FOURIER INVERSION AND PLANCHEREL FORMULA OF LIE GROUPS OF REAL TYPE $(A_1)^n$

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1. **Introduction.** In this paper, we are going to compute the Fourier transforms of the invariant integrals, and derive the Plancherel formula via the inversion formula for a class of Lie groups. The same problems were studied for the Lie groups of real rank one by Sally and Warner [9], and for the Lie groups of real rank two by Herb [6]. In this paper, we will consider the class of Lie groups specified as below.

Let G be a connected reductive Lie group with compact center. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_c$ its complexification. Let G_c be a complex analytic group with Lie algebra $\mathfrak{g}_c$. Assume that

(1.1)   The complex analytic subgroup of G_c corresponding to the derived subalgebra of $\mathfrak{g}_c$ is simply connected.

(1.2) G is the real analytic subgroup of G_c corresponding to $\mathfrak{g}$.

(1.3) G is acceptable.

(1.4) G has a compact Cartan subgroup.

(1.5) G is of real type $(A_1)^n$ for some positive integer n, that is, if $\mathfrak{a}$ is a split Cartan subalgebra of $\mathfrak{g}$, then the set of real roots of the pair $(\mathfrak{g}_c, \mathfrak{a}_c)$ forms a root system of type $(A_1)^n$.

Let T be a compact Cartan subgroup of G and $t \in T$ a regular element. For $f \in C_c^\infty(G)$, the invariant integral of f relative to T at t is denoted by $\Phi_f^T(t)$. Then the map

(1.6) $A_t : f \longrightarrow \Phi_f^T(t), \quad f \in C_c^\infty(G),$

defines a tempered invariant distribution on G, the Fourier transform of Λ_t is a linear functional $\hat{\Lambda}_t$ such that $\hat{\Lambda}_t(\hat{f}) = \Lambda_t(f)$ for all f in $C_c^\infty(G)$, where $\hat{f}$ is defined on the unitary dual of G.

In the general cases, the computation of the Fourier transform $\hat{\Lambda}_t$ of Λ_t has some technical difficulties: for example, the computation of the integer constants in the character formulas of the discrete series, the evaluation of some improper integrals or the determination of the closed forms of some multiply infinite sums. The Fourier inversion formulas in [6] contain certain complicated expressions involving some infinite sums which do not converge absolutely and have no obvious closed form. Thus, they cannot be differentiated term by term to get the Plancherel formula for G. Professor Sally suggested the idea of simplifying the problem of the Fourier inversion by using the stabilized distribution F_t, and it is the method by which Chao solved the problems for Lie groups of real rank two [1].

When the group G is of real type $(A_1)^n$, the explicit expression for the Fourier transform $\hat{\Lambda}_t$ can be obtained by using the techniques in [1] and [9]. Moreover, the Fourier transform of the distribution defined by the invariant integral relative to any noncompact Cartan subgroup of G can be obtained at the same time, these inversion formulas are proved in §8 of this paper.

A general Plancherel theorem has been proved by Harish-Chandra [5], but in this paper, we will derive the Plancherel formula via the Fourier inversion formula by using the following equation:

$$(1.7) \qquad f(e) = M_G^{-1} \lim_{t \to e} (\Pi^T \Phi_f^T)(t), \qquad f \in C_c^\infty(G).$$

Here, M_G is a constant, Π^T is a certain differential operator on T, and the limit is taken through regular points in T.

The Lie groups $SU(p, q)$ satisfy all the five conditions (1.1)–(1.5) and the Plancherel formulas for these groups have been obtained by Hirai [8]. Since the class of Lie groups specified as above contains not only groups of classical type, $SU(p, q)$, $Sp(p, q)$, etc., but also certain Lie groups of exceptional type, thus

the Plancherel formula obtained in this paper can be regarded as the generalization of Hirai's result.

The contents of §2–5 are the structure and the character theories of the group G, most of the results can be found in Harish-Chandra [3] and [4], Herb [7], and Sugiura [10]. The two theorems in §6 and §7 are the central parts of the computation of the Fourier transforms, and they are combined into the inversion formulas in §8. In §9, we obtain the Plancherel formula for G.

2. **Structure of $\mathfrak{g}$ and G.** A reductive Lie algebra $\mathfrak{g}$ over R is said to be of real type $(A_1)^n$ if for every split Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, the root system $\Phi_R(\mathfrak{g}_c, \mathfrak{a}_c)$ consisting of all real roots of the pair $(\mathfrak{g}_c, \mathfrak{a}_c)$ is of type $(A_1)^n$. This is equivalent to the following statement: If $\mathfrak{z}$ denotes the centralizer of the toroidal part of $\mathfrak{a}$ in $\mathfrak{g}$, then the derived subalgebra of $\mathfrak{z}$ is isomorphic to $(\mathrm{sl}\,(2, R))^n$. A reductive Lie group is of real type $(A_1)^n$ if its Lie algebra is so. The real simple Lie algebras which are of such real type are A_1, A III, C II, D III, E III, E VII, and F II. All these real simple Lie algebras have the common property: The Cartan subalgebras of the same split dimension are conjugate [10].

From now on, we assume that G and $\mathfrak{g}$ satisfy all assumptions in §1.

$\mathfrak{g}$ can be decomposed as follows: $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \cdots + \mathfrak{g}_s$, where $\mathfrak{g}_i$ is a simple ideal for $i = 2, \cdots, s$, and $\mathfrak{g}_1$ is a reductive ideal containing one and only one non-compact simple factor. Let n_i denote the real rank of $\mathfrak{g}_i$ for $i = 1, 2, \cdots, s$, then $n = n_1 + n_2 + \cdots + n_s$.

Fix once and for all a Cartan involution θ on $\mathfrak{g}$, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. We fix in $\mathfrak{g}$ a θ-stable split Cartan subalgebra $\mathfrak{a}$, then the pair $(\mathfrak{g}_c, \mathfrak{a}_c)$ has exactly $2n$ real roots. Let $\pm \alpha_{pq}$ denote these real roots, $q = 1, 2, \cdots, n_p$ and $p = 1, 2, \cdots, s$. For each α_{pq} in the root system, there exist X_{pq}, Y_{pq} and H_{pq}^* in $\mathfrak{g}$ such that X_{pq} and Y_{pq} belong to the root spaces of α_{pq} and $-\alpha_{pq}$, respectively; $[H_{pq}^*, X_{pq}] = 2X_{pq}$, $[H_{pq}^*, Y_{pq}] = -2Y_{pq}$, $[X_{pq}, Y_{pq}] = H_{pq}^*$. Moreover, we assume that $-Y_{pq}$ is

the conjugate of X_{pq} under the conjugation of $\mathfrak{g}_c$ relative to $\mathfrak{k} + \sqrt{-1}\,\mathfrak{p}$.

For each s-tuple $(i) = (i_1, \cdots, i_s)$ of nonnegative integers such that $i_p \leq n_p$ for $p = 1, \cdots, s$, there corresponds a θ-stable Cartan subalgebra $\mathfrak{j}^{(i)}$ defined as follows:

$$(2.1) \qquad \mathfrak{j}^{(i)} = \mathfrak{a}_{\mathfrak{k}} + \sum_{p=1}^{s} \sum_{q > i_p} \boldsymbol{R}(X_{pq} - Y_{pq}) + \sum_{p=1}^{s} \sum_{q \leq i_p} \boldsymbol{R}\, H_{pq}^{*}.$$

Then $\mathfrak{j}^{(n)} = \mathfrak{a}$ and $\mathfrak{t} = \mathfrak{j}^{(0)}$ is a compact Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. Let (i) and (k) be s-tuples, we denote by $(i) \leq (k)$ if $i_p \leq k_p$ for all $p = 1, \cdots, s$; and denote by $(i) < (k)$ if the strict inequality holds for at least one p. Thus, $(i) \leq (k)$ if and only if $\mathfrak{j}_p^{(i)} \subseteq \mathfrak{j}_p^{(k)}$.

$\{\mathfrak{j}^{(i)} \mid (i) \leq (n)\}$ is a complete set of representatives of conjugacy classes of Cartan subalgebras of $\mathfrak{g}$.

For each (i), we define an automorphism $y_{(i)}$ of $\mathfrak{g}_c$ as follows:

$$(2.2) \qquad y_{(i)} = \mathrm{Ad}\left[\prod_{p=1}^{s} \prod_{q > i_p} \exp\left(\frac{1}{4} \pi \sqrt{-1}\,(X_{pq} + Y_{pq}) \right) \right].$$

Then $y_{(i)}$ maps $\mathfrak{t}_c$ onto $\mathfrak{j}_c^{(i)}$. Fix a positive system $\Phi^{+}(\mathfrak{g}_c, \mathfrak{a}_c)$ of all the roots of the pair $(\mathfrak{g}_c, \mathfrak{a}_c)$ so that it is compatible with an ordered basis of $\mathfrak{a}_p + \sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$ defined as follows: $H_{11}^{*}, H_{12}^{*}, \cdots,$ $H_{1n_1}^{*}, H_{21}^{*}, \cdots, H_{sn_s}^{*}, B_1, \cdots, B_m$. Here, $\{B_1, B_2, \cdots, B_m\}$ is an ordered basis of $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$. Let $\Phi^{+}(\mathfrak{g}_c, \mathfrak{j}_c^{(i)})$ be the transport of $\Phi^{+}(\mathfrak{g}_c, \mathfrak{a}_c)$ via $y_{(i)}\, y_{(n)}^{-1}$ for each $(i) < (n)$. Thus, for each $(i) \leq (n)$, the complex conjugate of a non-imaginary positive root of the pair $(\mathfrak{g}_c, \mathfrak{j}_c^{(i)})$ is always positive.

