.9)$$

The main tool in the proof of Lemma 4.3 will be the following family of radial subsolutions. Let $R > 0$ and denote by

$$V_R(t, z) = U(t, z)\left((n-1)\frac{t}{R} + 1\right).$$

Then set

$$v_R(X) = V_R\left(R - \sqrt{|x'|^2 + (x_n - R)^2}, z\right), \quad (4.10)$$

that is we obtain the $n + 1$ -dimensional function v_R by rotating the 2-dimensional function V_R around $(0, R, z)$.

Proposition 4.5. *If R is large enough, the function $v_R(X)$ is a comparison subsolution to (2.1) in B_2 which is strictly monotone increasing in the e_n -direction in $B_2^+(v_R)$. Moreover, there exists a function \tilde{v}_R such that*

$$U(X) = v_R(X - \tilde{v}_R(X)e_n) \quad \text{in } B_1 \setminus P, \quad (4.11)$$

and

$$|\tilde{v}_R(X) - \gamma_R(X)| \leq \frac{C}{R^2}|X|^2, \quad \gamma_R(X) = -\frac{|x'|^2}{2R} + 2(n-1)\frac{x_n r}{R}, \quad (4.12)$$

with $r = \sqrt{x_n^2 + z^2}$ and C universal.

Proof. We divide the proof of this proposition in two steps.

Step 1. In this step we show that v_R is a comparison subsolution in B_2 which is monotone in the e_n -direction.

First we see that v_R is a strict subsolution to (4.9) in $B_2 \setminus \{z = 0\}$. One can easily compute that on such set,

$$\begin{aligned} \Delta v_R(X) + \beta \frac{(v_R)z(X)}{z} \\ = \Delta_{t,z} V_R(R - \rho, z) - \frac{n-1}{\rho} \partial_t V_R(R - \rho, z) + \beta \frac{\partial_z V_R(R - \rho, z)}{z}, \end{aligned}$$

where for simplicity we call

$$\rho := \sqrt{|x'|^2 + (x_n - R)^2}.$$

Also for (t, z) outside the set $\{(t, 0) : t \leq 0\}$

$$\begin{aligned}
& \Delta_{t,z}V_R(t, z) + \beta \frac{(V_R)_z(t, z)}{z} \\
&= (\partial_{tt} + \partial_{zz})V_R(t, z) + \beta \frac{(V_R)_z(t, z)}{z} \\
&= \frac{2(n-1)}{R} \partial_t U(t, z) + (1 + (n-1) \frac{t}{R}) (\Delta_{t,z}U(t, z) + \beta \frac{U_z(t, z)}{z}) \\
&= \frac{2(n-1)}{R} \partial_t U(t, z),
\end{aligned}$$

and

$$\partial_t V_R(t, z) = (1 + (n-1) \frac{t}{R}) \partial_t U(t, z) + \frac{n-1}{R} U(t, z). \quad (4.13)$$

Thus to show that v_R solves (4.9) in $B_2 \setminus \{z = 0\}$ we need to prove that in such set

$$\frac{2(n-1)}{R} \partial_t U - \frac{n-1}{\rho} [(1 + (n-1) \frac{R-\rho}{R}) \partial_t U + \frac{n-1}{R} U] \geq 0,$$

where U and $\partial_t U$ are evaluated at $(R - \rho, z)$.

Set $t = R - \rho$, then straightforward computations reduce the inequality above to

$$(n-1)[2(R-t) - R - (n-1)^2 t] \partial_t U(t, z) - (n-1)^2 U(t, z) \geq 0.$$

Using that $\partial_t U(t, z) = \alpha U(t, z)/r$ with $r^2 = t^2 + z^2$, this inequality becomes

$$R \geq 2t + (n-1)^2 t + \frac{(n-1)}{\alpha} r.$$

This last inequality is easily satisfied for R large enough, since $t, r \leq 3$.

Now we prove that v_R satisfies the free boundary condition in Definition 2.2. First observe that

$$F(v_R) = \partial \mathcal{B}_R(Re_n, 0) \cap \mathcal{B}_2,$$

and hence it is smooth. By the radial symmetry it is enough to show that the free boundary condition is satisfied at $0 \in F(v_R)$ that is

$$v_R(x, z) = aU(x_n, z) + o(|(x, z)|^\alpha), \quad \text{as } (x, z) \rightarrow (0, 0), \quad (4.14)$$

with $a \geq 1$.

