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A THEOREM ON PARTITION LATTICE AS A GEOMETRIC LATTICE

BY

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To the memory of H. C. Wang

1. **Statement of the result.** A set of points S consisting of at least $n+1$ points together with a family of subsets of S, named blocks, is called a partition of type $n(n \geq 1)$ if every n distinct points of S is contained in one and only one block and every block contains at least n distinct points [1]. The set $LP_n(S)$ of all partitions of type n on the same set S is known to be a complete, atomistic [1], meet-continuous [2] lattice called a partition lattice. The partial ordering $P_1 \leq P_2$ in $LP_n(S)$ for any two partitions P_1, P_2 is defined by the condition that every block of P_1 is contained in a block of P_2. Since every complete, atomistic and meet-continuous lattice is known to be a geometric lattice [3], the partition lattice $LP_n(S)$ is therefore a geometric lattice. That is, $LP_n(S)$ is a lattice isomorphic to the lattice of all subspaces (flats) of the merely finitary geometry $\langle S', C'\rangle$, where S' is the set of all atoms of $LP_n(S)$ and C' is the closure operation defined by C': $X \to C'(X) = \{P$: atom such that $p \leq \Sigma X$ (lattice sum)$\}$, for any subset X of S'.

The terms involved in here are defined as follows: A merely finitary geometry in the sense of Jonsson [4] is an ordered pair $\langle S, C\rangle$ consisting of a set S and a closure operation C which associates every subset X of S with another subset $C(X)$ such that the following conditions are satisfied:

(i) $X \subseteq C(X) = C(C(X))$ for every subset X of S,

(ii) $C(p) = p$ for every element $p \in S$,

(iii) $C(\phi) = \phi$, where ϕ is the empty set,

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(iv) for every subset X of S, $C(X)$ is the set union of all sets of the form $C(Y)$ with Y a finite subset of X.

A subset X of S is called a subspace (or flat) if $X = C(X)$ holds.

An atom of $LP_n(S)$ is a partition of type n on S which has only one block consisting of exactly $n+1$ distinct points of S, and every other block contains only n distinct points. If $\{x_1,\cdots, x_{n+1}\}$ is the $n+1$ distinct points of the unique block of the atom P_a, we denote this atom by $P_a = \{(x_1,\cdots, x_{n+1})\}$.

On the other hand, it is also shown by Hartmanis [1] that the partition lattice $LP_n(S)$ is isomorphic to the lattice $L(P_2G(S'))$ of all subspaces of the geometry $P_2G(S')$ in the sense of Hartmanis on the set S' of atoms of $LP_n(S)$. Actually. $P_2G(S')$ is a partition of type 2 on S' whose blocks are defined in the following way: For any two atoms $X = \{(x_1,\cdots, x_{n+1})\}$ and $Y = \{(y_1,\cdots, y_{n+1})\}$ of S', define the block (also called line in this case) $l(X, Y)$ in $P_2G(S')$ determined by X, Y to be the set of all atoms of $LP_n(S)$ which are $\leq X + Y$. Obviously.

$$X + Y = \begin{cases} \{(x_1,\cdots, x_n, x_{n+1}, y_{n+1})\} \\ \qquad \text{if } \{x_1,\cdots, x_n, x_{n+1}\} \cap \{y_1,\cdots, y_n, y_{n+1}\} \\ \qquad\qquad = \{x_1,\cdots, x_n\} = \{y_1,\cdots, y_n\} \\ \{(x_1,\cdots, x_n, x_{n+1}), (y_1,\cdots, y_n, y_{n+1})\} \\ \qquad \text{if } \{x_1,\cdots, x_n, x_{n+1}\} \cap \{y_1,\cdots, y_n, y_{n+1}\} \\ \qquad \text{contains less that } n \text{ points.} \end{cases}$$

The first case contains only one block which is non-trivial—consisting of at least $n+1$ distinct points. The second contains exactly two nontrival blocks.

A subset $T \subset S'$ is called a subspace if for any two atoms $X, Y \in T$, the line $l(X, Y) \subset T$.

It can be easily seen (proof is similar to the proof of the fact that a Wille geometry of grade n is a merely finitary geometry. [3, p. 16]) that $P_2G(S')$ is also a merely finitary geometry $\langle S', C\rangle$, where $C(X)$ is defined to be the least subspace containing the given subset X of S'.

