BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 9, Number 1, March 1981

PERMANENTAL AFFINE SUBSPACES OF GENERALIZED DOUBLY STOCHASTIC MATRICES

BY

KO-WEI LIH and EDWARD T. H. WANG[1]

Abstract. We construct, for each $n \geq 2$, an $n \times n$ matrix which has (i) $n-2$ independent variable entries; (ii) constant row and column sums; (iii) constant permanent. It is shown that this matrix generates an affine subspace of generalized doubly stochastic matrices on which the permanent is a constant. Specializing to nonnegative real entries with all row and column sums equal to one, we then obtain a simplex of $n \times n$ doubly stochastic matrices of dimension $n-2$ on which the permanent is a constant.

1. **Introduction.** Let Ω_n denote the convex polyhedron of all $n \times n$ doubly stochastic (d. s.) matrices; that is, $n \times n$ matrices with nonnegative real entries such that all row and column sums are equal to 1. The behaviour of the permanent function on Ω_n has been studied extensively over the past two decades. (See [4] for a comprehensive and up-to-date bibliography on this subject.) Recently, the second author [5] considered the problem of finding permanental pairs of d. s. matrices, i.e., distinct A and B in Ω_n such that $\operatorname{per}(\lambda A + (1-\lambda) B) = \text{constant}$ for all $\lambda \in [0, 1]$, and demonstrated some simple facts. A class of d. s. matrices which can not form a permanental pair with any other d. s. matrices was determined in [1]. Gibson [2] then generalized the notion to permanental polytopes and obtained some interesting results.

In the present paper, we consider generalized d. s. matrices; that is, matrices with real entries such that all row and column sums are equal to a constant. For $n \geq 2$, we shall construct a family of $(n-2)$-dimensional affine subspaces of $n \times n$ generalized d. s. matrices on each of which the permanent is a constant. This

Received by the editors March 21, 1980.

(1) Research supported by the National Research Council of Canada under grant NRC A-9121 and by the National Science Council of the Republic of China.

construction induces a simplex of $n \times n$ d.s. matrices which has dimension $n-2$ and on which the permanent is a constant.

Throughout, let R denote the real field and $\mathfrak{M}_n(R)$, the set of all real $n \times n$ matrices. For $A \in \mathfrak{M}_n(R)$, $A(i; j)$ is the submatrix obtained from A by deleting the ith row and the jth column. $\Lambda_n^r(R)$ is the set of all real $n \times n$ generalized d.s. matrices with constant row and column sums r. The Euclidean distance between two real matrices A and B is denoted by $d(A, B)$. Let $H(V_0, V_1, \cdots, V_k)$ be the convex polyhedron with vertices $V_0, V_1, \cdots, V_k$.

2. **The Matrices L_n.** In this section we define, for each $n \geq 2$, an $n \times n$ matrix L_n in $\mathfrak{M}_n(R)$ by an inductive procedure. All x_i's in the matrices to be defined are arbitrary but independent variables. First, let L_2 and L_3 be the matrices displayed below.

$$L_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} x_1 & 1 & 1-x_1 \\ 1-x_1 & 1 & x_1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Supposing that L_n has been defined, let L_n' be the matrix obtained from L_n by substituting $(1-x_n, 0, \cdots, 0, x_n)$ for the last row and let L_n'' be the matrix obtained from L_n by deleting the last row and adding $(x_n, 0, \cdots, 0, 1-x_n)$ as its first row. Thus the ith row of L_n'' is the $(i-1)$th row of L_n', $i = 2, 3, \cdots, n$. We display L_3' and L_3'' for illustration.

$$L_3' = \begin{pmatrix} x_1 & 1 & 1-x_1 \\ 1-x_1 & 1 & x_1 \\ 1-x_3 & 0 & x_3 \end{pmatrix}, \qquad L_3'' = \begin{pmatrix} x_3 & 0 & 1-x_3 \\ x_1 & 1 & 1-x_1 \\ 1-x_1 & 1 & x_1 \end{pmatrix}.$$

Now let L_{n+2} be the $(n+2) \times (n+2)$ matrix obtained by boarding L_n' on four sides as follows.

$$L_{n+2} = \left(\begin{array}{c|ccc|c} x_{n-1} & x_n & 0\cdots 0 & 1-x_n & 1-x_{n-1} \\ \hline 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & L_n' & & 0 \\ 1-x_{n-1} & & & & x_{n-1} \\ \hline 1 & 0 & 0\cdots 0 & 0 & 1 \end{array}\right).$$

The proof of the following lemma is trivial.