First notice since U is Hölder continuous with exponent α , it follows from the formula for V_R that

$$|V_R(t, z) - V_R(t_0, z)| \leq C|t - t_0|^\alpha \quad \text{for } |t - t_0| \leq 1.$$

Thus for $(x, z) \in B_s$, s small

$$|v_R(x, z) - V_R(x_n, z)| = |V_R(R - \rho, z) - V_R(x_n, z)| \leq C|R - \rho - x_n|^\alpha \leq Cs^{2\alpha},$$

where we have used that (recall that $\rho := \sqrt{|x'|^2 + (x_n - R)^2}$)

$$R - \rho - x_n = -\frac{|x'|^2}{R - x_n + \rho}. \quad (4.15)$$

It follows that for $(x, z) \in B_s$

$$\begin{aligned} |v_R(x, z) - U(x_n, z)| &\leq |v_R(x, z) - V_R(x_n, z)| + |V_R(x_n, z) - U(x_n, z)| \\ &\leq Cs^{2\alpha} + |V_R(x_n, z) - U(x_n, z)|. \end{aligned}$$

Thus from the formula for V_R

$$|v_R(x, z) - U(x_n, z)| \leq Cs^{2\alpha} + (n-1)\frac{|x_n|}{R}U(x_n, z) \leq C's^{2\alpha}, \quad (x, z) \in B_s$$

which gives the desired expansion (4.14) with $a = 1$.

Now, we show that v_R is strictly monotone increasing in the e_n -direction in $B_2^+(v_R)$. Outside of its zero plate,

$$\partial_{x_n} v_R(x) = -\frac{x_n - R}{\rho} \partial_t V_R(R - \rho, z).$$

Thus we only need to show that $V_R(t, z)$ is strictly monotone increasing in t outside $\{(t, 0) : t \leq 0\}$. This follows immediately from (4.13) and the formula for U .

Step 2. In this step we state the existence of \tilde{v}_R satisfying (4.11) and (4.12). Since we have a precise formula for v_R in terms of U , this is only a matter of straightforward (though tedious) computations which are carried on in [13]. Also, one needs to use Boundary Harnack inequality for

U and its derivatives, the fact that U is homogeneous of degree α and that the ratio $U_t/U = \alpha/r$ (with $\alpha = 1/2$ in [13].) All these are still valid in this context. \square

Then, one easily obtain the following Corollary.

Corollary 4.6. *There exist δ, c_0, C_0, C_1 universal constants, such that*

$$v_R(X + \frac{c_0}{R}e_n) \leq (1 + \frac{C_0}{R})U(X), \quad \text{in } \overline{B_1} \setminus B_{1/4}, \quad (4.16)$$

with strict inequality on $F(v_R(X + \frac{c_0}{R}e_n)) \cap \overline{B_1} \setminus B_{1/4}$,

$$v_R(X + \frac{c_0}{R}e_n) \geq U(X + \frac{c_0}{2R}e_n), \quad \text{in } B_\delta, \quad (4.17)$$

$$v_R(X - \frac{C_1}{R}e_n) \leq U(X), \quad \text{in } \overline{B_1}. \quad (4.18)$$

We are now ready to present the proof of Lemma 4.3.

Proof of Lemma 4.3. We prove the first statement. In view of (4.5)

$$g(\overline{X}) - U(\overline{X}) \geq U(\overline{X} + \epsilon e_n) - U(\overline{X}) = \partial_t U(\overline{X} + \lambda e_n)\epsilon \geq c\epsilon, \quad \lambda \in (0, \epsilon).$$

From Lemma 4.4 we then get

$$g(X) \geq (1 + c'\epsilon)U(X) \quad \text{in } \overline{B_{1/4}}. \quad (4.19)$$

Now let

$$R = \frac{C_0}{c'\epsilon},$$

where from now on the C_i, c_i are the constants in Corollary 4.6. Then, for ϵ small enough v_R is a subsolution to (2.1) in B_2 which is monotone increasing in the e_n - direction and it also satisfies (4.16)–(4.18). We now wish to apply the Comparison Principle as stated in Corollary 2.12. Let

$$v_R^t(X) = v_R(X + te_n), \quad X \in B_1,$$

then according to (4.18),

$$v_R^{t_0} \leq U \leq g \quad \text{in } \overline{B_{1/4}}, \quad \text{with } t_0 = -C_1/R.$$

Moreover, from (4.16) and (4.19) we get that for our choice of R ,

$$v_R^{t_1} \leq (1 + c'\epsilon)U \leq g \quad \text{on } \partial B_{1/4}, \text{ with } t_1 = c_0/R,$$

with strict inequality on $F(v_R^{t_1}) \cap \partial B_{1/4}$. In particular

$$g > 0 \quad \text{on } \mathcal{F}(v_R^{t_1}) \text{ in } \partial B_{1/4}.$$

Thus we can apply Corollary 2.12 in the ball $B_{1/4}$ to obtain

$$v_R^{t_1} \leq g \quad \text{in } B_{1/4}.$$

From (4.17) we have that

$$U(X + \frac{c_1}{R}e_n) \leq v_R^{t_1}(X) \leq g(X) \quad \text{on } B_\delta$$

which is the desired claim (4.6) with $\tau = \frac{c_1 c'}{C_0}$. □

5. Improvement of Flatness

In this section we state the improvement of flatness property for solutions to (2.1) and we provide its proof. Our main Theorem 1.1 follows from the Theorem below and Lemma 2.10.

