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HYPERSURFACES OF SYMMETRIC SPACES

BY

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To the Memory of Hsien-Chung Wang

Abstract. Let M be an irreducible symmetric space. If we assume that M admits a *single* submanifold with a particular property, how much can we say about the ambient space M? This is the general problem this paper being interested on. Several results will be obtained in this respect. In particular, we will prove that if an irreducible symmetric space M admits a hypersurface N with a constant principal curvature of multiplicity $\geqq \dim N-1$, then M must be a sphere, a real projective space, a complex projective space or one of their non-compact duals.

1. **Introduction.** For a long period, many differential geometers have been interested in and studied the theory of submanifolds in the following manner. Namely, they first assume that the submanifolds lie in a *fixed* model space, (e.g. a real- or complex—space—form, a conformally flat space, etc...), and then they try to investigate or classify such submanifolds with imposed conditions. Many interesting and beautiful results have been obtained in this direction.

In this paper, we would like to propose the following general problem in a "reversed" way as follows. "Let M be a Riemannian (or Kaehlerian, or symmetric, etc...) space. If we assume that M admits a single submanifold with a particular property, how much can we say about the ambient space?" In other words, "*we want to know the implications on the ambient space from the existence of a single submanifold.*" To quote just one example, we recall that if a Riemannian manifold admits an extrinsic n-sphere, then it admits a totally geodesic submanifold of dimension $n+1$ [3]. It seems to the authors that problems in this direction are not yet well-studied. However, they believe that many interesting results

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in this direction can be done.

In this paper, we shall treat this general problem among the class of (locally) symmetric spaces. In particular, we would like to ask and study the following problem.

PROBLEM. How many irreducible symmetric spaces admit a quasiumbilical hypersurface?

By a *quasiumbilical hypersurface* N we mean a hypersurface which has a principal curvature of multiplicity $\geqq \dim N - 1$. Our method in this paper works as follows. First, suppose that M is a symmetric space which admits a particular hypersurface. Then, we try to obtain some algebraic or geometric restrictions on the ambient space. Finally, we shall use these restrictions to find out all possible ambient spaces.

In §2, we shall state basic facts on symmetric spaces for later use and fix our notations.

In §3, we give some lemmas and state three main problems. Among them, Lemma 3.3 generalizes several results of various authors.

In §4, we shall prove that spheres, real projective spaces and their noncompact duals are the only irreducible symmetric spaces in which we can find hypercylinders. By a *hypercylinder* we mean a hypersurface N with 0 as its principal curvature with multiplicity $\geqq n-1$, $n = \dim N$.

In §5, we prove that spheres, real and complex projective spaces, and their noncompact duals are the only irreducible symmetric spaces in which we can find quasihyperspheres. By a *quasihypersphere* we mean a hypersurface with a constant principal curvature of multiplicity $\geqq n-1$.

In §6, we shall classify irreducible Hermitian symmetric spaces which admit $J\xi$-quasiumbilical hypersurfaces.

In §7, §8 §9, we shall classify irreducible symmetric spaces which admit locally symmetric, conformally flat, or Einstein quasiumbillical hypersurfaces. Several remarks will be given in the last section.

2. **Symmetric Spaces.** In this section we shall state basic facts on symmetric spaces for later use and fix our notations. Most of

these facts can be found in Helgason's book [9], (see also [5, 6, 11]).

An isometry s of a Riemannian manifold is called *involutive* if its iterate $s^2 = s \circ s$ is the identity map. A connected Riemannian manifold M is called a *symmetric space* if, for each point p of M, there exists an involutive isometry s_p of M such that p is an isolated fixed point of s_p. We call such s_p the *symmetry* of M at p. In the following we shall denote by G the closure of the group of isometries generated by $\{s_p : p \in M\}$ in the compact-open topology. Then G acts transitively on the symmetric space M; hence the typical isotropy subgroup H, say at o, is compact and $M = G/H$. We state some basic facts in the folowing lemmas without proof.

LEMMA 2.1. *s_0 induces -1 on the tangent space T_0M of M at o.*

LEMMA 2.2. *s_0 gives rise to an involutive automorphism σ of G by $\sigma(g) = s \circ g \circ s$. Moreover, σ induces an involutive automorphism of the Lie algebra $\mathfrak{G}$ of G which we denote by the same letter.*

Since σ is involutive on $\mathfrak{G}$, its eigenvalues are 1 and -1. It is clear that the Lie algebra $\mathfrak{H}$ of H is the eigenspace for $+1$. Let $\mathfrak{M}$ be the eigenspace for -1. Then we have

LEMMA 2.3. $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$, *and* $\mathfrak{H} = [\mathfrak{M}, \mathfrak{M}]$.

We call such decomposition the *Cartan decomposition.* It is clear that the members of $\mathfrak{H}$ vanish at o as vector fields on M.