Let $A = J_{(n)}$, $J_{(i)}$ and $T = J_{(0)}$ be the Cartan subgroups of G corresponding to the Cartan subalgebras $\mathfrak{a}$, $\mathfrak{j}^{(i)}$ and $\mathfrak{t}$, respectively. Then, T is a maximal torus of the maximal compact subgroup K corresponding to $\mathfrak{k}$, hence, T is connected and compact. For each $(n) \geq (i) > (0)$, let $Z(0, i) = K \cap \exp(\sqrt{-1}\,\mathfrak{j}_p^{(i)})$. Since $Z(0, i)$ is an elementary 2-group, so there exists a subgroup $\widetilde{Z}(0, i)$ of $Z(0, i)$ such that $Z(0, i)$ is equal to the direct product of $\widetilde{Z}(0, i)$ and $Z(0, i) \cap J_{(i)\mathfrak{k}}^{0}$, where $J_{(i)\mathfrak{k}}^{0}$ is the identity component of $J_{(i)\mathfrak{k}} = K \cap J_{(i)}$. Then, the Cartan subgroup $J_{(i)}$ has exactly

$|\widetilde{Z}(0, i)|$ connected components and $J_{(i)} = \widetilde{Z}(0, i) J^0_{(i)}$, $J_{(i)\mathfrak{k}}$ $= \widetilde{Z}(0, i) J^0_{(i)\mathfrak{k}}$.

Since there exists a fundamental system of $\Phi(\mathfrak{g}_c, \mathfrak{j}^{(i)}_c)$ containing all $\alpha^{(i)}_{pq} = \alpha_{pq} y_{(n)} y^{-1}_{(i)}$, $q = 1, 2, \cdots, i_p$, and $p = 1, 2, \cdots, s$, we see that every element of $Z(0, i)$ is of the following form:

$$\exp \left(\sum_{p=1}^{s} \sum_{q \leqslant i_p} m_{pq} \pi \sqrt{-1} \, H^*_{pq} \right), \qquad m_{pq} \in \mathbf{Z}.$$

Note that the expression is unique if we assume that $m_{pq} = 0$ or 1. For $q = 1, 2, \cdots, n_p$ and $p = 1, 2, \cdots, s$, let

(2.3) $r_{pq} = \exp (\pi \sqrt{-1} \, H^*_{pq})$.

Then $Z(0, i)$ is the subgroup of $J_{(i)\mathfrak{k}}$ generated by r_{pq}, $q = 1, \cdots, i_p$, and $p = 1, \cdots, s$. Thus, the order of $Z(0, i)$ is equal to $2^{|(i)|}$, where $|(i)| = i_1 + i_2 + \cdots + i_s$.

Let $P_{(i)}$ be the cuspidal parabolic subgroup of G associated to $J_{(i)}$ and $M_{(i)} J_{(i)\mathfrak{p}} N_{(i)}$ be the Langlands decomposition of $P_{(i)}$. Let $M^0_{(i)}$ be the identity component of $M_{(i)}$. Then $M^0_{(i)}$ is a connected reductive Lie group which satisfies the conditions (1.1)–(1.4) [3] (Lemma 28) and is of real type $(A_1)^{|(n)-(i)|}$, where $(k) - (i) = (k_1 - i_1, \cdots, k_s - i_s)$. Hence, the previous discussion about the structure of G can be applied to the group $M^0_{(i)}$. $J^0_{(i)\mathfrak{k}} = J_{(i)} \cap M^0_{(i)}$ is a compact Cartan subgroup of $M^0_{(i)}$, and the set $\{J_{(k)} \cap M^0_{(i)} \mid (i) \leq (k) \leq (n)\}$ is a complete set of representatives of conjugacy classes of Cartan subgroups of $M^0_{(i)}$.

All the connected components of $J_{(i)\mathfrak{k}}$ have the same centralizer in $\mathfrak{g}$ which we denote as $\mathfrak{z}^{(i)}$. Then, $\mathfrak{z}^{(i)}$ is a reductive Lie algebra of real type $(A_1)^{|(i)|}$; similarly, all components of $J_{(k)} \cap M^0_{(i)} \cap K$ have the same centralizer in $\mathfrak{m}^{(i)}$ for $(k) > (i)$, here $\mathfrak{m}^{(i)}$ is the Lie algebra of $M^0_{(i)}$. Let $\mathfrak{z}^{(i, k)}$ be the common centralizer. Then, $\mathfrak{z}^{(i, k)}$ is also a reductive Lie algebra.

For each (i), let $L_{(i)} = M_{(i)} J_{(i)\mathfrak{p}}$ and $L^0_{(i)} = M^0_{(i)} J_{(i)\mathfrak{p}}$. Let $\mathfrak{l}^{(i)}$ be the Lie algebra of $L^0_{(i)}$ and the positive system of the roots of the pair $(\mathfrak{l}^{(i)}_c, \mathfrak{j}^{(k)}_c)$ be fixed as the intersection of $\Phi^+(\mathfrak{g}_c, \mathfrak{j}^{(k)}_c)$ and $\Phi(\mathfrak{l}^{(i)}_c, \mathfrak{j}^{(k)}_c)$ for $(k) \geq (i)$. Let

(2.4) $\rho(i, k) = \dfrac{1}{2} \sum_{\alpha} \alpha$, $\qquad \alpha \in \Phi^+(\mathfrak{l}^{(i)}_c, \mathfrak{j}^{(k)}_c)$.

$$(2.5) \qquad \Delta_{(i,k)}(j) = \xi_{\rho(i,k)}(j) \prod_\alpha (1 - \xi_\alpha(j^{-1})) \,,$$

where

$$\alpha \in \Phi^+(\mathfrak{l}_c^{(i)}, \mathfrak{j}_c^{(k)}) \quad \text{and} \quad j \in J_{(k)} \cap L_{(i)}^0 \,.$$

$$(2.6) \qquad \epsilon_{(i,k)}(j) = \text{sgn} \prod_\alpha (1 - \xi_\alpha(j^{-1})) \,,$$

where

$$\alpha \in \Phi_R^+(\mathfrak{l}_c^{(i)}, \mathfrak{j}_c^{(k)}) \quad \text{and} \quad j \in J_{(k)} \cap L_{(i)}^0 \,.$$

Here, we understand that $L_{(0)}^0 = M_{(0)}^0 = G$, and if $\lambda : \mathfrak{j}_{(i)}^c \to C$ is a linear functional, then ξ_λ denotes the corresponding analytic homomorphism of $J_{(i)}$ if it exists.

The full Weyl group of $\mathfrak{l}_c^{(i)}$ with respect to $\mathfrak{j}_c^{(k)}$ will be denoted as $W_C(i, k)$. The subgroup of $W_C(i, k)$ generated by the reflections relative to the real roots will be denoted as $W_R(i, k)$. The factor group modulo $J_{(k)} \cap L_{(i)}^0$ of the normalizer in $L_{(i)}^0$ of $J_{(k)} \cap L_{(i)}^0$ can be embedded as a subgroup of $W_C(i, k)$ which is denoted as $W_K(i, k)$. For all $(k) \geq (i)$, we have $L_{(i)} = Z(0, k) L_{(i)}^0$. Hence $W_K(i, k)$ is isomorphic to the factor group module $J_{(k)}$ of the normalizer of $J_{(k)}$ in $L_{(i)}$. The groups $W_C(0, 0)$ and $W_K(0, 0)$ are also denoted as W_c and W_k, respectively.

3. **The characters of the discrete series.** The unitary character group of T may be identified with a lattice L_T in $\sqrt{-1}\, \mathfrak{t}^*$, the set of all purely imaginary R-linear functionals on $\mathfrak{t}$. $\tau \in L_T$ is said to be regular if $w\tau \neq \tau$ for all $w \neq 1$ in W_c. Let L_T' denote the set of all regular elements of L_T.

For each $\tau \in L_T'$, let $\epsilon(\tau) = \text{sgn} \prod_\alpha \tau(H_\alpha^*)$, where α runs over $\Phi^+(\mathfrak{g}_c, \mathfrak{t}_c)$. Then there is a tempered invariant eigendistribution θ_τ on G such that $(-1)^{(1/2)\dim(G/K)} \epsilon(\tau) \theta_\tau$ is the character of a discrete series representation of G and

$$(3.1) \qquad \theta_\tau = \Delta_{(0,0)}^{-1} \sum_{w \in W_k} \det(w) \xi_{w\tau} \qquad \text{on} \quad T' \,,$$

where T' is the set of all regular elements of T. For each (i), let

(3.2) $\quad J_{(i)\flat}^{+} = \left\{ \exp\left[\sum_{p=1}^{s} \sum_{q \leq i_p} x_{pq} H_{pq}^{*} \right] \,\middle|\, x_{pq} > 0 \ \text{ for all } \ p, \ q \right\}.$

LEMMA 3.1. *If* $j_k \in J_{(i)\mathfrak{k}}$ *and* $j_\flat = \exp\left[\sum_{p=1}^{s} \sum_{q \leq i_p} x_{pq} H_{pq}^{*}\right] \in J_{(i)\flat}^{+}$, *then*

(3.3) $\quad \Delta_{(0,i)}(j_k j_\flat)\, \theta_\tau(j_k j_\flat) = \sum_{m \in W_k}{}' \det(w)\, c(w\tau : (i))\, \xi_{w,\tau}^{*}(j_k j_\flat),$

where

(3.4) $\quad \xi_{w,\tau}^{*}(j_k j_\flat) = \xi_{w\tau}(j_k) \prod_{p=1}^{s} \prod_{q \leq i_p} \exp\left[-x_{pq} \mid (w\tau)(y_{(i)}^{-1} H_{pq}^{*})\mid\right] ;$

(3.5) $\quad c(\tau : (i)) = \begin{cases} 1, & \text{if } (-1)^{|(i)|} \prod_{p=1}^{s} \prod_{q \leq i_p} \tau(y_{(i)}^{-1} H_{pq}^{*}) > 0, \\[2mm] -1, & \text{if } (-1)^{|(i)|} \prod_{p=1}^{s} \prod_{q \leq i_p} \tau(y_{(i)}^{-1} H_{pq}^{*}) < 0, \\[2mm] 0, & \text{otherwise}. \end{cases}$

Proof. See [7].