Theorem 5.1 (Improvement of flatness). *There exist $\bar{\epsilon} > 0$ and $\rho > 0$ universal constants such that for all $0 < \epsilon \leq \bar{\epsilon}$ if g solves (2.1) with $0 \in F(g)$ and it satisfies*

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in } B_1, \tag{5.1}$$

then

$$U(x \cdot \nu - \frac{\epsilon}{2}\rho, z) \leq g(X) \leq U(x \cdot \nu + \frac{\epsilon}{2}\rho, z) \quad \text{in } B_\rho, \tag{5.2}$$

for some direction $\nu \in \mathbb{R}^n, |\nu| = 1$.

The proof of Theorem 5.1 is divided into the next four lemmas.

The following Lemma is contained in [13] (Lemma 7.2) and its proof remains unchanged, since it does not depend on the particular equation

satisfied by g but only on elementary considerations related to the definition of \tilde{g}_ϵ .

Lemma 5.2. *Let g be a solution to (2.1) with $0 \in F(g)$ and satisfying (5.1). Assume that the corresponding \tilde{g}_ϵ satisfies*

$$a_0 \cdot x' - \frac{1}{4}\rho \leq \tilde{g}_\epsilon(X) \leq a_0 \cdot x' + \frac{1}{4}\rho \quad \text{in } B_{2\rho} \setminus P, \quad (5.3)$$

for some $a_0 \in \mathbb{R}^{n-1}$. Then if $\epsilon \leq \bar{\epsilon}(a_0, \rho)$ g satisfies (5.2) in B_ρ .

The next lemma follows immediately from the Corollary 4.2 to Harnack inequality.

Lemma 5.3. *Let $\epsilon_k \rightarrow 0$ and let g_k be a sequence of solutions to (2.1) with $0 \in F(g_k)$ satisfying*

$$U(X - \epsilon_k e_n) \leq g_k(X) \leq U(X + \epsilon_k e_n) \quad \text{in } B_1. \quad (5.4)$$

Denote by \tilde{g}_k the ϵ_k -domain variation of g_k . Then the sequence of sets

$$A_k := \{(X, \tilde{g}_k(X)) : X \in B_{1-\epsilon_k} \setminus P\},$$

has a subsequence that converge uniformly (in Hausdorff distance) in $B_{1/2} \setminus P$ to the graph

$$A_\infty := \{(X, \tilde{g}_\infty(X)) : X \in B_{1/2} \setminus P\},$$

where \tilde{g}_∞ is a Holder continuous function.

From here on \tilde{g}_∞ will denote the function from Lemma 5.3.

Lemma 5.4. *The limiting function satisfies $\tilde{g}_\infty \in C_{loc}^{1,1}(B_{1/2} \setminus P)$.*

Proof. We fix a point $Y \in B_{1/2} \setminus P$, and let δ be the distance from Y to L . It suffices to show that the functions \tilde{g}_ϵ are uniformly $C^{1,1}$ in $B_{\delta/8}(Y)$. Indeed, since $g_\epsilon - U$ is an even function that solves the extension problem in $B_{\delta/2}(Y)$ we find

$$\|g_\epsilon - U\|_{C^{1,1}(B_{\delta/4}(Y))} \leq C \|g_\epsilon - U\|_{L^\infty(B_{\delta/2}(Y))} \leq C\epsilon,$$

and, by implicit function theorem it follows that

$$\|\tilde{g}_\epsilon\|_{C^{1,1}(B_{\delta/8})(Y)} \leq C.$$

Here the constants above depend on Y and δ as well. \square

Lemma 5.5. *The function \tilde{g}_∞ satisfies the linearized problem (3.8) in $B_{1/2}$.*

Proof. We start by showing that $U_n \tilde{g}_\infty$ satisfies (4.9) in $B_{1/2} \setminus \{z = 0\}$.

Let $\tilde{\varphi}$ be a C^2 function which touches \tilde{g}_∞ strictly by below at $X_0 = (x_0, z_0) \in B_{1/2} \setminus \{z = 0\}$. We need to show that

$$\Delta(U_n \tilde{\varphi})(X_0) + \beta \frac{(U_n \tilde{\varphi})_z(X_0)}{z_0} \leq 0. \quad (5.5)$$

Since by Lemma 5.3, the sequence A_k converges uniformly to A_∞ in $B_{1/2} \setminus P$ we conclude that there exist a sequence of constants $c_k \rightarrow 0$ and a sequence of points $X_k \in B_{1/2} \setminus \{z = 0\}$, $X_k \rightarrow X_0$ such that $\tilde{\varphi}_k := \tilde{\varphi} + c_k$ touches \tilde{g}_k by below at X_k for all k large enough.