LEMMA 2.4. *The space $\mathfrak{M}$ consists of the Killing vector fields X whose covariant derivative vanish at o; in particular, the evaluation map at o gives a linear isomorphism of $\mathfrak{M}$ onto $T_0M : X \mapsto X(0)$.*

From Lemma 2.4 follows immediately the next formula for the curvature tensor R at o.

LEMMA 2.5. $R(X, Y)Z = -[[X, Y], Z]$, *for* $X, Y, Z \in \mathfrak{M}$.

The famous criterion of E. Cartan is given by the following.

LEMMA 2.6. *A linear subspace L of the tangent space T_0M to a symmetric space M is the tangent space to some totally geodesic submanifold N of M if and only if L satisfies the condition* $[[\mathfrak{N}, \mathfrak{N}], \mathfrak{N}] \subset \mathfrak{N}$, *where*

$$\mathfrak{N} = \{X \in \mathfrak{M} : X(0) \in L\} .$$

A symmetric space $M = G/H$ is *irreducible* (referring to the restricted linear holonomy group) if and only if the identity component of the (linear) isotropy group H is irreducible, or equivalently, if and only if its Lie algebra $\mathfrak{H}$ is irreducible on $\mathfrak{M}$.

LEMMA 2.7. *If M is an irreducible symmetric space, then the sectional curvature $K(X, Y)$ for orthonormal vectors $X, Y \in \mathfrak{M}$ is given by*

$$K(X, Y) = \pm g([X, Y], [X, Y]),$$

where + or − is taken according as M is compact or not, and g is the metric tensor induced from the Killing-Cartan form. Moreover, M is Einsteinian.

The following result follows easily from the (M_+, M_-)-theory of [6].

LEMMA 2.8. *The only irreducible symmetric spaces which admit totally geodesic hypersurfaces are spheres, real projective spaces and their noncompact duals.*

Theorem 1 of [4] and a result of the first author give the following.

LEMMA 2.9. *The only irreducible Hermitian symmetric space which admits a totally umbilical hypersurface is the complex projective line. And the only irreducible symmetric spaces which admit totally umbilical hypersurfaces are spheres, real projective spaces and their noncompact duals.*

The following lemma is well-known.

LEMMA 3.10. *Every symmetric space is locally symmetric that is, its curvature tensor is parallel.*

We need also the following result [3] for later use.

LEMMA 2.11. *Let M be a symmetric space. If M admits an extrinsic n-sphere N, then M admits an $(n+1)$-dimensional totally geodesic submanifold $\tilde{N}$ of constant sectional curvature such that N is contained in N as an extrinsic hypersphere.*

By an *extrinsic sphere* we mean a totally umbilical submanifold with nonzero parallel mean curvature vector. A hypersurface is an extrinsic sphere if and only if it is totally umbilical with nonzero constant mean curvature.

The maximal dimensions of totally geodesic submanifolds of constant sectional curvature in symmetric spaces have been obtained in Table VI of [6].

3. **Lemmas and Problems.** Let N be an n-dimensional submanifold of an m-dimensional Riemannian manifold M. And let ∇ and ∇' be the covariant differentiations on N and M, respectively. Then the second fudamental form σ of the immersion is given by

$$\sigma(X, Y) = \nabla'_X Y - \nabla_X Y \tag{3.1}$$

for vector fields X, Y tangent to N, where σ is a normal-bundle-valued symmetric 2-form on N. For a vector field ξ normal to N, we write

$$\nabla'_X \xi = -A_\xi X + D_X \xi, \tag{3.2}$$

where $-A_\xi X$ and $D_X \xi$ denote the tangential and normal components of $\nabla'_X \xi$, respectively. For the second fundamental form σ we define the covariant derivative, denoted by $\overline{\nabla}_X \sigma$, to be

$$(\overline{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{3.3}$$

for vector fields X, Y, Z tangent to N. The equations of Gauss and Codazzi are then given respectively by [2],

$$\begin{aligned} R'(X, Y; Z, W) &= R(X, Y; Z, W) \\ &\quad + g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)), \end{aligned} \tag{3.4}$$

$$R(X, Y; Z, \xi) = g((\overline{\nabla}_X \sigma)(Y, Z), \xi) - g((\overline{\nabla}_Y \sigma)(X, Z), \xi), \tag{3.5}$$

for vector fields X, Y, Z, W tangent to N and ξ normal to N, where R' and R are the curvature tensors of N and M, respectively, and $R(X, Y; Z, W) = g(R(X, Y)Z, W)$. For orthonormal vectors X, Y in M, the *sectional curvature* $K(X, Y)$ of the plane section spanned by X, Y is given by

$$K(X, Y) = R(X, Y; Y, X). \tag{3.6}$$

We give the following general lemma for later use.