4. The characters of the non-degenerate series. Let $\hat{J}_{(i)\mathfrak{k}}$ be the unitary character group of $J_{(i)\mathfrak{k}}$, then $\chi \in \hat{J}_{(i)\mathfrak{k}}$ is said to be regular if $w\chi \neq \chi$ for all $w \neq 1$ in $W_C(i, i)$. If χ is a regular character in $\hat{J}_{(i)\mathfrak{k}}$, we set $\varepsilon(\chi) = \operatorname{sgn}\left[\prod_\alpha \log(\chi)(H_\alpha^{*})\right]$, where α runs over $\varPhi^{+}(\mathfrak{l}_c^{(i)}, \mathfrak{j}_c^{(i)})$. The unitary character group of $J_{(i)\flat}$ is identified with the dual space $\mathfrak{j}_\flat^{(i)*}$ of $\mathfrak{j}_\flat^{(i)}$ as follows: for each $\lambda \in \mathfrak{j}_\flat^{(i)*}$, we define the corresponding unitary character in $\hat{J}_{(i)\flat}$ by

(4.1) $\qquad j_\flat^{\sqrt{-1}\,\lambda} = \exp\left[\sqrt{-1}\,\lambda\,(\log j_\flat)\right], \qquad j_\flat \in J_{(i)\flat}.$

For each regular character χ in $\hat{J}_{(i)\mathfrak{k}}$, and each linear functional λ in $\mathfrak{j}_\flat^{(i)*}$, there is a tempered invariant eigendistribution $\theta_{\chi,\lambda}^{(i)}$ on G such that $(-1)^{(1/2)\dim(M_{(i)}^0/M_{(i)}^0 \cap K)}\,\varepsilon(\chi)\,\theta_{\chi,\lambda}^{(i)}$ is the character of a non-degenerate series representation of G relative to $J_{(i)}$, and the support of $\theta_{\chi,\lambda}^{(i)}$ is contained in the closure of the union of all conjugates of the subgroups $J_{(k)}$ with $(k) \geq (i)$ (see [12]). The expression of the eigendistribution $\theta_{\chi,\lambda}^{(i)}$ on $J_{(k)}$ is described as follows: if h is a regular element in $J_{(k)}$, let h_M (resp. $h_{J_\flat}$) be the "$M_{(i)}$-component" (resp. "$J_{(i)\flat}$-component") of h, and h_L be any of the "$L_{(i)}^0$-

components" of h, (note that the element h_L is unique up to $Z(0, i) \cap J^0_{(i)_\mathfrak{k}}$). Then

$$(4.2) \qquad \theta^{(i)}_{\chi, \lambda}(h) = \sum_{w \in W_K(0, k)} \frac{|\Delta_{(i, k)}((wh)_L)|}{|W_K(i, k)| |\Delta_{(0, k)}(h)|} \cdot \Psi_\chi((wh)_M)(wh)^{\sqrt{-1}\lambda}_{J_\mathfrak{p}}.$$

In particular, if $r \in Z(0, i)$, $j_k \in J^0_{(i)_\mathfrak{k}}$, and $j_\mathfrak{p} \in J_{(i)_\mathfrak{p}}$, then

$$(4.3) \qquad \theta^{(i)}_{\chi, \lambda}(r\, j_k\, j_\mathfrak{p}) = \sum_{w \in W_K(0, i)} \frac{|\Delta_{(i, i)}(j_k\, j_\mathfrak{p})|}{|W_K(i, i)| |\Delta_{(0, i)}(j_k\, j_\mathfrak{p})|} \cdot \Psi_\chi(w r\, j_k)(w\, j_\mathfrak{p})^{\sqrt{-1}\lambda}.$$

Here, Ψ_χ is a tempered invariant eigendistribution on $M_{(i)}$ such that the restriction of Ψ_χ to $M^0_{(i)}$ is defined as θ_r in §3 and

$$(4.4) \qquad \begin{aligned} \Psi_\chi(r\, j_k) &= \chi(r)\, \xi^{-1}_{\rho(i, i)}(r) \cdot \Psi_\chi(j_k)\,, \\ r &\in Z(0, i) \quad \text{and} \quad j_k \in J^0_{(i)_\mathfrak{k}}\,. \end{aligned}$$

Therefore, for any $M_{(i)}$-regular element j_k in $J_{(i)_\mathfrak{k}}$, we have

$$(4.5) \qquad \Psi_\chi(j_k) = \Delta_{(i, i)}(j_k)^{-1} \sum_{w \in W_K(i, i)}{}' \det(w)\, \chi(w\, j_k)\,.$$

For all regular χ in $\hat{J}_{(i)_\mathfrak{k}}$, λ in $\mathfrak{j}^{*}_\mathfrak{p}$, and u in $W_R(0, i)$, we have

$$(4.6) \qquad \theta^{(i)}_{u\chi, u\lambda} = \theta^{(i)}_{\chi, \lambda}\,.$$

5. **The invariant integrals of** G. To compute the Fourier transforms and the Plancherel formula for the Lie group G, we have to normalize various invariant measures. For each Cartan subgroup J of G, let $x \to \bar{x}$ be the canonical projection of G onto G/J. Normalize the G-invariant measure $d_{G/J}(\bar{x})$ on G/J as [11], Section 8.1. Let $d_T(t)$ be the Haar measure on T normalized so that the volume of T is one. Then the Haar measure on G is normalized by the following relations:

$$\int_G f(g)\, d_G(g) = \int_{G/T} \int_T f(xt)\, d_T(t)\, d_{G/T}(\bar{x})\,, \qquad f \in C_c(G)\,.$$

A Haar measure $d_J(j)$ for each Cartan subgroup J is then fixed by

$$\int_G f(g)\, d_G(g) = \int_{G/J} \int_J f(xj)\, d_J(j)\, d_{G/J}(\bar{x})\,, \qquad f \in C_c(G)\,.$$

If $J = J_{\mathfrak{k}} J_{\mathfrak{p}}$ is θ-stable, let $d_{J_{\mathfrak{p}}}(j_{\mathfrak{p}})$ be the Haar measure on $J_{\mathfrak{p}}$ which is the transport via the exponential map of the Haar measure on $j_{\mathfrak{p}}$ associated with the Euclidean structure derived from the Cartan-Killing form on $\mathfrak{g}$. Normalize a Haar measure on $J_{\mathfrak{k}}$ so that $d_J(j_k j_{\mathfrak{p}}) = d_{J_{\mathfrak{k}}}(j_k) \, d_{J_{\mathfrak{p}}}(j_{\mathfrak{p}})$.

For each (i), there is a unique positive number $c_{(i)}$ such that

$$(5.1) \qquad d_{J_{(i)\mathfrak{p}}}(j_{\mathfrak{p}}) = c_{(i)} \, dx_{11} \cdots dx_{1i_1} \, dx_{21} \cdots dx_{si_s} \, ,$$

where $j_{\mathfrak{p}} = \exp\left[\sum_{p=1}^{s} \sum_{q \le i_p} x_{pq} H_{pq}^*\right]$. On the other hand, $j_{\mathfrak{p}}^{(i)*}$ is identified with $\boldsymbol{R}^{|(i)|}$ as follows: if $\lambda \in j_{\mathfrak{p}}^{(i)*}$, then λ is identified with the $|(i)|$-tuple $(\mu)_{(i)} = (\mu_{11}, \cdots, \mu_{1i_1}, \mu_{21}, \cdots, \mu_{si_s})$ if

$$\lambda\left[\sum_{p=1}^{s} \sum_{q \le i_p} x_{pq} H_{pq}^*\right] = \sum_{p=1}^{s} \sum_{q \le i_p} \mu_{pq} \, x_{pq} \, .$$

Furthermore, the Haar measure on $j_{\mathfrak{p}}^{(i)*}$ is normalized by

$$(5.2) \qquad d\lambda = d(\mu)_{(i)} = d\mu_{11} \cdots d\mu_{1i_1} \, d\mu_{21} \cdots d\mu_{si_s} \, .$$

Let $f \in C_c^\infty(G)$ and $j \in J'_{(i)}$, then the invariant integral of f relative to $J_{(i)}$ at j is defined as follows:

$$(5.3) \qquad \Phi_f^{(i)}(j) = \varepsilon_{(0,i)}(j) \, \Delta_{(0,i)}(j) \int_{G/J_{(i)}} f(xj x^{-1}) \, d_{G/J_{(i)}}(\bar{x}) \, .$$

When $(i) = (0)$, then $\Phi_f^{(0)}$ is denoted as Φ_f^T. For each (i), the function $\Phi_f^{(i)}$ is integrable on $J_{(i)}$, C^∞ on $J'_{(i)}$, and compactly supported.

The Fourier coefficients of $\Phi_f^{(i)}$ are defined as follows:

$$(5.4) \qquad \hat{\Phi}_f^T(\tau) = \int_T \Phi_f^T(t) \, \xi_\tau(t) \, dt \, , \qquad \tau \in L_T \, .$$

$$(5.5) \qquad \hat{\Phi}_f^{(i)}(\chi, \lambda) = (2\pi)^{-|(i)|/2} \int_{J_{\mathfrak{k}}} \int_{J_{\mathfrak{p}}} \Phi_f^{(i)}(j_k j_{\mathfrak{p}}) \, \chi(j_k) \, j_{\mathfrak{p}}^{\sqrt{-1}\lambda} \, dj_k \, dj_{\mathfrak{p}} \, ,$$

$$\chi \in \hat{J}_{(i)\mathfrak{k}} \quad \text{and} \quad \lambda \in j_{\mathfrak{p}}^{(i)*} \, .$$

The proofs of the following lemmas can be found in [1].