Define the function φ_k by the following identity

$$\varphi_k(X - \epsilon_k \tilde{\varphi}_k(X) e_n) = U(X). \quad (5.6)$$

Then according to (3.7) φ_k touches g_k by below at $Y_k = X_k - \epsilon_k \tilde{\varphi}_k(X_k) e_n \in B_1 \setminus \{z = 0\}$, for k large enough. Thus, since g_k satisfies (4.9) in $B_1 \setminus \{z = 0\}$ it follows that

$$\Delta \varphi_k(Y_k) + \beta \frac{(\varphi_k)_{n+1}(Y_k)}{z_k} \leq 0, \quad (5.7)$$

with z_k denoting the $(n+1)$ -coordinate of X_k .

Let us compute $\Delta \varphi_k(Y_k)$ and $(\varphi_k)_{n+1}(Y_k)$. Since $\tilde{\varphi}$ is smooth, for any Y in a neighborhood of Y_k we can find a unique $X = X(Y)$ such that

$$Y = X - \epsilon_k \tilde{\varphi}_k(X) e_n. \quad (5.8)$$

Thus (5.6) reads

$$\varphi_k(Y) = U(X(Y)),$$

with $Y_i = X_i$ if $i \neq n$ and

$$\frac{\partial X_j}{\partial Y_i} = \delta_{ij}, \quad \text{when } j \neq n.$$

Using these identities we can compute that

$$\begin{aligned} \Delta \varphi_k(Y) &= U_n(X) \Delta X_n(Y) + \sum_{j \neq n} (U_{jj}(X) + 2U_{jn}(X)) \frac{\partial X_n}{\partial Y_j} \\ &\quad + U_{nn}(X) |\nabla X_n|^2(Y). \end{aligned} \quad (5.9)$$

From (5.8) we have that

$$D_X Y = I - \epsilon_k D_X (\tilde{\varphi}_k e_n).$$

Thus, since $\tilde{\varphi}_k = \tilde{\varphi} + c_k$

$$D_Y X = I + \epsilon_k D_X (\tilde{\varphi} e_n) + O(\epsilon_k^2),$$

with a constant depending only on the C^2 -norm of $\tilde{\varphi}$.

It follows that

$$\frac{\partial X_n}{\partial Y_j} = \delta_{jn} + \epsilon_k \partial_j \tilde{\varphi}(X) + O(\epsilon_k^2). \quad (5.10)$$

Hence

$$|\nabla X_n|^2(Y) = 1 + 2\epsilon_k \partial_n \tilde{\varphi}(X) + O(\epsilon_k^2), \quad (5.11)$$

and also,

$$\frac{\partial^2 X_n}{\partial Y_j^2} = \epsilon_k \sum_i \partial_{ji} \tilde{\varphi} \frac{\partial X_i}{\partial Y_j} + O(\epsilon_k^2) = \epsilon_k \sum_{i \neq n} \partial_{ji} \tilde{\varphi} \delta_{ij} + \epsilon_k \partial_{jn} \tilde{\varphi} \frac{\partial X_n}{\partial Y_j} + O(\epsilon_k^2),$$

from which we obtain that

$$\Delta X_n = \epsilon_k \Delta \tilde{\varphi} + O(\epsilon_k^2). \quad (5.12)$$

Combining (5.9) with (5.11) and (5.12) we get that

$$\Delta \varphi_k(Y) = \Delta U(X) + \epsilon_k U_n \Delta \tilde{\varphi} + 2\epsilon_k \nabla \tilde{\varphi} \cdot \nabla U_n + O(\epsilon_k^2)(U_n(X) + U_{nn}(X)).$$

From the computations above it also follows that,

$$\begin{aligned} (\varphi_k)_{n+1}(Y) &= U_n(X) \frac{\partial X_n}{\partial Y_{n+1}} + U_z(X) \frac{\partial X_{n+1}}{\partial Y_{n+1}} \\ &= U_n(X)(\epsilon_k \partial_{n+1} \tilde{\varphi}(X) + O(\epsilon_k^2)) + U_z(X). \end{aligned}$$

Using (5.7) together with the fact that U solves (4.9) at X_k we conclude that

$$0 \geq \Delta(U_n \tilde{\varphi})(X_k) + \beta \frac{(U_n \tilde{\varphi})z(X_k)}{z_k} + O(\epsilon_k)(U_n(X_k) + \beta \frac{U_n(X_k)}{z_k} + U_{nn}(X_k)).$$

The desired inequality (5.5) follows by letting $k \rightarrow +\infty$.