LEMMA 3.1. *Let N be a Riemannian manifold and η a unit*

1-form on N. We put

$$\mathcal{D} = \{X \in TN \mid \eta(X) = 0\}. \tag{3.7}$$

Then the distribution $\mathcal{D}$ *is integrable if and only if*

$$(\nabla_X \eta)(Y) = (\nabla_Y \eta)(X) \text{ for all } X, Y \in \mathcal{D}.$$

If $\mathcal{D}$ *is integrable, then the second fundamental form* $\tilde{\sigma}$ *of any maximal integral submanifold* $\tilde{N}$ *in* N *is given by*

$$\tilde{\sigma}(X, Y) = -(\nabla_X \eta)(Y)\bar{\eta}, \tag{3.8}$$

where $\eta(X) = g(X, \bar{\eta})$, *that is,* $\bar{\eta}$ *is the unit vector associated with the unit 1-form* η *on* N.

Proof. For any vector fields X, Y in $\mathcal{D}$, we have

$$0 = X\eta(Y) = (\nabla_X \eta)(Y) + \eta(\nabla_X Y),$$

from which we see that $[X, Y] \in \mathcal{D}$ if and only if $(\nabla_X \eta)(Y) = (\nabla_Y \eta)(Y)$. This proves the first part of the lemma.

If $\mathcal{D}$ is integrable, let $\tilde{N}$ be a maximal integral submanifold of $\mathcal{D}$. Then we have

$$(\nabla_X \eta)(Y) = -\eta(\nabla_X Y) = -\eta(\tilde{\sigma}(X, Y)) = -\tilde{h}(X, Y),$$

where $\tilde{\sigma}(X, Y) = \tilde{h}(X, Y)\bar{\eta}$. This shows (3.8). (Q. E. D.)

If the codimension of a submanifold N in an $(n+1)$-dimensional Riemannian manifold M is one, then the second fundamental form σ can be written as

$$\sigma(X, Y) = h(X, Y)\xi \tag{3.9}$$

where ξ is a unit vector field normal to N. In this case h is a scalar-valued symmetric 2-form on N. We shall also call it the second fundamental form for simplicity. Let $U = \{p \in N \mid A_\xi$ has at least two distinct eigenvalues at $p\}$. Then U is an open subset of N. If there exist, on U, two functions α, β and a unit 1-form ω such that

$$h = \alpha g + \beta\omega \otimes \omega, \tag{3.10}$$

on U, then N is called a *quasiumbilical hypersurface* [2, 7, 8]. In particular, if $\alpha = 0$ identically, N is called a *cylindrical hypersurface*, or simply, a *hypercylinder*. If $\alpha = \beta = 0$ identically, N is called a *totally geodesic hypersurface*. If $\beta = 0$ identically, N is called a *umbilical hypersurface*. A umbilical hypersurface with constant α is called an *extrinsic hypersphere*. Finally, a quasiumbilical hypersurface with constant α is called an *extrinsic quasihypersphere*, which will simply be called by *quasihypersphere* or *tube*

The following lemma is well-known.

LEMMA 3.2. *If M is an $(n+1)$-sphere S^{n+1}, a real projective $(n+1)$-space RP^{n+1}, or one of their noncompact duals, then for each $p \in M$ and each hyperplane V in $T_p M$, there exist a hypercylinder and a quasihypersphere in M through p such that their tangent spaces at p are V.*

Let $\bar{\omega}$ be the unit vector associated with the unit 1-form ω. We call $\bar{\omega}$ the *distinguished direction* of the quasiumbilical hypersurface N (with the second fundamental form given by (3.10)). Sometimes we call a quasiumbilical hypersurface with distinguished direction Z as a *Z-quasiumbilical hypersurface*.

Let (M, g, J) be a Kaehler manifold with complex structure J. For any unit vector X, the holomorphic sectional curvature $K(X, JX)$ is the sectional curvature of the holomorphic section $X \wedge JX$. A Kaehler manifold of constant holomorphic sectional curvature is called a *complex-space-form*. Complex projective spaces and their noncompact duals are complex-space-forms. It is proved in Theorem 6.1 that every $J\xi$-quasiumbilical hypersurface N in any irreducible Hermitian symmetric space of dimension >2 is a quasihypersphere, where ξ is a unit normal vector field of N. The following lemma generalizes main results of [12, 13, 14].

LEMMA 3.3. *If M is a complex-space-form, then for any point $p \in M$ and any hyperplane V of $T_p M$, there exists a $J\xi$-quasiumbilical hypersurface N through p with $T_p N = V$. Conversely, if M is a $2n$-dimensional $(n \geqq 3)$ Kaehler manifold such that, for any $p \in M$ and any hyperplane V of $T_p M$, there exists a $J\xi$-quasiumbilical hypersurface through p with V as its tangent space at p, then M is a complex-space-form.*