LEMMA 5.1. *Let* $f \in C_c^\infty(G)$ *and* t *in* T', *then*

$$(5.6) \qquad \Phi_f^T(t) = (-1)^r \sum_{\tau \in L_T} \overline{\xi_\tau(t)} \, \theta_\tau(f) + R_f^T(t) \, ,$$

where r *is the number of positive roots of the pair* $(\mathfrak{g}_c, \mathfrak{t}_c)$ *and*

$$(5.7) \qquad R_f^T(t) = (-1)^{r+1} \sum_{\tau \in L_T} \overline{\xi_\tau(t)} \sum_{(i) > (0)} \int_{G^{(i)}} f(g) \, \theta_\tau(g) \, dg \, .$$

Here, $G^{(i)} = \{xj\,x^{-1} \mid x \in G,\ j \in J_{(i)}\}$.

LEMMA 5.2. *Let* $f \in C_c^\infty(G)$ *and* $j_k j_p$ *in* $J'_{(i)}$, *then*

$$c_{(i)} \cdot \mathrm{vol}\,(J_{(i)\,\mathfrak{k}}) \cdot (2\pi)^{|(i)|} \cdot \Phi_f^{(i)}(j_k j_p)$$

$$(5.8) \qquad = (-1)^{r(i)} \sum_{\chi \in \hat{f}_{(i)\,\mathfrak{k}}} \overline{\chi(j_k)} \int_{\mathfrak{j}_p^{(i)*}} \theta_{\chi,\lambda}^{(i)}(f)\, j_p^{-\sqrt{-1}\,\lambda}\, d\lambda$$

$$\qquad\qquad\qquad\qquad + R_f^{(i)}(j_k j_p)\,,$$

where $r(i)$ *is the number of positive roots of the pair* $(\mathfrak{k}_c^{(i)},\ \mathfrak{j}_c^{(i)})$ *and*

$$R_f^{(i)}(j_k j_p)$$

$$(5.9) \qquad = (-1)^{r(i)+1} \sum_{\chi \in \hat{f}_{(i)\,\mathfrak{k}}} \overline{\chi(j_k)} \int_{\mathfrak{j}_p^{(i)*}} j_p^{-\sqrt{-1}\,\lambda}\, d\lambda$$

$$\qquad\qquad \cdot \sum_{(k)>(i)} \int_{G^{(k)}} f(g)\, \theta_{\chi,\lambda}^{(i)}(g)\, dg\,.$$

For fixed t in T' and $j_k j_p$ in $J'_{(i)}$, let

$$(5.10) \qquad\qquad \Lambda_t(f) = \Phi_f^T(t)\,, \qquad f \in C_c^\infty(G)\,.$$

$$(5.11) \qquad\qquad \Lambda_{j_k j_p}^{(i)}(f) = \Phi_f^{(i)}(j_k j_p)\,, \qquad f \in C_c^\infty(G)\,.$$

Then Λ_t and $\Lambda_{j_k j_p}^{(i)}$ are invariant distributions on G, and we are going to compute the Fourier transforms of these distributions. In the following two sections, we will compute some recurring formulas for the functions $R_f^{(i)}$ by which we will obtain the inversion formulas for Λ_t and $\Lambda_{j_k j_p}^{(i)}$.

6. Computation of $R_f^T(t)$. For every w in W_k, there exist $j_{(n)}(w) \in A_{\mathfrak{k}}^0$ and $-\pi < \phi_{pq}(w) < \pi$, $q = 1, 2, \cdots, n_p$ and $p = 1, 2, \cdots, s$, such that

$$(6.1) \qquad wt = j_{(n)}(w) \exp\left[\sum_{p=1}^s \sum_{q \leq n_p} \phi_{pq}(w)(X_{pq} - Y_{pq})\right].$$

For each (i), let

$$(6.2) \qquad j_{(i)}(w) = j_{(n)}(w) \exp\left[\sum_{p=1}^s \sum_{q > i_p} \phi_{pq}(w)(X_{pq} - Y_{pq})\right].$$

Then $j_{(i)}(w) \in J_{(i)\,\mathfrak{k}}^0$ and

$$(6.3) \qquad wt = j_{(i)}(w) \exp\left[\sum_{p=1}^s \sum_{q \leq i_p} \phi_{pq}(w)(X_{pq} - Y_{pq})\right].$$

This decomposition is unique up to $Z(0, i) \cap J_{(i)\,\mathfrak{k}}^0$. For each (i),

we will denote the element $(wt)[j_{(i)}(w)]^{-1}$ by $j'_{(i)}(w)$.

Throughout this paper, we will use the convergence over L_T and $\hat{J}_{(i)\,i}$ very often. For this, we must give the explicit form to the type of convergence which we use relative to L_T and $\hat{J}_{(i)\,i}$.

Fix a basis B in the center of $\mathfrak{g}$ so that an element H in the center of $\mathfrak{g}$ satisfies $\exp(H) = e$ if and only if H is an integral linear combination of elements in $2\pi\sqrt{-1}\,B$.

For any fundamental system S of $\Phi(\mathfrak{g}_c, \mathfrak{t}_c)$. Let S^* be the dual basis of $B \cup \{H^*_\alpha \mid \alpha \in S\}$. Then L_T is the set of all integral linear combinations of elements in S^*.

For any positive integer m, we define a subset L_T^m of L_T as follows: an element $\tau \in L_T$ is in L_T^m if there is a fundamental system S of $\Phi(\mathfrak{g}_c, \mathfrak{t}_c)$ such that $\tau = \sum a_\lambda \lambda$, where λ runs over S^* and $|a_\lambda| \leq m$ for each λ. Then we define the convergence over L_T by

$$(6.4) \qquad \sum_{\tau \in L_T} = \lim_{m \to \infty} \sum_{\tau \in L_T^m}.$$

Since any two fundamental systems of $\Phi(\mathfrak{g}_c, \mathfrak{t}_c)$ are W_c-conjugate, thus L_T^m is invariant under W_c.

If J is a Cartan subgroup of G and L_J is the lattice in $\sqrt{-1}\,\mathfrak{j}^*_i$ which is identified with the unitary character group of J^0_i. Define the subset L_J^m as above, and let $\hat{J}^m_i$ be the subset of $\hat{J}_i$ consisting of the elements $\chi \in \hat{J}_i$ such that $\log(\chi)$ is in L_J^m. Then the convergence over $\hat{J}_i$ is defined by

$$(6.5) \qquad \sum_{\chi \in \hat{J}_i} = \lim_{m \to \infty} \sum_{\chi \in \hat{J}^m_i}$$

Consider the following series

$$(6.6) \qquad R_f^\tau(t, (i)) = (-1)^{r+1} \sum_{\tau \in L_T} \overline{\xi_\tau(t)} \int_{G^{(i)}} f(g)\, \theta_\tau(g)\, dg.$$

By using the Weyl integral formula and considering the transformations s on $J_{(i)\,\mathfrak{p}}$ for all s in $W_R(0, i)$, it is easy to see that

$$(6.7) \qquad \begin{aligned} &\int_{G^{(i)}} f(g)\, \theta_\tau(g)\, dg \\ &\quad = \frac{(-1)^{r(i)} \cdot 2^{|(i)|}}{|W_K(0, i)|} \int_{J_{(i)\,i}} \int_{J_{(i)\,\mathfrak{p}}^+} \Phi_f^{(i)}(j_k j_\mathfrak{p}) \\ &\qquad \cdot \sum_{w \in W_k} \det(w)\, c(w\tau : (i))\, \xi^*_{w,\tau}(j_k j_\mathfrak{p})\, dj_k\, dj_\mathfrak{p}. \end{aligned}$$

Let h be a function on an open dense subset of $J_{(i)}$ such that

(6.8)   The complement of the domain of h in $J_{(i)}$ is of measure zero.

(6.9) h is integrable on $J_{(i)}$.

(6.10) h is C^∞ on its domain.

(6.11) h vanishes off a compact set.

For a function h as above, define $T(G; t; (i); h)$ as follows:

(6.12)
$$\frac{(-1)^{r(i)}}{(4\pi)^{|(i)|}} \cdot T(G; t; (i); h)$$
$$= \sum_{\tau \in I_T} \sum_{w \in W_k} \det(w) \, \overline{\xi_\tau(wt)}$$
$$\cdot \int_{J_{(i)\,i}} \int_{J_{(i)\,p}^+} h(j_k j_p) \, c(\tau : (i)) \, \xi_{1,\tau}^*(j_k j_p) \, dj_k \, dj_p.$$

Throughout this paper, the notation "$\gamma^m \in Z(i, k)$" will mean

(6.13)
$$\gamma^m = \prod_{p=1}^{s} \prod_{i_p < q \le k_p} (\gamma_{pq})^{m_{pq}},$$

$m_{pq} = 0$ or 1 for all p, q.

PROPOSITION 6.1. *Retain the above notations, we have*

(6.14)
$$\frac{(-1)^{r(i)}}{(2\pi)^{|(i)|} c_{(i)} \operatorname{vol}(J_{(i)\,i})} T(G; t; (i); h)$$
$$= \sum_{\gamma^m \in Z(G,i)} \sum_{w \in W_k} \det(w)$$
$$\cdot \int_{J_{(i)\,p}^+} h(\gamma^m j_{(i)}(w) j_p) \cdot S(\gamma^m j_{(i)}^{\cdot}(w); j_p) \, dj_p,$$

where, if j_p corresponds to the $|(i)|$-tuple $(x)_{(i)}$, then

(6.15)
$$S(\gamma^m j_{(i)}^{\cdot}(w); j_p)$$
$$= \frac{1}{c_{(i)}} \prod_{p=1}^{s} \prod_{q \le i_p}$$
$$\cdot [-2\sqrt{-1}\,(-x_{pq}) \sin(\phi_{pq}(w) \pm m_{pq}\pi)]$$
$$/ [1 - 2\exp(-x_{pq}) \cos(\phi_{pq}(w) \pm m_{pq}\pi) + \exp(-2x_{pq})],$$

the signs $\pm$ are chosen so that $|\phi_{pq}(w) \pm m_{pq}\pi| < \pi$ for all p, q.

