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A SUFFICIENT CONDITION FOR CONVERGENCE OF $\lim_{t\to\pm\infty} t^{-1}\int_0^t \exp(-\tau T) A \exp(\tau S)\, d\tau$ FOR UNBOUNDED OPERATORS S AND T

BY

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To the memory of H. C. Wang

Abstract. Let T, S and A be linear operators in a Banach space X. In our work [1] we studied the existence of

$$\lim_{t\to\pm\infty} t^{-1}\int_0^t \exp(-\tau T) A \exp(\tau S)\, d\tau = \pi_\pm(A)$$

for bounded operators T, S and A. In this paper we continue our study of the same operator limits for unbounded operators T and S.

Introduction. In [1] we were able to characterize the domain, the range and the null space of the mapping $\pi_\pm$ under suitable assumptions on the bounded operators S and T. We then deduced the existence of operator limits

$$\lim_{t\to\pm\infty} \exp(-tT)\exp(tT+A), \tag{1}$$

as a consequence of ours by setting $T=S$. The limits in (1) has been studied by Ellis and Pinsky [2] when X is finite dimensional and T. Kato [3] when X is infinite dimensional.

Let $B(X)$ be the algebra of all bounded linear operators in X.

DEFINITION 1. Let M, α and β be real numbers with $M \geqq 1$. An operator T in X is said to belong to the class $G_+(M, \beta)$ if T generates a semigroup of operators $\exp(tT)$, $0 \leqq t < \infty$, of class C_0 and if $\|\exp(tT)\| \leqq Me^{\beta t}$, for all $t \in [0, \infty)$. Similarly, we say that T belongs to the class $G_-(M, \alpha)$ if the operator

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$-T \in G_+(M, -\alpha)$. We let $G(M, \alpha, \beta)$ denote the intersection of $G_+(M, \beta)$ and $G_-(M, \alpha)$.

We refer the readers to Hille and Phillip [4, Chapter 10] for precise definitions of semigroup of operators, its type, infinitesimal generator and their properties.

DEFINITION 2. Let $S \in G_+(M, \beta)$ and $T \in G_-(M, \alpha)$. Let $\Gamma_\pm = \{A \in B(X) : A_\pm \text{ exists}\}$, where $A_\pm$ denote the limits $\lim t^{-1} \int_0^t \exp(-\tau T) A \exp(\tau S)\, d\tau$ as $t \to \pm\infty$ in the uniform operator topology. $\pi_\pm$ denote the mappings from $\Gamma_\pm$ to $B(X)$ defined by $\pi_\pm(A) = A_\pm$. It is clear that $\Gamma_\pm$ are linear, subspaces of $B(X)$. $\Gamma_+(\Gamma_-)$ is closed if $\beta \leqq \alpha (\alpha \leqq \beta)$.

Our main results in [1] are as follows.

THEOREM. *Let $S \in G_+(M, \beta) \cap B(X)$ and $T \in G_-(M, \alpha) \cap B(X)$. Let $E(t)A = \exp(-tT) A \exp(tS)$ and $\Delta_{S,T}(A) = AS - TA$ for $A \in B(X)$. Let $R(Q)$ and $N(Q)$ be the range and the null space of an operator Q, respectively. Suppose $\beta \leqq \alpha$. Then we have*

(a) *Γ_+ is a closed subspace of $B(X)$.*

(b) *$E(t)\Gamma_+ \subset \Gamma_+$ and $\pi_+ E(t) = E(t)\pi_+ = \pi_+$ in Γ_+ for any $t \geqq 0$.*

(c) *π_+ is a bounded linear projection in Γ_+ with $\|\pi_+\| \leqq M^2$. Moreover, $R(\pi_+) = N(\Delta_{S,T})$ and $N(\pi_+) = R(\Delta_{S,T})^-$, where $R(\Delta_{S,T})^-$ is the closure of $R(\Delta_{S,T})$ in $B(X)$.*

(d) *$\Gamma_+ = N(\Delta_{S,T}) \oplus R(\Delta_{S,T})^-$.*

Similarly, if $S \in G_-(M, \alpha) \cap B(X)$ and $T \in G_+(M, \beta) \cap B(X)$ with $\alpha \geqq \beta$, then the results of our theorem remain true with all "+" signs replaced by "−" signs.

COROLLARY. *If $S, T \in G(M, \alpha, \alpha) \cap B(X)$, then we have $\Gamma_+ = \Gamma_- = N(\Delta_{S,T}) \oplus R(\Delta_{S,T})^-$ and $\pi_+ = \pi_-$. Moreover, if X is finite dimensional, then $\Gamma_\pm = B(X)$.*

The main results. When S and T are not bounded, the operator $AS - TA$ $(A \in B(X))$ is not defined for all $x \in X$, the natural domain of $AS - TA$ is $D(S) \cap A^{-1}(D(T))$, and the operator $AS - TA$ is in general unbounded ($D(S)$ denotes the domain of S, etc. $\cdots$). We are only interested in bounded operators which can be expressed in the form $AS - TA$, thus we shall restrict ourself to the case that $D(S) \cap A^{-1}(D(T))$ is dense in X and the closure $\overline{AS - TA} \in B(X)$. This leads to the following