Proof. The existence was proved in [13]. Now suppose that M is a Kaehler manifold such that for each $p \in M$ and each unit vector ξ in $T_p M$, there exists a $J\xi$-quasiumbilical hypersurface N with ξ as its normal at p and whose second fundamental form h satisfies

$$h = \alpha g + \beta \omega \otimes \omega \tag{3.11}$$

with $\bar{\omega} = J\xi$. Thus equation (3.5) of Codazzi reduces to

$$\begin{aligned} R(X, Y; Z, \xi) = (X\alpha)g(Y, Z) &+ (X\beta)\omega(Y)\omega(Z) \\ &+ \beta(\nabla_X \omega)(Y)\omega(Z) + \beta\omega(Y)(\nabla_X \omega)(Z) \\ &- (Y\alpha)g(X, Z) - (Y\beta)\omega(X)\omega(Z) \\ &- \beta(\nabla_Y \omega)(X)\omega(Z) - \beta\omega(X)(\nabla_Y \omega)(Z). \end{aligned} \tag{3.12}$$

Hence, for vectors X, Y, Z perpendicular to ξ and $J\xi$ with $g(X, Z) = g(Y, Z) = 0$, we have

$$R(X, Y; Z, \xi) = 0. \tag{3.13}$$

In particular, if $\{X, Z\}$ is totally real, that is, $g(X, Z) = g(JX, Z) = 0$, we have

$$R(X, JX; Z, \xi) = 0. \tag{3.14}$$

Since this is true for any $\xi \in T_p M$, (3.14) implies

$$R(X, JX; \xi + Z, J\xi - JZ) = 0, \tag{3.15}$$

from which we find that the bisectional curvature $B(X, \xi) = R(X, JX; \xi, J\xi)$ satisfies

$$R(X, JX; \xi, J\xi) = R(X, JX; Z, JZ), \tag{3.16}$$

for totally real sections $X \wedge \xi$ and $X \wedge Z$. Since the complex dimension of M is $\geqq 3$, (3.16) implies that the totally real bisectional curvatures of totally real sections are independent of the choice of totally real sections at each point. Thus, by a theorem of Houh [10], we know that M is a complex-space-form. (Q.E.D.)

From Lemmas 3.2 and 3.3 we know that spheres, real and complex projective spaces, and their noncompact duals are irreducible symmetric spaces which admit many quasiumbilical hypersurfaces. Therefore it seems to be natural and interesting to ask the following problems.

PROBLEM 1. *How many irreducible symmetric spaces contain hypercylinders?*

PROBLEM 2. *How many irreducible symmetric spaces contain quasihyperspheres?*

PROBLEM 3. *How many irreducible symmetric spaces contain quasiumbilical hypersurfaces?*

In §4 and §5 we shall give complete answers to Problems 1 and 2. At this moment we cannot solve Problem 3. However we believe that the following conjecture is true.

CONJECTURE. *Spheres, real projective spaces, complex projective spaces and their noncompact duals are the only irreducible symmetric spaces in which we can find quasiumbilical hypersurfaces.*

In §§6, 7, 8 and 9 we shall give many supports to this conjecture.

For the later use we shall also give the following intrinsic characterization of hypercylinder.

LEMMA 3.4. *Let $\tilde{N}$ be a hypersurface in a Riemannian manifold N. Then $\tilde{N}$ is a hypercylinder in N if and only if the curvature tensors R and $\tilde{R}$ of N and $\tilde{N}$ satisfy $R=\tilde{R}$ on $T\tilde{N}$, that is $R(X, Y; Z, W)=\tilde{R}(X, Y; Z, W)$ for all vectors X, Y, Z, W tangent to $\tilde{N}$.*

This lemma follows trivially from equation (3.4) of Gauss.

4. **Answer to Problem 1.** The main purpose of this section is to prove the following *classification theorem* which gives a complete answer to Problem 1.

THEOREM 4.1. *Spheres, real projective spaces and their noncompact duals are the only irreducible symmetric spaces in which we can find hypercylinders.*

Let M be a symmetric space. Suppose that N is a hypercylinder in M. Then either N is totally geodesic in M or there exists a nonempty subset U such that the second fundamental form h of N in M is given by

$$h=\beta\omega\otimes\omega \tag{4.1}$$

where $\beta\neq 0$ on U and ω is a unit 1-form. If the first case occurs and M is irreducible, then Lemma 2.8 says that M is a sphere or a real projective space or one of their noncompact duals. Consequently, we may consider only the second case. Since our study is local, we may also assume that $U=N$ and β is *nowhere zero*. The equations (3.4) and (3.5) then reduce respectively to

$$R'(X, Y; Z, W) = R(X, Y; Z, W), \tag{4.2}$$

$$\begin{aligned} R(X, Y; Z, \xi) = (X\beta)\omega(Y)\omega(Z) &+ \beta(\nabla_X \omega)(Y)\omega(Z) \\ &+ \beta\omega(Y)(\nabla_X \omega)(Z) - (Y\beta)\omega(X)\omega(Z) \\ &- \beta(\nabla_Y \omega)(X)\omega(Z) - \beta\omega(X)(\nabla_Y \omega)(Z), \end{aligned} \tag{4.3}$$

for vector fields X, Y, Z, W tangent to N and ξ normal to N.