Proof. Let S be any fundamental system of $\Phi(\mathfrak{g}_c, \mathfrak{t}_c)$ containing the roots $\alpha_{pq}^{(0)}$, $q = 1, 2, \cdots, i_p$ and $p = 1, 2, \cdots, s$. Let S^* be the

dual basis of $B \cup \{H_{\alpha}^* \mid \alpha \in S\}$, where B is defined as above. Let τ_{pq} be the element in S^* with $\tau_{pq}(X_{pq} - Y_{pq}) = -\sqrt{-1}$, $q = 1, 2, \cdots, i_p$ and $p = 1, 2, \cdots, s$. Let

$$L_T^* = \{\tau \in L_T \mid \tau(X_{pq} - Y_{pq}) = 0, \quad q = 1, \cdots, i_p, \quad p = 1, \cdots, s\}.$$

$$L_T^{(i)} = \sum_{p=1}^{s} \sum_{q \leq i_p} Z \tau_{pq}.$$

Then $L_T = L_T^* + L_T^{(i)}$ and this is a direct sum. Furthermore, if τ is an element of $L_T^{(i)}$, then 2τ is an integral linear combination of the roots of $(\mathfrak{g}_c, \mathfrak{t}_c)$, then $\xi_{2\tau} = 1$ on $J_{(i)\,i}$. Let

(6.16) $\qquad h^*(j_k, j_p, \tau) = \sum_{\gamma^m \in Z(0,i)} \xi_\tau(\gamma^m j_k) \, h(\gamma^m j_k j_p),$

where $j_k \in J_{(i)\,i}$, $j_p \in J_{(i)\,p}$ and $\tau \in L_T$. For any fixed j_p and τ, the function $j_k \to h^*(j_k, j_p, \tau)$ can be regarded as a function on the connected compact abelian Lie group $J_{(i)\,i}/Z(0, i)$. Since the function is piecewise smooth, integrable, and compactly supported, the following equation follows from the elementary Fourier analysis on $J_{(i)\,i}/Z(0, i)$.

(6.17)
$$\sum_{\sigma \in L_T^*} \overline{\xi_{\sigma+\tau}(j_k')} \int_{J_{(i)\,i}} \xi_{\sigma+\tau}(j_k) \, h(j_k j_p) \, dj_k$$
$$= \frac{\mathrm{vol}\,(J_{(i)\,i})}{2^{|(i)|}} \sum_{\gamma^m \in Z(0,i)} \xi_\tau(\gamma^m) \, h(\gamma^m j_k' j_p),$$

for all $j_k' \in J_{(i)\,i}$, $j_p \in J_{(i)\,p}$, $\tau \in L_T$. Moreover, the series on the left-hand side of (6.17) converges absolutely and uniformly with respect to j_k' and the sum has compact support relative to $J_{(i)\,p}$. Note also that the sum on the left-hand side of (6.17) is constant on the cosets of L_T relative to the sublattice $L_T^* + 2L_T^{(i)}$.

If $\sigma \in L_T^*$ and $\tau \in L_T^{(i)}$, we have $c(\sigma + \tau : (i)) = c(\tau : (i))$ and $\xi_{1,\sigma+\tau}^*(j_p) = \xi_{1,\tau}^*(j_p)$. For $w \in W_k$ and $b_{pq} = 0$ or 1, $q = 1, \cdots, i_p$, $p = 1, \cdots, s$, let

(6.18)
$$S(w : (b_{pq}))$$
$$= \frac{1}{c_{(i)}} \sum_{\tau \in 2L_T^{(i)} + \Sigma_{p,q} b_{pq}\tau_{pq}} \overline{\xi_\tau(j_{(i)}'(w))} \, c(\tau : (i)) \, \xi_{1,\tau}^*(j_p).$$

When j_p lies in $J_{(i)\,p}^+$, the series (6.18) converges absolutely. By using the inequality $|1 - \exp(x + \sqrt{-1}\, y)| \geq 1 - |\cos(y)|$ for

$y \in \mathbf{R}$ and $x < 0$, it is easy to see that the partial sums are uniformly bounded in $J_{(i)\mathfrak{p}}^{+}$. Since the series (6.17) converges absolutely and uniformly with respect to j_k', and the sum has compact support relative to $J_{(i)\mathfrak{p}}$, the following result follows from the bounded convergence theorem.

$$\frac{(-1)^{r(i)}}{(2\pi)^{|(i)|} c_{(i)} \operatorname{vol}(J_{(i)t})} \cdot T(G;\ t;\ (i);\ h)$$

(6.19)
$$= \sum_{\gamma^m \in Z(0,i)} \sum_{w \in W_k} \det(w)$$

$$\cdot \int_{J_{(i)\mathfrak{p}}^{+}} h(\gamma^m j_{(i)}(w) j_{\mathfrak{p}})\, S(\gamma^m j_{(i)}'(w);\ j_{\mathfrak{p}})\, dj_{\mathfrak{p}},$$

where

(6.20)
$$S(\gamma^m j_{(i)}'(w);\ j_{\mathfrak{p}})$$
$$= \frac{1}{c_{(i)}} \sum_{\tau \in L_T^{(i)}} \overline{\xi_\tau(\gamma^m j_{(i)}'(w))}\, c(\tau : (i))\, \xi_{1,\tau}^{*}(j_{\mathfrak{p}}).$$

It is easy to express the sum $S(\gamma^m j_{(i)}'(w);\ j_{\mathfrak{p}})$ in the form (6.15) and the proof is completed.

THEOREM 6.2. *For each (i), we have*

$$R_f^{\tau}(t,\ (i))$$

(6.21)
$$= \frac{(-1)^{|(i)|+1}}{(2\pi)^{|(i)|}\, |W_K(0,\ i)|} \cdot R_{t,(i)}(f)$$

$$+ \frac{\sqrt{-1}^{|(i)|}(-1)^{r+1}}{2^{|(i)|}\, |W_K(0,\ i)|}$$

$$\cdot \sum_{\chi \in \hat{J}_{(i)t}} \int_{\mathbf{R}^{|(i)|}} \theta_{\chi,(\mu)}^{(i)}(f)\, K_{t,(i)}(\chi,\ (\mu))\, d(\mu)_{(i)},$$

where

$$K_{t,(i)}(\chi,\ (\mu))$$

(6.22)
$$= \sum_{\gamma^m \in Z(0,i)} \sum_{w \in W_k} \det(w)\, \overline{\chi(\gamma^m j_{(i)}(w))}$$

$$\cdot \prod_{p=1}^{s} \prod_{q \leq i_p} \frac{\sinh \mu_{pq}[\phi_{pq}(w) \pm (1 - m_{pq})\pi]}{\sinh(\mu_{pq}\pi)}.$$

The signs $\pm$ are chosen so that $|\phi_{pq}(w) \pm (1 - m_{pq})\pi| < \pi$ for all p, q. And

$$(6.23) \quad R_{t,(i)}(f) = \sum_{\gamma^m \in Z(0,i)} \sum_{w \in W_k} \det(w)$$

$$\cdot \int_{J^+_{(i)\,\mathfrak{p}}} S(\gamma^m j'_{(i)}(w); \, j_{\mathfrak{p}}) \, R_f^{(i)}(\gamma^m j_{(i)}(w) \, j_{\mathfrak{p}}) \, dj_{\mathfrak{p}}.$$

Proof. By (6.6), (6.7) and (6.12), we have

$$(6.24) \quad R_f^T(t, (i)) = \frac{(-1)^{r+1}}{(2\pi)^{|(i)|} \, |W_K(0, i)|} \cdot T(G; \, t; \, (i); \, \Phi_f^{(t)}).$$

Hence, the first term of (6.21) follows from Lemma 5.2 and Proposition 6.1. The second term of (6.21) follows from (4.6), the absolute convergence of the first series on the right-hand side of (5.8), and also the following improper integral [2]:

$$\int_0^\infty \frac{x^{\sqrt{-1}\,\mu}}{1 + 2x \cos(t) + x^2} \, dx = \frac{\pi \sinh(\mu t)}{\sin(t) \sinh(\mu \pi)},$$
$$0 < |t| < \pi.$$

7. **Computation of** $R_{t,(i)}(f)$. For $(k) > (i)$, $\delta \in \tilde{Z}(0, i)$, $j_k \in J^0_{(i)\,t}$ and $j_{\mathfrak{p}} \in J_{(i)\,\mathfrak{p}}$, let

$$R_f^{(i)}(\delta j_k j_{\mathfrak{p}}, \, (k))$$

$$(7.1) \qquad = (-1)^{r(i)+1} \sum_{\chi \in \hat{J}_{(i)\,t}} \overline{\chi(\delta j_k)}$$

$$\cdot \int_{i^{(i)}_{\mathfrak{p}}*} j_{\mathfrak{p}}^{-\sqrt{-1}\,\lambda} \, d\lambda \int_{G^{(k)}} f(g) \, \theta^{(i)}_{\chi, \lambda}(g) \, dg.$$