Thus there are two seemingly different representations of $LP_n(S)$ as a geometric lattice; one through the geometry $\langle S', C'\rangle$ and the other through $P_2 G(S') = \langle S', C\rangle$. It is natural to raise the question: How are these two geometries related? It is intended in this note to show the following result concerning this question.

THEOREM. *The two seemingly different representations of the partition lattice $LP_n(S)$ as a geometric lattice are actually the same. That is, the two geometries $\langle S', C'\rangle$ and $P_2 G(S')=\langle S', C\rangle$ are the same.*

2. **Blocks of the partition $Q + P_a$.** As the preparation for the proof of this theorem, we need to study the blocks of the partition $Q + P_a$, where Q is any partition of type n on S and P_a is an atom of $LP_n(S)$ such that $P_a \not\leq Q$. Let $P_a = \{(x_1,\cdots, x_n, x_{n+1})\}$, and let $B'(y_1,\cdots, y_n)$ be the block of Q determined by the n distinct points $\{y_1,\cdots, y_n\}$. Define

$$B_1 = \bigcup_{i=1}^{n+1} B'(x_1,\cdots, \hat{x}_i,\cdots, x_{n+1}) \text{ (set union).}$$

where $\{x_1,\cdots, \hat{x}_i, \cdots, x_{n+1}\} = \{x_1, \cdots, x_{i-1}, x_{i+1},\cdots, x_{n+1}\}$. Then define

$$B_{i+1} = \bigcup B'(z_1,\cdots, z_n)$$

for all distinct $\{z_1,\cdots, z_n\} \subset B_i$ provided that B_i is already defined. It is obvious now that

$$\{x_1,\cdots, x_{n+1}\} \subset B_1 \subset B_2 \subset \cdots \subset B_i \subset B_{i+1} \subset \cdots.$$

Let

$$B = \bigcup_i B_i \text{ (set union).}$$

Now, for any n distinct points $\{y_1,\cdots, y_n\}$ of S we define

$$B(y_1,\cdots, y_n) = \begin{cases} B \text{ if } \{y_1,\cdots, y_n\} \subset B \\ B'(y_1,\cdots, y_n) \text{ if } \{y_1,\cdots, y_n\} \not\subset B. \end{cases}$$

Let P be the collection of all $B(y_1,\cdots, y_n)$ defined this way. Then P is a partition of type n on S.

In fact, let $\{z_1,\cdots, z_n\} \subset B(y_1,\cdots, y_n)$. Case 1. If $B(y_1,\cdots, y_n) = B$, then $\{z_1,\cdots, z_n\} \subset B$, so $B(z_1,\cdots, z_n) = B = B(y_1,\cdots, y_n)$. Case 2. If $B(y_1,\cdots, y_n) = B'(y_1,\cdots, y_n)$, then $\{z_1,\cdots, z_n\} \subset B'(y_1,\cdots, y_n)$, hence $B'(z_1,\cdots, z_n) = B'(y_1,\cdots, y_n)$. Furthermore, $\{z_1,\cdots, z_n\} \not\subset B$, sine otherwise, $\{z_1,\cdots, z_n\} \subset B$ would imply that there is a B_{i_0} such that $\{z_1,\cdots, z_n\} \subset B_{i_0}$ and $B'(y_1,\cdots, y_n)$

$= B'(z_1,\cdots, z_n) \subset B_{i_0} \subset B$. As such it contradicts the fact that $\{y_1,\cdots, y_n\} \not\subset B$. Since $\{z_1,\cdots, z_n\} \not\subset B$, we have $B(z_1,\cdots, z_n) = B'(z_1,\cdots, z_n) = B'(y_1,\cdots, y_n) = B(y_1,\cdots, y_n)$. Thus P is a partition of type n on S.