Now, let $\mathcal{D}$ be the distribution which annihilates ω, that is,

$$\mathcal{D} = \{X \in TN \mid \omega(X) = 0\}. \tag{4.4}$$

Then for vector fields X, Y, Z in $\mathcal{D}$, (4.3) gives

$$R(X, Y; Z, \xi) = 0. \tag{4.5}$$

Since M is symmetric, for any vector W tangent to N, Lemma 2.7, (4.1) and (4.5) give

$$\begin{aligned} &R(\nabla_W X, Y; Z, \xi) + R(X, \nabla_W Y; Z, \xi) \\ &\quad + R(X, Y; \nabla_W Z, \xi) - R(X, Y; Z, A_\xi W) = 0. \end{aligned} \tag{4.6}$$

If X, Y, Z, W are in $\mathcal{D}$, (4.1), (4.3) and (4.6) imply

$$\begin{aligned} &\omega(\nabla_W X)(\nabla_Y \omega)(Z) - \omega(\nabla_W Y)(\nabla_X \omega)(Z) \\ &\quad + (\nabla_Y \omega)(X)\omega(\nabla_W Z) \\ &\quad - (\nabla_X \omega)(Y)\omega(\nabla_W Z) = 0. \end{aligned} \tag{4.7}$$

Since $0 = W\omega(X) = \omega(\nabla_W X) + (\nabla_W \omega)(X)$, (4.7) gives

$$\begin{aligned} &(\nabla_X \omega)(Y)(\nabla_W \omega)(Z) + (\nabla_X \omega)(Z)(\nabla_W \omega)(Y) \\ &\quad = (\nabla_Y \omega)(X)(\nabla_W \omega)(Z) + (\nabla_Y \omega)(Z)(\nabla_W \omega)(X). \end{aligned} \tag{4.8}$$

Let $Y = Z = W$. (4.8) shows that

$$(\nabla_Y \omega)(Y)[(\nabla_X \omega)(Y) - (\nabla_Y \omega)(X)] = 0. \tag{4.9}$$

If there exists X, Y in $\mathcal{D}$ with $(\nabla_X \omega)(Y) \neq (\nabla_Y \omega)(X)$, then (4.9) gives

$$(\nabla_Y \omega)(Y) = 0. \tag{4.10}$$

Thus, by putting $X = W$ and $Y = Z$ in (4.8), we have

$$(\nabla_X \omega)(Y)[2(\nabla_X \omega)(Y) - (\nabla_Y \omega)(X)] = 0. \tag{4.11}$$

Similarly, we also have

$$(\nabla_Y \omega)(X)[2(\nabla_Y \omega)(X) - (\nabla_X \omega)(Y)] = 0 . \tag{4.12}$$

Combining (4.11) and (4.12) we have $(\nabla_X \omega)(Y) = (\nabla_Y \omega)(X) = 0$. Thus we have $(\nabla_X \omega)(Y) = (\nabla_Y \omega)(X)$ for all $X, Y \in \mathcal{D}$. Consequently, Lemma 3.1 implies the following.

LEMMA 4.2. *Let N be a hypercylinder in a symmetric space M with $\beta \neq 0$. Then the distribution $\mathcal{D}$ (given by (4.4)) is integrable. Moreover, the second fundamental form $\tilde{h}$ of any maximal integral submanifold $\tilde{N}$ in N satisfies*

$$\tilde{h}(X, Y) = -(\nabla_X \omega)(Y) = -(\nabla_Y \omega)(X) \tag{4.13}$$

for all X, Y tangent to $\tilde{N}$.

From (4.8) and (4.13), we have

$$\tilde{h}(X, Z)\tilde{h}(W, Y) = \tilde{h}(Y, Z)\tilde{h}(X, W)$$

for all vector fields X, Y, Z, W tangent to the maximal integral submanifold $\tilde{N}$. Hence, from (3.4) and Lemma 3.4 we see that $\tilde{N}$ is a hypercylinder in N. Consequently we have the following.

LEMMA 4.3. *Under the hypotheses of Lemma 4.2, every maximal integral submanifold $\tilde{N}$ of $\mathcal{D}$ is a hypercylinder in N.*

Let X, Y, Z, W, T be vectors tangent to the hypercylinder N in M. Then from (4.2) we obtain

$$\begin{aligned}(\nabla_T R')&(X, Y; Z, W) \\ &= \beta\omega(T)\{\omega(X)\omega(Z)[(\nabla_Y \omega)(W) + (\nabla_W \omega)(Y)] \\ &\qquad - \omega(Y)\omega(Z)[(\nabla_X \omega)(W) + (\nabla_W \omega)(X)] \\ &\qquad + \omega(Y)\omega(W)[(\nabla_X \omega)(Z) + (\nabla_Z \omega)(X)] \\ &\qquad - \omega(X)\omega(W)[(\nabla_Y \omega)(Z) + (\nabla_Z \omega)(Y)]\}.\end{aligned} \tag{4.14}$$