LEMMA 1. $L_n \in \Lambda_n^2(R)$ *for all* $n \geq 2$.

LEMMA 2. $\operatorname{per}(L_n') = \operatorname{per}(L_n'') = 1$ *for all* $n \geq 2$.

Proof. We employ induction on n. That $\operatorname{per}(L_2') = \operatorname{per}(L_2'') = \operatorname{per}(L_3') = \operatorname{per}(L_3'') = 1$ is trivial. Now suppose $\operatorname{per}(L_n') = \operatorname{per}(L_n'') = 1$. Then expanding along the last row of L_{n+2}', we get (cf. L_{n+2}' and L_{n+2}'' displayed below)

$$\begin{aligned} \operatorname{per}(L_{n+2}') &= (1-x_{n+2}) \operatorname{per}(L_{n+2}'(n+2;\ 1)) \\ &\qquad + x_{n+2} \operatorname{per}(L_{n+2}'(n+2;\ n+2)) \\ &= (1-x_{n+2})((1-x_{n-1}) \operatorname{per}(L_n') + x_{n-1} \operatorname{per}(L_n'')) \\ &\qquad + x_{n+2}(x_{n-1} \operatorname{per}(L_n') + (1-x_{n-1}) \operatorname{per}(L_n'')) \\ &= 1 . \end{aligned}$$

Similarly,

$$\begin{aligned} \operatorname{per}(L_{n+2}'') &= x_{n+2} \operatorname{per}(L_{n+2}''(1;\ 1)) \\ &\qquad + (1-x_{n+2}) \operatorname{per}(L_{n+2}''(1;\ n+2)) \\ &= x_{n+2}((1-x_{n-1}) \operatorname{per}(L_n') + x_{n-1} \operatorname{per}(L_n'')) \\ &\qquad + (1-x_{n+2})(x_{n-1} \operatorname{per}(L_n') \\ &\qquad + (1-x_{n-1}) \operatorname{per}(L_n'')) \\ &= 1 . \end{aligned}$$

The induction is thus complete.

$$L_{n+2}' = \begin{pmatrix} x_{n-1} & x_n & 0\cdots 0 & 1-x_n & 1-x_{n-1} \\ 0 & & & & 0 \\ \vdots & & L_n' & & \vdots \\ 0 & & & & 0 \\ 1-x_{n-1} & & & & x_{n-1} \\ 1-x_{n+2} & 0 & 0\cdots 0 & 0 & x_{n+2} \end{pmatrix},$$

$$L_{n+2}'' = \begin{pmatrix} x_{n+2} & 0 & 0\cdots 0 & 0 & 1-x_{n+2} \\ x_{n-1} & x_n & 0\cdots 0 & 1-x_n & 1-x_{n-1} \\ 0 & & & & 0 \\ \vdots & & L_n' & & \vdots \\ 0 & & & & 0 \\ 1-x_{n-1} & & & & x_{n-1} \end{pmatrix}.$$

LEMMA 3. $\operatorname{per}(L_n) = 2$ *for all* $n \geq 2$.

Proof. Straightforward computations show that $\text{per}(L_2) = \text{per}(L_3) = 2$. Now suppose $\text{per}(L_n) = 2$. Then, by Lemma 2,

$$\begin{aligned}\text{per}(L_{n+2}) &= \text{per}(L_{n+2}(n+2;\ 1)) + \text{per}(L_{n+2}(n+2;\ n+2)) \\ &= (1 - x_{n-1})\,\text{per}(L_n') + x_{n-1}\,\text{per}(L_n'') \\ &\qquad + x_{n-1}\,\text{per}(L_n') + (1 - x_{n-1})\,\text{per}(L_n'') \\ &= 2\,.\end{aligned}$$

The proof is thus complete by induction.

3. **Permanental affine subspaces.** Call an affine subspace of $\mathfrak{M}_n(R)$ a permanental affine subspace if the permanent function is a constant on it.

MAIN THEOREM. *For each* $n \geq 2$, *there is a permanental affine subspace* $S \subseteq \Lambda_n^2(R)$ *of dimension* $n-2$ *such that* $\text{per}(M) = 2$ *for all* $M \in S$.