Next we need to show that

$$|\nabla_r \tilde{g}_\infty|(X_0) = 0, \quad X_0 = (x'_0, 0, 0) \in B_{1/2} \cap L,$$

in the viscosity sense of Definition 3.2. The proof is the same as in the case $\alpha = 1/2$, once the properties of the function v_R defined in Proposition 4.5 have been established. For convenience of the reader, we present the details.

Assume by contradiction that there exists a function ϕ which touches \tilde{g}_∞ by below at $X_0 = (x'_0, 0, 0) \in B_{1/2} \cap L$ and such that

$$\phi(X) = \phi(X_0) + a(X_0) \cdot (x' - x'_0) + b(X_0)r + O(|x' - x'_0|^2 + r^{1+\gamma}),$$

for some $\gamma > 0$, with

$$b(X_0) > 0.$$

Then we can find constants α, δ, \bar{r} and a point $Y' = (y'_0, 0, 0) \in B_2$ depending on ϕ such that the polynomial

$$q(X) = \phi(X_0) - \frac{\alpha}{2}|x' - y'_0|^2 + 2\alpha(n-1)x_n r$$

touches ϕ by below at X_0 in a tubular neighborhood $N_{\bar{r}} = \{|x' - x'_0| \leq \bar{r}, r \leq \bar{r}\}$ of X_0 , with

$$\phi - q \geq \delta > 0, \quad \text{on } N_{\bar{r}} \setminus N_{\bar{r}/2}.$$

This implies that

$$\tilde{g}_\infty - q \geq \delta > 0, \quad \text{on } N_{\bar{r}} \setminus N_{\bar{r}/2}, \quad (5.13)$$

and

$$\tilde{g}_\infty(X_0) - q(X_0) = 0. \quad (5.14)$$

In particular,

$$|\tilde{g}_\infty(X_k) - q(X_k)| \rightarrow 0, \quad X_k \in N_{\bar{r}} \setminus P, X_k \rightarrow X_0. \quad (5.15)$$

Now, let us choose $R_k = 1/(\alpha\epsilon_k)$ and let us define

$$w_k(X) = v_{R_k}(X - Y' + \epsilon_k\phi(X_0)e_n), \quad Y' = (y'_0, 0, 0),$$

with v_R the function defined in Proposition 4.5. Then the ϵ_k -domain variation of w_k , which we call \tilde{w}_k , can be easily computed from the definition

$$w_k(X - \epsilon_k\tilde{w}_k(X)e_n) = U(X).$$

Indeed, since U is constant in the x' -direction, this identity is equivalent to

$$v_{R_k}(X - Y' + \epsilon_k\phi(X_0)e_n - \epsilon_k\tilde{w}_k(X)e_n) = U(X - Y'),$$

which in view of Proposition 4.5 gives us

$$\tilde{v}_{R_k}(X - Y') = \epsilon_k(\tilde{w}_k(X) - \phi(X_0)).$$

From the choice of R_k , the formula for q and (4.12), we then conclude that

$$\tilde{w}_k(X) = q(X) + \alpha^2\epsilon_k O(|X - Y'|^2),$$

and hence

$$|\tilde{w}_k - q| \leq C\epsilon_k \quad \text{in } N_{\bar{r}} \setminus P. \quad (5.16)$$

Thus, from the uniform convergence of A_k to A_∞ and (5.13)-(5.16) we get that for all k large enough

$$\tilde{g}_k - \tilde{w}_k \geq \frac{\delta}{2} \quad \text{in } (N_{\bar{r}} \setminus N_{\bar{r}/2}) \setminus P. \quad (5.17)$$

Similarly, from the uniform convergence of A_k to A_∞ and (5.16)-(5.15) we get that for k large

$$\tilde{g}_k(X_k) - \tilde{w}_k(X_k) \leq \frac{\delta}{4}, \quad \text{for some sequence } X_k \in N_{\bar{r}} \setminus P, X_k \rightarrow X_0. \quad (5.18)$$

On the other hand, it follows from Lemma 3.1 and (5.17) that

$$\tilde{g}_k - \tilde{w}_k \geq \frac{\delta}{2} \quad \text{in } N_{\bar{r}} \setminus P,$$

which contradicts (5.18). \square

The main Theorem now follows combining all of the lemmas above with the regularity result for the linearized problem, as in the case $\alpha = 1/2$. For completeness we present the details.