By using the Weyl integral formula, the expression (4.2) and considering the transformations s on $J_{(k)\,\mathfrak{p}}$ for all s in $W_R(0, k)$, we have

$$\int_{G^{(k)}} f(g) \, \theta^{(i)}_{\chi, \lambda}(g) \, dg$$

$$(7.2) \qquad = \frac{(-1)^{r(k)}}{|W_K(i, k)|} \sum_{\eta \in \tilde{Z}(0,i)} \chi(\eta)$$

$$\cdot \int_{L^0_{(i)} \cap J_{(k)}} \varepsilon_{(i,k)}(h) \, \Phi_f^{(k)}(\eta h) \, \Delta_{(i,k)}(h) \, \Psi_\chi(h_M) \, h_{J_\mathfrak{p}}^{\sqrt{-1}\,\lambda} \, dh.$$

Here, we have used the fact that every element of $J_{(k)}$ can be expressed uniquely as ηh, $\eta \in \tilde{Z}(0, i)$ and $h \in L^0_{(i)} \cap J_{(k)}$. For an element of the form, the following relation is always true:

$$(7.3) \qquad \frac{\varepsilon_{(0,k)}(\eta h)\, \Delta_{(0,k)}(\eta h)}{|\Delta_{(0,k)}(\eta h)|} = \xi_{\rho(i,i)}(\eta) \cdot \frac{\varepsilon_{(i,k)}(h)\, \Delta_{(i,k)}(h)}{|\Delta_{(i,k)}(h)|} \, .$$

By the orthogonality relations for finite groups, we have

$$(-1)^{r(i)+1} \cdot R_f^{(i)}(\delta\, j_k\, j_p,\ (k))$$

$$(7.4) \qquad = \frac{(-1)^{r(k)}\, |\tilde{Z}(0,i)|}{|W_K(i,k)|} \sum_{\tau \in L_{J_{(i)}}} \overline{\xi_\tau(j_k)}$$

$$\cdot \int_{i_p^{(i)*}} j_p^{-\sqrt{-1}\,\lambda}\, d\lambda \int_{J_{(i)\,p}} \P(j_p')(j_p')^{\sqrt{-1}\,\lambda}\, dj_p' \,,$$

where

$$(7.5) \qquad \P(j_p') = \int_{M_{(i)}^0 \cap J_{(k)}} \varepsilon_{(i,k)}(h_M)\, \Phi_f^{(k)}(\delta\, h_M\, j_p')\, \Delta_{(i,k)}(h_M)\, \Psi_x(h_M)\, dh_M \,.$$

Since the function $\P$ is compactly supported, and C^∞ on an open dense set whose complement in $J_{(i)\,p}$ is of measure zero, by the elementary Fourier analysis on Euclidean spaces, we see that the following equation is true for almost all j_p in $J_{(i)\,p}$:

$$\int_{i_p^{(i)*}} j_p^{-\sqrt{-1}\,\lambda}\, d\lambda \int_{J_{(i)\,p}} \P(j_p')(j_p')^{\sqrt{-1}\,\lambda}\, dj_p'$$

$$(7.6) \qquad = (2\pi)^{|(i)|}\, c_{(i)} \int_{M_{(i)}^0 \cap J_{(k)}} \varepsilon_{(i,k)}(h_M)\, \Phi_f^{(k)}(\delta\, h_M\, j_p)$$

$$\cdot \Delta_{(i,k)}(h_M)\, \Psi_x(h_M)\, d h_M \,.$$

Compare (7.6) with (6.7), it is easy to see that the integral on the right-hand side of (7.6) is a constant multiple of $T(M_{(i)}^0;\, j_k;\, (k);\, h)$ with h equal to the translate of $\Phi_f^{(k)}$ by δj_p. For each $u \in W_K(i,i)$, we write $u\, j_k = j_{(i,k)}(u)\, j_{(i,k)}'(u)$ as (6.3), where $j_{(i,k)}(u) \in J_{(k)\,t}^0$ and

$$(7.7) \qquad j_{(i,k)}'(u) = \exp\left[\sum_{p=1}^{s} \sum_{i_p < q \le k_p} \psi_{pq}(u)(X_{pq} - Y_{pq})\right],$$

$$|\psi_{pq}(u)| < \pi \,.$$

PROPOSITION 7.1. *Retain the above notations, then for almost all* j_p, *we have*

$$\frac{(-1)^{|(l)|-|(i)|+1}\, |W_K(i,l)|}{(2\pi)^{|(i)|}\, c_{(l)}\, \mathrm{vol}\,(J_{(l)\,t})} \cdot R_f^{(i)}(\delta\, j_k\, j_p,\ (l))$$

$$(7.8) \qquad = \sum_{r^m \in Z(i,l)} \sum_{u \in W_K^{(i,i)}} \det(u)$$

$$\cdot \int_X \Phi_f^{(l)}(\delta\, r^m\, j_{(i,l)}(u)\, h_p\, j_p) \cdot S(r^m\, j_{(i,l)}'(u);\, h_p)\, d h_p \,,$$

where $X = (M^0_{(i)} \cap J_{(l)})^+_p$ *and* $S(\gamma^m j'_{(i,l)}(u); h_p)$ *is defined as* (6.15).

Proof. It follows immediately from Proposition 6.1 and the previous remark. Note also that $\mathrm{vol}\,(J_{(l)\,t}) = |\tilde{Z}(0, i)|\,\mathrm{vol}\,(M^0_{(i)} \cap J_{(l)\,t})$.

THEOREM 7.2. *For each* $(i) > (0)$, *we have*

$$R_{t,(i)}(f)$$

$$(7.9)\qquad \begin{aligned} &= \sum_{(l)>(i)} \frac{(-1)^{|(l)-(i)|+1}\,|W_K(i, i)|}{(2\pi)^{|(l)-(i)|}\,|W_K(i, l)|} \cdot R_{t,(l)}(f) \\ &\quad + \sum_{(l)>(i)} \frac{(2\pi)^{|(i)|}\,(-1)^{r(i)+1}\,|W_K(i, i)|}{(-2\sqrt{-1})^{|(i)|}\,|W_K(i, l)|} \\ &\qquad \times \sum_{\chi \in \hat{\mathcal{J}}_{(l)\,t}} \int_{R^{|(l)|}} \theta^{(l)}_{\chi,(\mu)}(f) \cdot K_{t,(l)}(\chi, (\mu))\,d(\mu)_{(l)}. \end{aligned}$$

Proof. It follows immediately from (6.22), (6.23), Theorem 6.2, and Proposition 7.1.

8. **The Fourier transforms of the invariant integrals.** Using the results of Theorem 6.2 and Theorem 7.2, we can compute the Fourier transform of the distribution Λ_t by an induction process.

THEOREM 8.1. (*Fourier inversion of* Λ_t) *Let* t *be a regular element of* T. *Then for* $f \in C^\infty_c(G)$,

$$(8.1)\qquad \begin{aligned} \Phi^T_f(t) &= (-1)^r \sum_{\tau \in L_T} \overline{\xi_\tau(t)}\,\theta_\tau(f) \\ &\quad + \sum_{(i)>(0)} \frac{(\sqrt{-1}/2)^{|(i)|}\,(-1)^{r(i)}}{|W_K(0, i)|} \\ &\qquad \cdot \sum_{\chi \in \hat{\mathcal{J}}_{(i)\,t}} \int_{R^{|(i)|}} \theta^{(i)}_{\chi,(\mu)}(f)\,K_{t,(i)}(\chi, (\mu))\,d(\mu)_{(i)}. \end{aligned}$$

Proof. By Theorem 6.2 and Theorem 7.2, we see that

$$\begin{aligned} \Phi^T_f(t) &= (-1)^r \sum_{\tau \in L_T} \overline{\xi_\tau(t)}\,\theta_\tau(f) \\ &\quad + \sum_{(i)>(0)} a(0, i) \\ &\qquad \cdot \sum_{\chi \in \hat{\mathcal{J}}_{(i)\,t}} \int_{R^{|(i)|}} \theta^{(i)}_{\chi,(\mu)}(f) \cdot K_{t,(i)}(\chi, (\mu))\,d(\mu)_{(i)}, \end{aligned}$$

where, for $(k) > (i)$,

$$a(i, k) = \frac{(-1)^{r(k)} (-\sqrt{-1})^{|(k)-(i)|}}{2^{|(k)-(i)|}}$$

(8.2)

$$\times \sum_{\substack{(i)=(i^0)<\cdots<(i^p)=(k) \\ 1 \le p \le |(k)-(i)|}} \frac{(-1)^p \prod_{m=1}^{p-1} |W_K(i^m, i^m)|}{\prod_{m=1}^{p} |W_K(i^{m-1}, i^m)|}.$$

To complete the proof, we have to show that

(8.3) $$\sum_{\substack{(i)=(i^0)<\cdots<(i^p)=(k) \\ 1 \le p \le |(k)-(i)|}} \frac{(-1)^p \prod_{m=1}^{p-1} |W_K(i^m, i^m)|}{\prod_{m=1}^{p} |W_K(i^{m-1}, i^m)|} = \frac{(-1)^{|(k)-(i)|}}{W_K(i, k)|}.$$

Let $C(i, k)$ denote the product of $|W_K(i, k)|$ and the left-hand side of (8.3), we need two lemmas to prove that $C(i, k) = (-1)^{|(k)-(i)|}$.