Proof. Let S denote the set of all real $n \times n$ matrices of the form given by L_n in Section 2. Since $\sum \lambda_i(1 - x_i) = 1 - \sum \lambda_i x_i$ for all real λ_i's such that $\sum \lambda_i = 1$, S is an affine subspace of $\Lambda_n^2(R)$ in view of Lemma 1. Since $\text{per}(M) = 2$ for all $M \in S$ by Lemma 3, it remains to show that $\dim(S) = n-2$. First note that the entries of L_n include $n-2$ independent variables x_i's. For each $i = 1, 2, \cdots, n-2$, let W_i be the matrix obtained from L_n by setting $x_i = 1$ and $x_j = 0$ for all $j \neq i$, and let W_0 be the one obtained by setting $x_i = 0$ for all $i = 1, 2, \cdots, n-2$. Then it is clear that $W_0, W_1, \cdots, W_{n-2}$ are linearly independent. Furthermore, they span S since it is easy to see that $L_n = (1 - \sum_{i=1}^{n-2} x_i)\, W_0 + \sum_{i=1}^{n-2} x_i W_i$. Thus $\dim(S) = n-2$.

Let $r \in R$. Since $M \in \Lambda_n^2(R)$ if and only if $(r/2)M \in \Lambda_n^r(R)$, the following corollary is obvious.

COROLLARY 1. *For each* $n \geq 2$ *and any fixed* $r \in R$, *there is a permanental affine subspace* $S \subseteq \Lambda_n^r(R)$ *of dimension* $n-2$ *such that* $\text{per}(M) = r^n/2^{n-1}$ *for all* $M \in S$.

Call a simplex in $\mathfrak{M}_n(R)$ a permanental simplex if the permanent is a constant on it.

COROLLARY 2. *For each* $n \geq 2$, *there is a permanental simplex* $\mathfrak{H} = H(V_0, V_1, \cdots, V_{n-2})$ *of d.s. matrices such that* (i) $\dim(\mathfrak{H}) = n-2$; (ii) $\operatorname{per}(M) = 1/2^{n-1}$ *for all* $M \in \mathfrak{H}$; (iii) $d(V_i, V_j) = \sqrt{2}$ *for all* $i \neq j$, $i, j = 1, 2, \cdots, n-2$ *and* $d(V_0, V_i) = 1$ *for all* $i = 1, 2, \cdots, n-2$.

Proof. Since $L_n \in \Lambda_n^2(R)$, $L_n/2 \in \Omega_n$ if we restrict the values of the variables in L_n to the interval $[0, 1]$. Let $V_i = W_i/2$ for $i = 0, 1, \cdots, n-2$, and let $\mathfrak{H} = H(V_0, V_1, \cdots, V_{n-2})$. Since $V_1 - V_0$, $V_2 - V_0, \cdots, V_{n-2} - V_0$ are linearly independent, $\dim(\mathfrak{H}) = n-2$ follows from the Main Theorem. Since $\operatorname{per}(L_n/2) = \operatorname{per}(L_n)/2^n = 1/2^{n-1}$, (ii) is obvious. To see (iii), note that from the construction of W_i, and hence of V_i, it is clear that for distinct i and j, $i, j = 1, 2, \cdots, n-2$, V_i and V_j have exactly eight unequal entries while for each $i = 1, 2, \cdots, n-2$, V_0 and V_i have exactly four unequal entries. In a position where unequal entries appear, one of the entries is $1/2$ while the other is 0. (iii) now follows by simple computations.

Acknowledgment. This work is done when the second author is on sabbatical leave at the Institute of Mathematics, Academia Sinica, Republic of China. The hospitality of the Institute is greatly appreciated.

REFERENCES

1. J. Brenner and E. T. H. Wang, *Permanental pairs of doubly stochastic matrices*, II, Linear Algebra Appl., **28** (1979), 39–41.

2. P. M. Gibson, *Permanental polytopes of doubly stochastic matrices*, Linear Algebra Appl., **32** (1980), 87–111.

3. M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Prindle, Weber and Schmidt, Boston, 1969.

4. H. Minc, *Permanent*, Encyclopedia of Mathematics and Its Applications, **6**, Addison-Wesley, Reading, 1978.

5. E. T. H. Wang, *Permanental pairs of doubly stochastic matrices*, Amer. Math. Monthly, **85** (1978), 188–190.

ACADEMIA SINICA, TAIPEI, TAIWAN, R.O.C.
WILFRID LAURIER UNIVERSITY, WATERLOO, ONTARIO, CANADA.