If N is locally symmetric, $\nabla_T R' = 0$. By choosing $X = W$, $Y = Z$, such that X, Y are in $\mathcal{D}$ and $\omega(T) \neq 0$, we find $(\nabla_X \omega)(X) = 0$. Conversely, if we have $(\nabla_X \omega)(X) = 0$ for all X in $\mathcal{D}$, then by linearization, we find $(\nabla_X \omega)(Y) + (\nabla_Y \omega)(X) = 0$ for all X, Y in $\mathcal{D}$. Thus, (4.14) gives $(\nabla_T R')(X, Y; Z, W) = 0$ not only for X, Y, Z, W, T in $\mathcal{D}$ but also for X, Y, Z, W, T in TN. This shows that the hypercylinder N is locally symmetric if and only if we have $(\nabla_X \omega)(X) = 0$ for all X in $\mathcal{D}$. On the other hand, Lemma 4.2 shows that the later condition is equivalent to that every maximal integral submanifold is totally geodesic in N. Consequently, we have the following

LEMMA 4.4. *Under the hypothesis of Lemma 4.2, N is locally symmetric if and only if maximal integral submanifolds of $\mathcal{D}$ are totally geodesic in N (and hence in M).*

Now we shall give the following *obstruction* for a symmetric space to admit a hypercylinder.

THEOREM 4.5. *Let M be any symmetric space (not necessary irreducible). If M admits a hypercylinder N, then M admits a totally geodesic submanifold of codimension $\leqq 3$.*

Proof. If N is totally geodesic, then nothing needs to be proved. If N is not totally geodesic, without loss of generality, we may assume that the second fundamental form h is nonzero everywhere. Use the same notations as before. For vectors X, Y, Z in $\mathcal{D}$, we have

$$R(X, Y; Z, \xi) = 0. \tag{4.15}$$

Let $\tilde{N}$ be a maximal integral submanifold of $\mathcal{D}$. If $\tilde{N}$ is totally geodesic in N, then it is totally geodesic in M. Thus M admits a totally geodesic submanifold of codimension 2. If $\tilde{N}$ is not totally geodesic in N, then Lemma 4.3 says that $\tilde{N}$ is a hypercylinder in N. Therefore, there exist a unit 1-form η (at least on a nonempty subset of $\tilde{N}$) and a nonzero function λ on $\tilde{N}$ such that the second fundamental form $\tilde{h}$ of $\tilde{N}$ in N is given by

$$\tilde{h} = \lambda \eta \otimes \eta. \tag{4.16}$$

Let $\bar{\eta}$ be the unit vector tangent to $\tilde{N}$ associated with the unit 1-form η. Then we have

$$(\nabla_X \omega)(Y) = -\tilde{h}(X, Y) = -\lambda \eta(X)\eta(Y) \tag{4.17}$$

for X, Y tangent to $\tilde{N}$. We put

$$\tilde{\mathcal{D}} = \{X \in T\tilde{N} \mid \eta(X) = 0\}. \tag{4.18}$$

From (4.6) we have

(4.19) $$R(\nabla_{\bar{\omega}} X, Y; Z, \xi) + R(X, \nabla_{\bar{\omega}} Y; Z, \xi) + R(X, Y; \nabla_{\bar{\omega}} Z, \xi) = \beta R(X, Y; Z, \bar{\omega})$$

for X, Y, Z in $\mathcal{D}$. By using (4.3) we have

$$R(\nabla_{\bar{\omega}} X, Y; Z, \xi) = -\beta\omega(\nabla_{\bar{\omega}} X)(\nabla_Y \omega)(Z).$$

Thus, for Y, Z in $\tilde{\mathcal{D}}$, (4.17) implies $R(\nabla_{\bar{\omega}} X, Y; Z, \xi) = 0$. Similar arguments give $R(X, \nabla_{\bar{\omega}} Y; Z, \xi) = R(X, Y; \nabla_{\bar{\omega}} Z, \xi) = 0$ for X, Y, Z in $\tilde{\mathcal{D}}$. Consequently, (4.19) implies

(4.20) $$R(X, Y; Z, \bar{\omega}) = 0$$

for all X, Y, Z in $\tilde{\mathcal{D}}$. Taking differentiation of (4.20) with respect to $\bar{\eta}$, we find

(4.21) $$R(\nabla_{\bar{\eta}} X, Y; Z, \bar{\omega}) + R(X, \nabla_{\bar{\eta}} Y; Z, \bar{\omega}) + R(X, Y; \nabla_{\bar{\eta}} Z, \bar{\omega}) + R(X, Y; Z, \nabla_{\bar{\eta}} \bar{\omega}) = 0$$

for X, Y, Z in $\tilde{\mathcal{D}}$. On the other hand, (4.1), (4.16) and (4.20) tell us that

$$R(\nabla_{\bar{\eta}} X, Y; Z, \bar{\omega}) = R(X, \nabla_{\bar{\eta}} Y; Z, \bar{\omega}) = R(X, Y; \nabla_{\bar{\eta}} Z, \bar{\omega}) = 0.$$