LEMMA 8.2. *For* $(m) \ge (k) > (i)$, *we have*

(8.4) $$\frac{|W_K(i, m)|}{|W_K(k, m)|} = 2^{|(k)-(i)|} \cdot \frac{\prod_{p=1}^{s} (m_p - i_p)!}{\prod_{p=1}^{s} (m_p - k_p)!}.$$

Proof. Note that the group $W_K(i, k)$ is equal to the direct product of its subgroups corresponding to the simple factors of $\mathfrak{g}$ or G. Hence, it suffices to prove this lemma for the case that $\mathfrak{g}$ is simple. Recall that the real simple Lie algebras of real type $(A_1)^n$ are the following: A_1 I, A III, C II, D III, E III, E VII, and F II. These real Lie algebras have a common property [10]: the full Weyl group of the pair $(\mathfrak{g}_c, \mathfrak{a}_c)$ contains a subset consisting of elements which stabilize $\mathfrak{a}_\mathfrak{p}$, and the restrictions to $\mathfrak{a}_\mathfrak{p}$ of this subset is the symmetric group of the set $\{\alpha_1, \cdots, \alpha_n\}$. We have to show that for any one-to-one mapping σ of the set $\{1, \cdots, k\}$ into the set $\{1, \cdots, m\}$ with $\sigma(q) = q$ for $q = 1, \cdots, i$, there exists $s \in W_K(i, m)$ such that $s\alpha_q = \alpha_{\sigma(q)}$ for all $q = 1, \cdots, k$. Thus, $|W_K(i, m)|/|W_K(k, m)|$ is equal to the product of $2^{|(k)-(i)|}$ and the number of the mappings σ with the previous property. Here, the factor $2^{|(k)-(i)|}$ comes from the sign changes of the roots of the pair $(\mathfrak{l}_c^{(i)}, \mathfrak{j}_c^{(m)})$. Now, let σ be any mapping of $\{1, \cdots, k\}$ into $\{1, \cdots, m\}$ which is one-to-one and $\sigma(q) = q$ for $q = 1, \cdots, i$. By the previous remark on the full Weyl group of $(\mathfrak{g}_c, \mathfrak{a}_c)$, we see

that there exists w in $W_C(\mathfrak{g}, \mathfrak{a})$ such that $w(\mathfrak{a}_\mathfrak{p}) = \mathfrak{a}_\mathfrak{p}$, $w(\mathfrak{j}_\mathfrak{p}^{(m)}) = \mathfrak{j}_\mathfrak{p}^{(m)}$, and $w\alpha_q = \alpha_{\sigma(q)}$ for $q = 1, \cdots, k$. Since $w(\mathfrak{a}_\mathfrak{p}) = \mathfrak{a}_\mathfrak{p}$ implies that $w(\mathfrak{a}_i) = \mathfrak{a}_i$, by a theorem of Satake [11] (Proposition 1.1.3.3), there exists $x \in G$ such that $w = \text{Ad}(x)$ on $\mathfrak{a}_\mathfrak{p}$. Thus, $\text{Ad}(x)(\mathfrak{j}_i^{(m)})$ is a compact Cartan subalgebra of the reductive Lie algebra $\mathfrak{m}^{(m)}$, and hence there exists $y \in K \cap M_{(m)}^0$ such that $\text{Ad}(yx)(\mathfrak{j}_i^{(m)}) = \mathfrak{j}_i^{(m)}$. Obviously, yx normalizes $J_{(m)}$, and defines an element s in $W_K(0, m)$ which has the same action as w on $\mathfrak{j}_\mathfrak{p}^{(m)}$. Moreover, since s is the identity on $\mathfrak{j}_\mathfrak{p}^{(i)}$, it belongs, in fact, to $W_K(i, m)$.

LEMMA 8.3. $C(i, k) = (-1)^{|(k)-(i)|}$ *for all* $(k) > (i)$.

Proof.   For $(i) = (i^0) < (i^1) < \cdots < (i^p) = (k)$, we have

$$(8.5) \qquad \frac{|W_K(i^m, i^m)|}{|W_K(i^{m-1}, i^m)|} = \frac{1}{2^{|(i^m)-(i^{m-1})|} \prod\limits_{q=1}^{s} (i_q^m - i_q^{m-1})!}$$

$$\text{for } m = 1, \cdots, p\,;$$

$$(8.6) \qquad \frac{|W_K(i, k)|}{|W_K(i^{p-1}, i^p)|} = 2^{|(i^{p-1})-(i)|} \cdot \frac{\prod\limits_{q=1}^{s} (k_q - i_q)!}{\prod\limits_{q=1}^{s} (i_q^p - i_q^{p-1})!}.$$

Thus, we have

$$(8.7) \qquad C(i, k) = \sum_{\substack{(i)=(i^0)<\cdots<(i^p)=(k) \\ 1 \le p \le |(k)-(i)|}} (-1)^p$$

$$\cdot \left[\prod_{q=1}^{s} \frac{(k_q - i_q)!}{(i_q^1 - i_q^0)! \, (i_q^2 - i_q^1)! \cdots (i_q^p - i_q^{p-1})!} \right].$$

For each $1 \le p \le |(k) - (i)|$, the term in $C(i, k)$ corresponding to p is the coefficient of $x_1^{k_1 - i_1} x_2^{k_2 - i_2} \cdots x_s^{k_s - i_s}$ in the power series of the analytic function $(k_1 - i_1)! \, (k_2 - i_2)! \cdots (k_s - i_s)!$ $\cdot [1 - \exp(x_1 + x_2 + \cdots + x_s)]^p$. Therefore, $C(i, k)$ is equal to the coefficient of $x_1^{k_1 - i_1} x_2^{k_2 - i_2} \cdots x_s^{k_s - i_s}$ in the power series of the following analytic function

$$F(x_1, x_2, \cdots, x_s)$$
$$= (k_1 - i_1)! \, (k_2 - i_2)! \cdots (k_s - i_s)!$$
$$\cdot \exp(-x_1 - \cdots - x_s)$$
$$\cdot [1 - (1 - \exp(x_1 + \cdots + x_s))^{|(k)-(i)|+1}].$$

Thus, it is easy to show that $C(i, k) = (-1)^{|(k)-(i)|}$ by differentiation.

Now, we turn to the Fourier transform of the distribution $A^{(i)}_{\delta j_k j_p}$.

THEOREM 8.4. (*Fourier inversion of* $A^{(i)}_{\delta j_k j_p}$) *For* $\delta \in \widetilde{Z}(0, i)$, $j_k \in J^0_{(i)t}$, *and* $j_p \in J_{(i)p}$ *such that the inversion formula* (7.6) *holds, we have*

$$
\begin{aligned}
c_{(i)} &\cdot \mathrm{vol}\,(J_{(i)t}) \cdot (2\pi)^{|(i)|} \cdot \Phi^{(i)}_f(\delta j_k j_p) \\
&= (-1)^{r(i)} \cdot \sum_{\chi \in \hat{f}_{(i)t}} \overline{\chi(\delta j_k)} \int_{\mathrm{i}^{(i)}_p{}^*} \theta^{(i)}_{\chi,\lambda}(f) \cdot j_p^{-\sqrt{-1}\,\lambda}\, d\lambda \\
&\quad + \sum_{(l)>(i)} \left(\frac{\sqrt{-1}}{2} \right)^{|(l)-(i)|} \cdot \frac{(-1)^{r(l)}}{|W_K(i,l)|} \\
&\quad\quad \cdot \sum_{\chi \in \hat{f}_{(l)t}} \int_{\mathrm{j}^{(l)}_p{}^*} \theta^{(l)}_{\chi,(\mu)}(f) \cdot K_{\delta j_k j_p,(i),(l)}(\chi,(\mu))\, d(\mu)_{(l)},
\end{aligned}
\tag{8.8}
$$

where, if j_p *corresponds to the* $|(i)|$-*tuple* (x^0_{pq}), *then*

$$
\begin{aligned}
K_{\delta j_k j_p,(i),(l)}&(\chi,(\mu)) \\
&= \sum_{r^m \in Z(i,l)} \sum_{u \in W_K(i,i)} \det(u)\, \overline{\chi(\delta\, r^m\, j_{(i,l)}(u))} \\
&\quad \cdot \prod_{p=1}^{s} \left[\prod_{q \le i_p} \exp\left(-\sqrt{-1}\,\mu_{pq}\, x^0_{pq}\right) \right] \\
&\quad \cdot \left[\prod_{i_p < q \le l_p} \frac{\sinh \mu_{pq}(\psi_{pq}(u) \pm (1-m_{pq})\,\pi)}{\sinh(\mu_{pq}\,\pi)} \right].
\end{aligned}
\tag{8.9}
$$

Proof. For each $(l) > (i)$, we define $D^{(i)}_{\delta j_k j_p,(l)}$ and $R^{(i)}_{\delta j_k j_p,(l)}$ as follow:

$$
\begin{aligned}
c_{(i)} &\cdot \mathrm{vol}\,(J_{(i)t}) \cdot (2\pi)^{|(i)|} \cdot D^{(i)}_{\delta j_k j_p,(l)}(f) \\
&= \sum_{\chi \in \hat{f}_{(l)t}} \int_{\mathrm{i}^{(l)}_p{}^*} \theta^{(l)}_{\chi,(\mu)}(f) \cdot K_{\delta j_k j_p,(i),(l)}(\chi,(\mu))\, d(\mu)_{(l)}.
\end{aligned}
\tag{8.10}
$$

$$
\begin{aligned}
c_{(i)} &\cdot \mathrm{vol}\,(J_{(i)t}) \cdot (2\pi)^{|(i)|} \cdot R^{(i)}_{\delta j_k j_p,(l)}(f) \\
&= \sum_{r^m \in Z(i,l)} \sum_{u \in W_K(i,i)} \det(u) \\
&\quad \cdot \int_{(M^0_{(i)} \cap J_{(l)})^+_p} S(r^m\, j'_{(i,l)}(u);\, h_p) \\
&\quad \cdot R^{(l)}_f(\delta\, r^m\, j_{(i,l)}(u)\, h_p\, j_p)\, dh_p.
\end{aligned}
\tag{8.11}
$$