Proof of Theorem 5.1. Let ρ be the universal constant from Lemma 3.3 and assume by contradiction that we can find a sequence $\epsilon_k \rightarrow 0$ and a sequence g_k of solutions to (2.1) in B_1 such that g_k satisfies (5.1), i.e.

$$U(X - \epsilon_k e_n) \leq g_k(X) \leq U(X + \epsilon_k e_n) \quad \text{in } B_1, \quad (5.19)$$

but it does not satisfy the conclusion of the Theorem.

Denote by \tilde{g}_k the ϵ_k -domain variation of g_k . Then by Lemma 5.3 the sequence of sets

$$A_k := \{(X, \tilde{g}_k(X)) : X \in B_{1-\epsilon_k} \setminus P\},$$

converges uniformly (up to extracting a subsequence) in $B_{1/2} \setminus P$ to the graph

$$A_\infty := \{(X, \tilde{g}_\infty(X)) : X \in B_{1/2} \setminus P\},$$

where \tilde{g}_∞ is a Holder continuous function in $B_{1/2}$. By Lemma 5.5, the function \tilde{g}_∞ solves the linearized problem (3.8) and hence by Corollary 3.3 \tilde{g}_∞ satisfies

$$a_0 \cdot x' - \frac{1}{8}\rho \leq \tilde{g}_\infty(X) \leq a_0 \cdot x' + \frac{1}{8}\rho \quad \text{in } B_{2\rho}, \quad (5.20)$$

with $a_0 \in \mathbb{R}^{n-1}$.

From the uniform convergence of A_k to A_∞ , we get that for all k large enough

$$a_0 \cdot x' - \frac{1}{4}\rho \leq \tilde{g}_k(X) \leq a_0 \cdot x' + \frac{1}{4}\rho \quad \text{in } B_{2\rho} \setminus P, \quad (5.21)$$

and hence from Lemma 5.2, the g_k satisfy the conclusion of our Theorem (for k large). We have thus reached a contradiction. \square

6. The Regularity of the Linearized Problem

The purpose of this section is to prove an improvement of flatness result for viscosity solutions to the linearized problem associated to (2.1), that is

$$\begin{cases} \operatorname{div}(|z|^\beta \nabla(U_n w)) = 0, & \text{in } B_1 \setminus P, \\ |\nabla_r w| = 0, & \text{on } B_1 \cap L, \end{cases} \quad (6.1)$$

where we recall that for $X_0 = (x'_0, 0, 0) \in B_1 \cap L$, we set

$$|\nabla_r w|(X_0) := \lim_{(x_n, z) \rightarrow (0,0)} \frac{w(x'_0, x_n, z) - w(x'_0, 0, 0)}{r}, \quad r^2 = x_n^2 + z^2.$$

The following is our main theorem.

Theorem 6.1. *Given a boundary data $\bar{h} \in C(\partial B_1)$, $|\bar{h}| \leq 1$, which is even with respect to $\{z = 0\}$, there exists a unique classical solution h to (6.1) such that $h \in C(\bar{B}_1)$, $h = \bar{h}$ on ∂B_1 , h is even with respect to $\{z = 0\}$ and it satisfies*

$$|h(X) - h(X_0) - a' \cdot (x' - x'_0)| \leq C(|x' - x'_0|^2 + r^{1+\gamma}), \quad X_0 \in B_{1/2} \cap L, \quad (6.2)$$

for universal constants C, γ and a vector $a' \in \mathbb{R}^{n-1}$ depending on X_0 .

As a corollary of the theorem above we obtain the desired regularity result, as stated also in Section 3.

Theorem 6.2 (Improvement of flatness). *There exists a universal constant C such that if w is a viscosity solution to (6.1) in B_1 with*

$$-1 \leq w(X) \leq 1 \quad \text{in } B_1,$$

then

$$a_0 \cdot x' - C|X|^{1+\gamma} \leq w(X) - w(0) \leq a_0 \cdot x' + C|X|^{1+\gamma}, \quad (6.3)$$

for some vector $a_0 \in \mathbb{R}^{n-1}$.

The existence of the classical solution of Theorem 6.1 will be achieved via a variational approach in the appropriate weighted Sobolev space. The advantage of working in the variational setting is that the difference of two solutions remains a solution. This is not obvious if we work directly with viscosity solutions.