DEFINITION 3. Let S, T be densely defined closed operators. Denote by $\hat{\Delta}_{S,T}$ the operator with maximal domain $D(\hat{\Delta}_{S,T}) = \{A \in B(X) : AS - TA$ is bounded with dense domain $D(S) \cap A^{-1}(D(T))$ in $X\}$, and define

$$\hat{\Delta}_{S,T}(A) = \overline{AS - TA} \in B(X).$$

DEFINITION 4. Let $\Delta_{S,T}$ be the restriction of $\hat{\Delta}_{S,T}$ to the domain

$$D(\Delta_{S,T}) = \{A \in D(\hat{\Delta}_{S,T}) : AD(S) \subset D(T)\}.$$

THEOREM 1. *If $S \in G_+(M, B)$ and $T \in G_-(M, \alpha)$ and if $\alpha \geq \beta$, then $R(\Delta_{S,T})^- \subset N(\pi_+)$; if $S \in G_-(M, \alpha)$ and $T \in G_+(M, \delta)$ and if $\alpha \geqq \delta$, then $R(\Delta_{S,T})^- \subset N(\pi_-)$.*

COROLLARY 2. *Let $S, T \in G(M, \alpha, \alpha)$. Then we have $R(\Delta_{S,T})^- \oplus N(\Delta_{S,T}) \subset \Gamma_+ \cap \Gamma_-$, and π_+ coincides with π_- there.*

Proof. Let $B \in R(\Delta_{S,T})$. Then $B = AS - TA$ for some $A \in D(\Delta_{S,T})$. Since for each fix $x \in D(S)$, $Ae^{tS}x \in AD(S) \subset D(T)$, we see that the derivative of $e^{-\tau T}Ae^{\tau S}x$ with respect to τ evaluated at $\tau = t$ is equal to

$$\begin{aligned}
&\lim_{\tau \to t} (\tau - t)^{-1} \{e^{-\tau T} A\, e^{\tau S} x - e^{-tT} A\, e^{tS} x\} \\
&\quad = \lim_{\tau \to t} \left\{ e^{-\tau T} A \frac{e^{\tau S} - e^{tS}}{\tau - t} x + \frac{e^{-\tau T} - e^{-tT}}{\tau - t} A\, e^{tS} x \right\} \\
&\quad = e^{-tT} AS\, e^{tS} x + e^{-tT}(-T)\, A\, e^{tS} x \\
&\quad = e^{-tT} B\, e^{tS} x.
\end{aligned}$$

Thus, by integration from 0 to t, we obtain

$$\frac{1}{t} \int_0^t e^{-\tau T} B\, e^{\tau S} x\, d\tau = \frac{1}{t} (e^{-tT} A\, e^{tS} x - Ax).$$

It follows that, for every $x \in D(S)$,

$$\left\| \frac{1}{t} \int_0^t e^{-\tau T} B\, e^{\tau S} x\, d\tau \right\| \leqq \frac{1}{t} (M^2 + 1) \|A\| \, \|x\|.$$

Now to each element $y \in X$, we can choose a sequence $\{x_n\}$ in $D(S)$ such that $\|x_n - y\| \leqq 1/n$ and $\|x_n\| \leqq 2\,\|y\|$. Then

$$\begin{aligned}&\left\|\frac{1}{t}\int_0^t e^{-\tau T}B\,e^{\tau S}\,y\,d\tau\right\|\\&\quad\leqq\left\|\frac{1}{t}\int_0^t e^{-\tau T}B\,e^{\tau S}\,x_n\,d\tau\right\|+\left\|\frac{1}{t}\int_0^t e^{-\tau T}B\,e^{\tau S}(y-x_n)\,d\tau\right\|\\&\quad\leqq\frac{1}{t}\,(M^2+1)\,\|A\|\,\|x_n\|+M^2\,\|B\|\,\|y-x_n\|\\&\quad\leqq\frac{2}{t}\,(M^2+1)\,\|A\|\,\|y\|+M^2\,\|B\|/n\end{aligned}$$

for all $n=1,2,\cdots$. By letting $n\to\infty$, we get

$$\left\|\frac{1}{t}\int_0^t e^{-\tau T}B\,e^{\tau S}\,y\,d\tau\right\|\leqq\frac{2}{t}\,(M^2+1)\,\|A\|\,\|y\|\,.$$

This implies that $\pi_+(B)=u-\lim_{t\to\infty}1/t\int_0^t e^{-\tau T}B\,e^{\tau S}\,d\tau=0$. That is $B\in N(\pi_+)$, hence $R(\Delta_{S,T})^-\subset N(\pi_+)$. The second part of Theorem 1 can be proved similarly. This completes the proof of Theorem 1. Corollary 2 follows immediately.