Now, *it can be shown that* $P = Q + P_a$. Obviously, $\{x_1,\cdots, x_{n+1}\} \subset B_i \subset B$. For any n distinct points $\{y_1,\cdots, y_n\}$, if $\{y_1,\cdots, y_n\} \subset B$, then as shown above $B'(y_1,\cdots, y_n) \subset B(y_1,\cdots, y_n) = B$. If $\{y_1,\cdots, y_n\} \not\subset B$, then $B'(y_1,\cdots, y_n) = B(y_1,\cdots, y_n)$. Thus P_a, $Q \le P$. Hence $P_a + Q \le P$. On the other hand, let $P_a + Q \le R$, and let $C(y_1,\cdots, y_n)$ be the block of R determined by the n distinct points $\{y_1,\cdots, y_n\}$. Since $P_a \le R$, $C(x_1,\cdots, x_n) \supset \{x_1,\cdots,x_n, x_{n+1}\}$, and since $Q \le R$ implies that $C(x_1,\cdots, x_n) = C(x_1,\cdots, \hat{x}_i,\cdots, x_{n+1}) \supset B'(x_1,\cdots, \hat{x}_i,\cdots, x_{n+1})$, we have $C(x_1,\cdots, x_n) \supset B_1$. Now, if $\{z_1,\cdots, z_n\} \subset B_1$, then $C(x_1,\cdots, x_n) = C(z_1,\cdots, z_n) \supset B'(z_1,\cdots, z_n)$, hence $C(x_1,\cdots, x_n) \supset B_2$. In this way we can show that $C(x_1,\cdots, x_n) \supset B_1, B_2,\cdots$ and hence $C(x_1,\cdots, x_n) \supset B$. Thus, if $\{y_1,\cdots, y_n\} \subset B$, then $B(y_1,\cdots, y_n) = B \subset C(x_1,\cdots, x_n)$. On the other hand, if $\{y_1,\cdots, y_n\} \not\subset B$, then $B(y_1,\cdots, y_n) = B'(y_1,\cdots, y_n) \subset C(y_1,\cdots, y_n)$, since $Q \le R$. Therefore, it is shown that $P \le R$. Thus $P = Q + P_a$.

3. **Equivalency of the two geometries.** For the proof of the theorem, we show that the two geometries $\langle S', C'\rangle$ and $P_2G(S') = \langle S', C\rangle$ are the same. To show this, we need the following:

LEMMA. *If $\langle S', C\rangle$ and $\langle S', C'\rangle$ are two merely finitary geometries, then $\langle S', C\rangle = \langle S', C'\rangle$ if and only if $C(K) = C'(K)$ for every finite subset K of S'.*

Proof. Let H be any subset of S'. Since $\langle S', C\rangle$ is a merely finitary geometry, $C(H) = \bigcup_\tau C(K_\tau)$ for all finite subsets K_τ of H. But, since $C(K_\tau) = C'(K_\tau)$ we have $C(H) = \bigcup_\tau C(K) = \bigcup_\tau C'(K) = C'(H)$. Q. E. D.

Proof of the equivalency of $\langle S', C'\rangle$ and $P_2G(S')$. Let $P \in LP_n(S)$ be a lattice join of finite number of atoms $P_\alpha (\alpha = 1,\cdots, k)$ such that $P_s \not\le \sum_{h=1}^{s-1} P_h$, $s = 2,\cdots, k$. Let $K = \{P_1,\cdots, P_k\}$ be the set of these atoms. Obviously, $C'(K) = \{P_a\text{: atom such that } P_a \le P = \sum_{\alpha=1}^{k} P_\alpha\} \supseteq C(K)$, since $C'(K)$ is a subspace of $P_2G(S')$. Thus, by the above lemma, we need only to show the proposition that *every atom* P_α *under* P *is contained in* $C(K)$. This can be done by induction on the number k of the atoms in K.

This proposition holds for $k=2$ by the definition of line in $P_2G(S')$. Assume now that it holds for $k-1$. Let $P_k \not\leq Q = \sum_{\beta=1}^{k-1} P_\beta$, then $P = Q + P_k$. Also, let $K' = \{P_1,\cdots, P_{k-1}\}$. Suppose that the atom $P_\alpha = \{(y_1,\cdots, y_{n+1})\} \leq P$. Then there is a block B_0 of the partition P which satisfies $B_0 \supset \{y_1,\cdots, y_{n+1}\}$. As a block of P, B_0 is either one of the following two types: 1) $B_0 = B'(y_1,\cdots, y_n)$, a block of Q not contained in B, or 2) $B_0 = B = \bigcup B_i$. In the case 1), $\{(y_1,\cdots, y_{n+1})\} \leq Q = \sum_{\beta=1}^{k-1} P_\beta$, so by the induction hypothesis, $P_\alpha = \{(y_1,\cdots, y_{n+1})\} \in C(K') \subset C(K)$. In the case 2), there is a B_{i_0} such that $\{y_1,\cdots, y_{n+1}\} \subset B_{i_0}$. Thus it suffices to show that *if* $\{y_1,\cdots, y_{n+1}\} \subset B_i$ *for any* $i = 1, 2,\cdots$, *then* $P_\alpha = \{(y_1,\cdots, y_{n+1})\} \subset C(K)$. This proposition can be shown by induction on i.