We say that $h \in H^1(U_n^2 dX, B_1)$ is a minimizer to the energy functional

$$J(h) := \int_{B_1} |z|^\beta U_n^2 |\nabla h|^2 dX,$$

if

$$J(h) \leq J(h + \phi), \quad \forall \phi \in C_0^\infty(B_1).$$

Since J is strictly convex this is equivalent to

$$\lim_{\epsilon \rightarrow 0} \frac{J(h) - J(h + \epsilon\phi)}{\epsilon} = 0, \quad \forall \phi \in C_0^\infty(B_1),$$

which is satisfied if and only if

$$\int_{B_1} |z|^\beta U_n^2 \nabla h \cdot \nabla \phi dX = 0, \quad \forall \phi \in C_0^\infty(B_1).$$

Below, we briefly describe the relation between minimizers and viscosity solutions. First, a minimizer h solves the equation

$$\operatorname{div}(|z|^\beta U_n^2 \nabla h) = 0 \quad \text{in } B_1,$$

which in $B_1 \setminus P$ is equivalent to solving

$$\operatorname{div}(|z|^\beta \nabla(U_n h)) = 0 \quad \text{in } B_1 \setminus P. \quad (6.4)$$

Indeed, if $\phi \in C_0^\infty(B_1 \setminus P)$ then

$$\int |z|^\beta U_n^2 \nabla h \nabla \left(\frac{\phi}{U_n} \right) dX = 0.$$

This implies,

$$\int |z|^\beta (U_n \nabla h \nabla \phi - \nabla h \phi \nabla U_n) dX = 0.$$

Hence,

$$\int |z|^\beta (\nabla(U_n h) \nabla \phi - \nabla U_n \nabla(h\phi)) = 0.$$

The second integral is zero, since U_n is a solution of the equation $\operatorname{div}(|z|^\beta \nabla U_n) = 0$. Thus, our conclusion follows.

Moreover, we claim that if $h \in C(B_1)$ is a solution to (6.4), such that

$$\lim_{r \rightarrow 0} h_r(x', x_n, z) = b(x'), \quad (6.5)$$

with $b(x')$ a continuous function, then h is a minimizer to J in B_1 if and only if $b \equiv 0$.

Proof of the claim. By integration by parts and the computation above the identity

$$\int_{B_1} |z|^\beta U_n^2 \nabla h \cdot \nabla \phi dX = 0, \quad \forall \phi \in C_0^\infty(B_1),$$

is equivalent to the following two conditions

$$\operatorname{div}(|z|^\beta \nabla(U_n h)) = 0 \quad \text{in } B_1 \setminus P, \quad (6.6)$$

and

$$\lim_{\delta \rightarrow 0} \int_{\partial C_\delta \cap B_1} |z|^\beta U_n^2 \phi \nabla h \cdot \nu d\sigma = 0, \quad (6.7)$$

where C_δ is the cylinder $\{r \leq \delta\}$ and ν the inward unit normal to C_δ .

Here we use that

$$\lim_{\epsilon \rightarrow 0} \int_{\{|z|=\epsilon\} \cap (B_1 \setminus C_\delta)} |z|^\beta U_n^2 \phi h_\nu d\sigma = 0.$$

Indeed, in the set $\{|z| = \epsilon\} \cap (B_1 \setminus C_\delta)$ we have, (for some C independent of ϵ)

$$U_n \leq C|z|^{1-\beta},$$

and

$$|\nabla(U_n h)|, |\nabla U_n| \leq C|z|^\beta,$$

from which it follows that

$$|\nabla h| \leq C|z|^{-1}.$$

In conclusion we need to show that (6.7) is equivalent to $b(x') = 0$.

This follows, after an easy computation showing that

$$\lim_{\delta \rightarrow 0} \int_{\partial C_\delta \cap B_1} |z|^\beta U_n^2 \phi \nabla h \cdot \nu d\sigma = C_\alpha \int_L b(x') \phi(x', 0, 0) dx'$$

with

$$C_\alpha = \alpha^2 \int_{-\pi}^{\pi} (\cos \theta)^\beta (\cos \frac{\theta}{2})^{2-2\beta} d\theta. \quad \square$$

From the claim it follows that the function

$$v(X) := -\frac{|x'|^2}{n-1} + 2x_n r,$$

is a minimizer of J . Using as comparison functions the translations of the function v above we obtain as in Lemma 4.3 that minimizers h satisfy Harnack inequality.

Since our linear problem is invariant under translations in the x' -direction, we see that discrete differences of the form

$$h(X + \tau) - h(X),$$

with τ in the x' -direction are also minimizers. Now by standard arguments we obtain the following regularity result.

Lemma 6.3. *Let h be a minimizer to J in B_1 which is even with respect to $\{z = 0\}$. Then $D_{x'}^k h \in C^\gamma(B_{1/2})$ and*

$$[D_{x'}^k h]_{C^\gamma(B_{1/2})} \leq C \|h\|_{L^\infty(B_1)},$$

with C depending on the index $k = (k_1, \dots, k_{n-1})$.

We are now ready to prove our main theorem.

Proof of Theorem 6.1. It suffices to show that minimizers h with smooth boundary data on ∂B_1 achieve the boundary data continuously and satisfy the conclusion of our theorem. Then the general case follows by approximation.

First we show that h achieves the boundary data continuously. At points on $\partial B_1 \setminus P$ this follows from the continuity of $U_n h$, since $U_n \neq 0$.