Thus we have $R(X, Y; Z, \nabla_{\bar{\eta}} \bar{\omega}) = 0$. Since (4.16) gives $\nabla_{\bar{\eta}} \bar{\omega} = \lambda\bar{\eta}$, we have

(4.22) $$R(X, Y; Z, \bar{\eta}) = 0$$

for any point p in $\tilde{N}$ with $\lambda(p) \neq 0$. From (4.15), (4.20) and (4.22), we see that

(4.23) $$R(\tilde{\mathcal{D}}_p, \tilde{\mathcal{D}}_p)\tilde{\mathcal{D}}_p \subset \tilde{\mathcal{D}}_p$$

for any point p with $\lambda(p) \neq 0$. Since $M = G/H$ is a symmetric space and G acts on M transitively, we may assume p is the origin o (fixed by H). From (4.23) and Lemma 2.5, we have

$$[[\tilde{\mathcal{D}}_p, \tilde{\mathcal{D}}_p], \tilde{\mathcal{D}}_p] \subset \tilde{\mathcal{D}}_p.$$

Consequently, Lemma 2.6 implies that M admits a totally geodesic submanifold B of codimension 3 whose tangent space at p is $\tilde{\mathcal{D}}_p$. This completes the proof of the theorem.

Proof of Theorem 4.1. We assume that M is an irreducible symmetric space which admits a hypercylinder N. By Lemma 2.7, M is Einsteinian. Let S be the Ricci tensor of M. Then we have

$$S(X, \xi) = 0 \tag{4.24}$$

for all X in TN.

In N is totally geodesic, Lemma 2.8 shows that M is a sphere, a real projective space or one of their noncompact duals.

If N is not totally geodesic, we have from (4.3) that

$$R(X, Y; Z, \xi) = 0 \tag{4.25}$$

for X, Y, Z in $\mathcal{D}$. For orthonormal basis $X_1, \cdots, X_{n+1}$ in $T_p M$ with $X_{n+1} = \xi$, $X_n = \bar{\omega}$, we have

$$0 = S(X, \xi) = \sum_{i=1}^{n+1} R(X, X_i; X_i, \xi) = R(X, \bar{\omega}; \bar{\omega}, \xi) \tag{4.26}$$

for X in $\mathcal{D}$. From Lemma 4.2 and equation (4.3) of Codazzi we obtain

$$R(X, Y; \bar{\omega}, \xi) = 0 \tag{4.27}$$

for all X, Y in $\mathcal{D}$. By (4.3) we also have

$$R(X, \bar{\omega}; Z, \xi) = \beta(\nabla_X \omega)(Z) = \beta(\nabla_Z \omega)(X). \tag{4.28}$$

Since M is Einsteinian, $S(\bar{\omega}, \xi) = 0$. Thus from Lemma 4.2 and (4.28), we find

$$0 = \sum_{i=1}^{n+1} R(\bar{\omega}, X_i; X_i, \xi) = -\beta \operatorname{trace} \tilde{h}, \tag{4.29}$$

where $\tilde{h}$ is the second fundamental form of a maximal integral submanifold $\tilde{N}$ of $\mathcal{D}$. Since $\tilde{N}$ is a hypercylinder in N (Lemma 4.3), (4.29) is equivalent to say that $\tilde{N}$ is totally geodesic in N, from which we have $(\nabla_X \omega)(Y) = 0$ for all X, Y in $\mathcal{D}$. Substituting this into (4.28), we obtain

$$R(X, \bar{\omega}; Z, \xi) = 0 \tag{4.30}$$

for all X, Z in $\mathcal{D}$. Consequently, (4.25), (4.26), (4.27) and (4.30) give

$$R(X, Y; Z, \xi) = 0 \tag{4.31}$$

for all X, Y, Z in TN. Therefore we have

$$R(TN, TN)TN \subset TN.$$

By a similar argument as the last part of the proof of Theorem 4.5

we see that M admits a totally geodesic hypersurface. Theorem 4.1 then follows from Lemma 2.8. (Q. E. D.)

5. **Answer to Problem 2.** The main purpose of this section is to prove the following *classification theorem* which gives a complete answer to Problem 2.