For $i=1$, $\{y_1,\cdots, y_{n+1}\} \subset B_1 = \bigcup_i B'(x_1,\cdots, \hat{x}_i,\cdots, x_{n+1})$. Without loss of generality, we can assume that $x_1 = y_1,\cdots, x_l = y_l$ and that $x_1,\cdots, x_l, x_{l+1},\cdots, x_{n+1}, y_{l+1},\cdots, y_{n+1}$ are distinct. For a fixed j between $l+1$ and $n+1$, let $y_j \in B'(x_1,\cdots, \hat{x}_i,\cdots, x_{n+1})$. Then $R_j = \{(x_1,\cdots, \hat{x}_i,\cdots, x_{n+1}, y_j)\} \leq Q$ hence $R_j \in C(K') \subset C(K)$ by the induction hypothesis on $k-1$. Since $P_k = \{(x_1,\cdots, x_{n+1})\} \in C(K)$, the line determined by P_k and R_j is of the form $\{(x_1,\cdots, x_{n+1}, y_j)\}$, and is contained in $C(K)$. Thus the atom $\{(x_1,\cdots, x_n, y_j)\} \in C(K)$, where j can be any integer in between $l+1$ and $n+1$. It then follows that the line $\{(x_1,\cdots, x_n, y_{n+1}, y_j)\} \subset C(K)$, and hence $\{(x_1,\cdots, x_{n-1}, y_{n+1}, y_j)\} \in C(K)$ for $j = l+1,\cdots, n$. By continuing similar argument, we can conclude that $\{(x_1,\cdots, x_{n-2}, y_{n+1}, y_n, y_k)\} \in C(K)$, $(k = l+1,\cdots, n-1),\cdots,(x_1,\cdots, x_l, y_{n+1},\cdots, y_{l+3}, y_m) \in C(K)$, $(m = l+1, l+2)$, and finally $\{(x_1,\cdots, x_l, y_{n+1},\cdots, y_{l+1})\} = \{(y_1,\cdots, y_n, y_{n+1})\} \in C(K)$.

Next, assume that $\{y_1,\cdots, y_{n+1}\} \subset B_i$ implies $\{(y_1,\cdots, y_{n+1})\} \subset C(K)$. Suppose now that the $n+1$ distinct points $\{y_1,\cdots, y_{n+1}\} \subset B_{i+1}$. We assume as above that $x_1 = y_1,\cdots, x_l = y_l$ and that $\{x_1,\cdots, x_n, y_{l+1},\cdots, y_{n+1}\}$ are distinct. Further, suppose that for a fixed j in $l+1 \leq j \leq n+1$, $y_j \in B'(z_1,\cdots, z_n)$ with $\{z_1,\cdots, z_n\} \subset B_i$. Then, if $y_j = z_h$ for an h $(h = 1,\cdots, n)$, we have $\{(x_1,\cdots, x_n, y_j)\} \subset C(K)$, by the induction hypothesis on i, since $\{x_1,\cdots, x_n, y_j = z_h\}$

$\subset B_i$. Suppose next that y_j is distinct from $z_1, \cdots, z_n$. Then, since $\{z_1, \cdots, z_n, y_i\} \subset B'(z_1, \cdots, z_n)$, a block of Q, we have $\{(z_1, \cdots, z_n, y_j)\} \leq Q$. Hence $\{(z_1, \cdots, z_n, y_j)\} \in C(K)$ by the inducton hypothesis on $k-1$. Now assume that $x_{1'} = z_1, \cdots, x_{m'} = z_m$ and that $\{z_1, \cdots, z_n, x_{(m+1)'}, \cdots, x_{n'}\}$ are distinct. Then, since these points are contained in B_i, by the induction hypothesis on i, the atoms $\{(z_1, \cdots, z_n, x_{(m+1)'})\}, \cdots, \{(z_1, \cdots, z_n, x_{n'})\} \in C(K)$. Also shown above is the fact that $\{(z_1, \cdots, z_n, y_i)\} \in C(K)$. By using the same argument as above in showing $\{(y_1, \cdots, y_n, y_{n+1})\} \in C(K)$, we can conclude that $\{(x_1, \cdots, x_n, y_i)\} \in C(K)$. Thus $\{(x_1, \cdots, x_n, y_i)\} \in C(K)$ for $j = l+1, \cdots, n+1$. From there, it follows that $\{(y_1, \cdots, y_n, y_{n+1})\} \in C(K)$.

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