For points $X_0 \in \partial B_1 \cap P$ we need to construct local barriers for h which vanish at X_0 and are positive in \bar{B}_1 near X_0 . If $X_0 \notin L$ then we consider barriers of the form

$$z^{1-\beta} W(x) / U_n$$

with W harmonic in x . If $X_0 \in L$ then the barrier is given by

$$(x' - x'_0) \cdot x'_0.$$

By Lemma 6.3 and (6.5), it remains to prove that

$$|h(x', x_n, x) - h(x', 0, 0) - b(x')r| \leq Cr^{1+\gamma}, \quad (x', 0, 0) \in B_{1/2} \cap L, \quad (6.8)$$

$$|h_r(x', x_n, z) - b(x')| \leq Cr^\gamma, \quad (x', 0, 0) \in B_{1/2} \cap L, \quad (6.9)$$

with C, γ universal and $b(x')$ a continuous function.

Indeed, h solves

$$\operatorname{div}(|z|^\beta \nabla(U_n h)) = 0 \quad \text{in } B_1 \setminus P.$$

Since U_n is independent on x' we can rewrite this equation as

$$\operatorname{div}_{x_n, z}(|z|^\beta \nabla(U_n h)) = -|z|^\beta U_n \Delta_{x'} h, \quad (6.10)$$

and according to Lemma 6.3 we have that

$$\Delta_{x'} h \in L^\infty(B_{1/2}).$$

Thus, for each fixed x' , we need to investigate the 2-dimensional problem (in the (t, z) -variables)

$$\operatorname{div}(|z|^\beta \nabla(U_t h)) = |z|^\beta U_t f, \quad \text{in } B_{1/2} \setminus \{t \leq 0, z = 0\}$$

with f bounded.

After fixing x' , say $x' = 0$, we may subtract a constant and assume $h(0, 0, 0) = 0$. Then $U_t h$ is continuous at the origin and coincides with the solution $H(t, z)$ to the problem

$$\operatorname{div}(|z|^\beta \nabla H) = |z|^\beta U_t f, \quad \text{in } B_{1/2} \setminus \{t \leq 0, z = 0\}, \quad (6.11)$$

such that

$$H = U_t h \quad \text{on } \partial B_{1/2}, \quad H = 0 \quad \text{on } B_{1/2} \cap \{t \leq 0, z = 0\}.$$

The fact that $U_t h = H$ follows from standard arguments by comparing $H - U_t h$ with $\pm \epsilon U_t$ and then letting $\epsilon \rightarrow 0$.

Using that U is a positive solution to the homogenous equation (6.11) we may apply boundary Harnack estimate (see Remark 6.4) and obtain that H/U is a C^γ function in a neighborhood of the origin. Thus

$$|H - aU| \leq C_0 r^\gamma U, \quad r^2 = t^2 + z^2, \quad C_0 \text{ universal,}$$

for some $a \in \mathbb{R}$. Since $U/U_t = r/\alpha$ we obtain (6.8) with $b = a/\alpha$.

We show that (6.9) follows from (6.8) and the derivative estimates for the extension equation. Indeed, the function $\bar{H} := H - aU$ above satisfies

$$|\operatorname{div}(|z|^\beta \nabla \bar{H})| \leq C r^{-\alpha}, \quad \|\bar{H}\|_{L^\infty(B_{2r} \setminus B_r)} \leq C r^\gamma U,$$

and the derivative estimates for the rescaled function $\bar{H}(r(t, z))$ imply

$$|\bar{H}_r| \leq C r^{\gamma-1} U = C r^\gamma U_t.$$

Using that

$$U_t h_r = H_r + (1 - \alpha) \frac{H}{r},$$

we easily obtain (6.9).

Finally we remark that $b(x')$ is a smooth function since by the translation invariance of our equation in the x' direction, the derivatives of b are the corresponding functions in (6.8) for the derivatives $\partial_{x_i} h, i = 1, \dots, n-1$. \square

Remark 6.4. In general boundary Harnack estimate is stated for the quotient v/u of two solutions (and u positive) to a homogenous equation $Lu = 0$.

The result remains valid if v solves the equation $Lv = g$ for a right hand side g that is not too degenerate near the boundary. In fact we only need to find an explicit barrier w such that $Lw \geq |g|$ and w/u is Hölder continuous at 0. Then the strategy of trapping v in dyadic balls between multiples $a_k u$ and $b_k u$ can be carried out by trapping v between functions of the type $a_k u + w$ and $b_k u - w$.

In the case of equation (6.11) an explicit w is given by $w := rU$ and it is easy to check that

$$\operatorname{div}(|z|^\beta \nabla w) \geq c_0 |z|^\beta U/r,$$

for some positive constant c_0 .

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