Next, we shall denote by $D_\lambda(\Delta_{S,T})$ the set

$$\{(\lambda-T)^{-1}A(\lambda-S)^{-1}:A\in B(X)\}\qquad\text{for each }\ \lambda\in\rho(S)\cap\rho(T)\,.$$

It can be checked easily that $D_\lambda(\Delta_{S,T})$ consists precisely of those $Q\in B(X)$ such that $R(Q)\subset D(T)$ and such that the operator $(\lambda-T)\,Q(\lambda-S)$ is bounded on $D(S)$. We have

THEOREM 3. *Let S and T be densely defined closed operators in X, and let $\lambda\in\rho(S)\cap\rho(T)$. Then $D_\lambda(\Delta_{S,T})\subset D(\Delta_{S,T})$ and*

$$R(\Delta_{(\lambda-S)^{-1},(\lambda-T)^{-1}})=R(\Delta_{S,T}\mid D_\lambda(\Delta_{S,T}))\,.$$

In particular, when S and T are bounded, we have $D_\lambda(\Delta_{S,T})=D(\Delta_{S,T})=B(X)$, and in this case $R(\Delta_{S,T})=R(\Delta_{(\lambda-S)^{-1},(\lambda-T)^{-1}})$.

REMARK. It follows that $D(\Delta_{S,T})$ contains the linear span of the set $\bigcup\{D_\lambda(\Delta_{S,T}):\lambda\in\rho(S)\cap\rho(T)\}$. Since for any T, closed densely defined, $D(\Delta_T)$ is an algebra with

$$\Delta_T(AB)=A\,\Delta_T(B)+\Delta_T(A)\,B\,,\qquad\text{for }\ A,B\in D(\Delta_T)\,.$$

Thus $D(\Delta_T)$ contains the linear algebra generated by $\bigcup\{D_\lambda(\Delta_T):\lambda\in\rho(S)\cap\rho(T)\}$. We also have

$$\operatorname{Span}\bigcup\{R(\Delta_{(\lambda-S)^{-1},(\lambda-T)^{-1}}):\lambda\in\rho(S)\cap\rho(T)\}\subset R(\Delta_{S,T})\,.$$

Theorem 3 is a consequence of the following

LEMMA 4. *Let S and T be densely defined closed operators in X and $\lambda \in \rho(S) \cap \rho(T)$. We have:*

(i) *If $A \in B(X)$, then $B = (\lambda - T)^{-1} A (\lambda - S)^{-1} \in D(\Delta_{S,T})$ with $R(B) \subset D(T)$ and $\Delta_{S,T}(B) = [\Delta_{(\lambda-S)^{-1},(\lambda-T)^{-1}}](A)$.*

(ii) *If $A \in D(\Delta_{S,T})$, then $\Delta_{S,T}(B) = (\lambda - T)^{-1} \Delta_{S,T}(A)(\lambda - S)^{-1}$.*

Proof. To each $x \in D(S)$, we have

$$\begin{aligned}\Delta_{S,T}(B)\,x &= (\lambda - T)^{-1} A (\lambda - S)^{-1} Sx \\ &\qquad - T(\lambda - T)^{-1} A (\lambda - S)^{-1} x \\ &= [\lambda(\lambda - T)^{-1} A(\lambda - S)^{-1} - (\lambda - T)^{-1} A]\,x \\ &\qquad - [\lambda(\lambda - T)^{-1} A (\lambda - S)^{-1} - A(\lambda - S)^{-1}]\,x \\ &= [A(\lambda - S)^{-1} - (\lambda - T)^{-1} A]\,x \\ &= [\Delta_{(\lambda-S)^{-1},(\lambda-T)^{-1}}]\,(A)\,x\,.\end{aligned}$$

Since $D(S)$ is dense in X and the right hand side of the identity is a bounded operator, we see that $B \in D(\Delta_{S,T})$ and $\Delta_{S,T}(B) = [\Delta_{(\lambda-S)^{-1},(\lambda-T)^{-1}}](A)$. This proves (i). Next, let $A \in D(\Delta_{S,T})$. Then $AD(S) \subset D(T)$ and $\Delta_{S,T}(A) = AS - TA$ on $D(S)$. To each $x \in D(S)$, we have

$$\begin{aligned}&(\lambda - T)^{-1} \Delta_{S,T}(A)(\lambda - S)^{-1} x \\ &\quad = (\lambda - T)^{-1} (AS - TA)(\lambda - S)^{-1} x \\ &\quad = (\lambda - T)^{-1} AS(\lambda - S)^{-1} x \\ &\qquad\quad - (\lambda - T)^{-1} TA(\lambda - S)^{-1} x \\ &\quad = (\lambda - T)^{-1} A(\lambda - S)^{-1} Sx \\ &\qquad\quad - T(\lambda - T)^{-1} A(\lambda - S)^{-1} x \\ &\quad = \Delta_{S,T}[(\lambda - T)^{-1} A(\lambda - S)^{-1}]\,x \\ &\quad = \Delta_{S,T}(B)\,x\,.\end{aligned}$$

Again, since both sides of the identity are bounded operators and $D(S)$ is dense in X, the identity holds for all $x \in X$. (ii) follows.

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