By (5.8), (7.1), (7.8), and using the techniques which we were employed in the proof of Theorem 6.2, we have the following recurring formulas:

$$R_f^{(i)}(\delta\, j_k\, j_p,\ (k))$$

$$(8.12) \qquad = \left(\frac{\sqrt{-1}}{2}\right)^{|(k)-(i)|} \cdot \frac{(-1)^{r(i)+1}}{|W_K(i,\ k)|} \cdot D_{\delta\, j_k\, j_p,\ (k)}^{(i)}(f)$$

$$+ \frac{(-1)^{|(k)-(i)|+1}}{(2\pi)^{|(k)-(i)|}\,|W_K(i,\ k)|} \cdot R_{\delta\, j_k\, j_p,\ (k)}^{(i)}(f)\,.$$

$$R_{\delta\, j_k\, j_p,\ (k)}^{(i)}(f)$$

$$(8.13) \qquad = \sum_{(l)>(k)} \frac{(2\pi)^{|(k)-(i)|}(-1)^{r(k)+1}\,|W_K(k,\ k)|}{(-2\sqrt{-1}\,)^{|(l)-(i)|}\,|W_K(k,\ l)|} \cdot D_{\delta\, j_k\, j_p,\ (l)}^{(i)}(f)$$

$$+ \sum_{(l)>(k)} \frac{(-1)^{|(l)-(k)|+1}\,|W_K(k,\ k)|}{(2\pi)^{|(l)-(k)|}\,|W_K(k,\ l)|} \cdot R_{\delta\, j_k\, j_p,\ (l)}^{(i)}(f)\,.$$

Using the recurring formulas (8.12) and (8.13), we see that

$$c_{(i)} \cdot \operatorname{vol}(J_{(i)\,t}) \cdot (2\pi)^{|(i)|} \cdot \Phi_f^{(i)}(\delta\, j_k\, j_p)$$

$$(8.14) \qquad = (-1)^{r(i)} \sum_{\chi\in\hat{J}_{(i)\,t}} \overline{\chi(\delta\, j_k)} \int_{\mathfrak{i}\,\mathfrak{p}^{(i)*}} \theta_{\chi,\lambda}^{(i)}(f) \cdot j_p^{-\sqrt{-1}\,\lambda}\, d\lambda$$

$$+ \sum_{(l)>(i)} a(i,\ l) \cdot D_{\delta\, j_k\, j_p,\ (l)}^{(i)}(f)\,.$$

Thus, the Theorem follows immediately from (8.2) and (8.3).

9. Plancherel formula for G.

For $f \in C_c^\infty(G)$, we have

$$(9.1) \qquad f(e) = M_G^{-1} \lim_{\substack{t\to e \\ t\in T'}} (\Pi^T\, \Phi_f^T)(t)\,,$$

where $M_G = (-1)^{(1/2)\dim(G/K)} \cdot (2\pi)^r$ and $\Pi^T = \prod_\alpha H_\alpha,\ \alpha\in\Phi^+(\mathfrak{g}_c,\ \mathfrak{t}_c)$. Thus, we apply Π^T to the function Φ_f^T at a point $t\in T'$ and compute the limit of $(\Pi^T\,\Phi_f^T)(t)$ as t approaches e through the regular elements in T.

THEOREM 9.1. (*Plancherel formula*) *If* $f\in C_c^\infty(G)$, *then*

$$M_G \cdot f(e)$$

$$= \sum_{\tau\in L_T'} \left[\prod_{\alpha\in\Phi^+(\mathfrak{g}_c,\ \mathfrak{t}_c)} (\tau,\ \alpha)\right] \theta_\tau(f)$$

$$(9.2) \qquad + \sum_{(i)>(0)} \left(\frac{\sqrt{-1}}{2}\right)^{|(i)|} \cdot \frac{|W_K(0,\ 0)|}{|W_K(0,\ i)|}$$

$$\cdot \sum_{\chi\in\hat{J}_{(i)\,t}} \int_{R^{|(i)|}} \theta_{\chi,(\mu)}^{(i)}(f)\, P_{(i)}(\chi,\ (\mu))\, Q_{(i)}(\chi,\ (\mu))\, d(\mu)\,,$$

where

$$
P_{(i)}(\chi, (\mu))
$$

(9.3)
$$
= \prod_{p=1}^{s} \prod_{q \leq i_p} \left[\frac{1}{2} (1 + \chi(r_{pq})) \coth \left(\frac{1}{2} \pi \mu_{pq} \right) \right.
$$
$$
\left. + \frac{1}{2} (1 - \chi(r_{pq})) \tanh \left(\frac{1}{2} \pi \mu_{pq} \right) \right].
$$

$$
Q_{(i)}(\chi, (\mu))
$$

(9.4)
$$
= \prod_{\alpha \in \Phi^+(\mathfrak{s}_c, \, i_c^{(i)})} \left[\log(\chi) + \frac{\sqrt{-1}}{2} \sum_{p=1}^{s} \sum_{q \leq i_p} \mu_{pq} \, \alpha_{pq}^{(i)}, \, H_\alpha \right].
$$

Proof. The contribution of the discrete series representations of G to the Plancherel formula for G is the first term on the right-hand side of (9.2) (see [9]).

For each $r^m \in Z(0, i)$ and each $|(i)|$-tuple (ε_{pq}), $\varepsilon_{pq} = 1$ or -1 for all p, q, let

(9.5)
$$
S_{m, \varepsilon}(t) = \prod_{p=1}^{s} \prod_{q \leq i_p} \overline{\chi(j_{(i)}(w))} \cdot \varepsilon_{pq}
$$
$$
\cdot \exp \left[\varepsilon_{pq} \, \mu_{pq} (\phi_{pq}(w) \pm (1 - m_{pq}) \pi) \right].
$$

Then it is easy to see that

(9.6)
$$
\left[\prod_{p=1}^{s} \prod_{q \leq i_p} \sinh (\pi \mu_{pq}) \right] K_{t, (i)}(\chi, (\mu))
$$
$$
= 2^{-|(i)|} \sum_{r^m \in Z(0, i)} \sum_{\varepsilon} \sum_{w \in W_k} \det(w) \, \chi(r^m) \, S_{m, \varepsilon}(t).
$$

Thus, we have to compute the differentiation of $S_{m, \varepsilon}$ under Π^T.

Let c be a positive number less than $\pi/2$ such that the only element in $J^0_{(i) \, t}$ which can be expressed in the form

$$
\exp \left[\sum_{p=1}^{s} \sum_{q \leq i_p} x_{pq} (X_{pq} - Y_{pq}) \right], \quad |x_{pq}| < 2c \quad \text{for all} \quad p, q,
$$

is the identity element. Then, let

$$
U = \left\{ j_k \exp \left[\sum_{p=1}^{s} \sum_{q \leq i_p} x_{pq} (X_{pq} - Y_{pq}) \right] \, \middle| \, j_k \in J^0_{(i) \, t}, \right.
$$
$$
\left. |x_{pq}| < c \quad \text{for all} \quad p, q \right\}.
$$

Then U is an open neighborhood of $J^0_{(i) \, t}$ and it is easy to see that every element of U is uniquely expressed in the above form. Since

we are interested in t in a small neighborhood of e, we assume that wt belongs to U for all w in W_k. Extending χ to U by $\chi(j_k\, j_k') = \chi(j_k)$, where $j_k\, j_k' \in U$ with j_k as its "$J^0_{(i)\,t}$-component", let $\log(\chi)$ be the differential of this extension. Let α'_{pq} be the transport to t of α_{pq} via $y_{(n)}$ for all p and q, then we have

$$
(9.7) \qquad S_{m,\varepsilon}(t) = \prod_{p=1}^{s} \prod_{q \leq i_p} \varepsilon_{pq}\, \overline{\chi(wt)}
$$

$$
\cdot \exp\left[\pm \varepsilon_{pq}\, \mu_{pq}(1 - m_{pq})\, \pi\right]\left[\xi_{\alpha'_{pq}}(wt)\right]^{\sqrt{-1}/2(\varepsilon_{pq}\mu_{pq})}.
$$

If $\alpha \in \Phi^+(\mathfrak{g}_c,\, \mathfrak{t}_c)$, then

$$
(9.8) \qquad (H_\alpha\, S_{m,\varepsilon})(t)
$$

$$
= \left[-\log(\chi) + \frac{\sqrt{-1}}{2} \sum_{p=1}^{s} \sum_{q \leq i_p} \varepsilon_{pq}\, \mu_{pq}\, \alpha'_{pq},\ w H_\alpha\right] \cdot S_{m,\varepsilon}(t).
$$

This implies that

$$
(9.9) \qquad
\begin{aligned}
&(\Pi^T S_{m,\varepsilon})(e) \\
&\qquad = (-1)^{r(i)} \det(w)\, Q_{(i)}(\chi,\, (\mu)) \\
&\qquad\qquad \cdot \prod_{p=1}^{s} \prod_{q \leq i_p} \left[\exp(\pm \varepsilon_{pq}\, \mu_{pq}(1 - m_{pq})\, \pi)\right].
\end{aligned}
$$

Therefore,

$$
(9.10) \qquad
\begin{aligned}
&\Pi^T[K_{\cdot,(i)}(\chi,\, (\mu))](e) \\
&\qquad = (-1)^{r(i)}\, |W_k|\, Q_{(i)}(\chi,\, (\mu)) \\
&\qquad\qquad \cdot \sum_{r^m \in Z(0,i)} \left[\prod_{p=1}^{s} \prod_{q \leq i_p} \frac{\cosh(\mu_{pq}(1 - m_{pq})\, \pi)}{\sinh(\pi\, \mu_{pq})}\right] \\
&\qquad = (-1)^{r(i)}\, |W_k|\, Q_{(i)}(\chi,\, (\mu)) \\
&\qquad\qquad \cdot \prod_{p=1}^{s} \prod_{q \leq i_p} \left[\coth(\pi\, \mu_{pq}) + \chi(r_{pq})\, \operatorname{csch}(\pi\, \mu_{pq})\right] \\
&\qquad = (-1)^{r(i)}\, |W_k|\, Q_{(i)}(\chi,\, (\mu))\, P_{(i)}(\chi,\, (\mu)).
\end{aligned}
$$

Thus, the Theorem follows from Theorem 8.1.

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