THEOREM 5.1. *Spheres, real projective spaces, complex projective spaces and their noncompact duals are the only irreducible symmetric spaces in which we can find quasihyperspheres.*

Let M be a symmetric space (not necessary irreducible) and let N be a quasihypersphere in M. Then N is a hypersurface whose second fundamental form h in M satisfies

$$h = \alpha g + \beta \omega \otimes \omega \tag{5.1}$$

where α is a constant. If $\alpha = 0$ identically, N is a hypercylinder. This case has been considered in §4. If $\beta = 0$ identically, N is an extrinsic hypersphere. Lemma 2.11 says that the ambient space then must have constant sectional curvature. Thus M is either a sphere, a real projective space or one of their concompact duals if M is not Euclidean. Since our study is local, we may assume in the remaining part of this section that both α and β are nonzero everywhere. As before, we put $\mathcal{D} = \{X \in TN \mid \omega(X) = 0\}$. Since α is constant, equation (3.3) of Codazzi gives

$$\begin{aligned} R(X, Y; Z, \xi) &= (X\beta)\omega(Y)\omega(Z) + \beta(\nabla_X \omega)(Y)\omega(Z) \\ &\quad + \beta\omega(Y)(\nabla_X \omega)(Z) - (Y\beta)\omega(X)\omega(Z) \\ &\quad - \beta(\nabla_Y \omega)(X)\omega(Z) - \beta\omega(X)(\nabla_Y \omega)(Z), \end{aligned} \tag{5.2}$$

for X, Y, Z in TN. In particular, if X, Y, Z are vector fields in $\mathcal{D}$, we have

$$R(X, Y; Z, \xi) = 0. \tag{5.3}$$

Let W be a vector field in $\mathcal{D}$. Then, by taking differentiation of (5.3) with respect to W and by applying (5.2), we obtain

$$\begin{aligned} &\alpha\{g(W, X)R(\xi, Y; Z, \xi) \\ &\qquad - g(W, Y)R(\xi, X; Z, \xi) - R(X, Y; Z, W)\} \\ &= \beta\{(\nabla_X \omega)(Y)(\nabla_W \omega)(Z) - (\nabla_Y \omega)(X)(\nabla_W \omega)(Z) \\ &\qquad + (\nabla_W \omega)(Y)(\nabla_X \omega)(Z) - (\nabla_W \omega)(X)(\nabla_Y \omega)(Z)\}, \end{aligned} \tag{5.4}$$

for X, Y, Z in $\mathcal{D}$. Let $X = Z = W$ and Y be orthonormal. Then (5.4) gives

$$\alpha R(\xi, Y; X, \xi) = 2\beta(\nabla_X \omega)(X)\{(\nabla_X \omega)(Y) - (\nabla_Y \omega)(X)\}. \tag{5.5}$$

Since $R(\xi, Y; X, \xi) = R(\xi, X; Y, \xi)$, (5.5) implies

$$[(\nabla_X \omega)(X) + (\nabla_Y \omega)(Y)][(\nabla_X \omega)(Y) - (\nabla_Y \omega)(X)] = 0 \tag{5.6}$$

for orthonormal vectors X, Y in $\mathcal{D}$. If we put $X = W$ and $Y = Z$ and assume X, Y are orthonormal, then (5.4) gives

$$\begin{aligned}\alpha[K(\xi, Y) - K(X, Y)] &\\ = \beta\{2(\nabla_X \omega)(Y)^2 - (\nabla_X \omega)(Y)(\nabla_Y \omega)(X) &\\ - (\nabla_X \omega)(X)(\nabla_Y \omega)(Y)\}, &\end{aligned}$$

where $K(X, Y)$ denotes the sectional curvature of the section $X \wedge Y$ in M. Since $K(X, Y) = K(Y, X)$, this implies

$$\alpha[K(\xi, Y) - K(\xi, X)] = 2\beta[(\nabla_X \omega)(Y)^2 - (\nabla_Y \omega)(X)^2] \tag{5.7}$$

for orthonormal vectors X, Y in $\mathcal{D}$.

LEMMA 5.2. *If $(\nabla_X \omega)(Y) = (\nabla_Y \omega)(X)$ for all X, Y in $\mathcal{D}$, then the sectional curvature $K(\xi, X)$ is independent of the choice of X in $\mathcal{D}_p$, $p \in M$, and $R(\xi, Y; Z, \xi) = 0$ for orthogonal vectors Y, Z in $\mathcal{D}$.*

This lemma follows immediately from equations (5.5) and (5.7).

LEMMA 5.3. *If there exist orthonormal vectors X, Y in $\mathcal{D}_p$ such that*

$$(\nabla_X \omega)(X) + (\nabla_Y \omega)(Y) \neq 0, \tag{5.8}$$

then there exist orthonormal vectors $X_1, \cdots, X_{n-1}$ in $\mathcal{D}_p$ such that

$$(\nabla_{X_i} \omega)(X_i) + (\nabla_{X_j} \omega)(X_j) \neq 0, \quad i \neq j. \tag{5.9}$$

Proof. We shall prove this lemma by induction. Let Z be a unit vector in $\mathcal{D}$ perpendicular to X and Y. If we have

$$(\nabla_X \omega)(X) + (\nabla_Z \omega)(Z) = 0 \tag{5.10}$$

or

$$(\nabla_Y \omega)(Y) + (\nabla_Z \omega)(Z) = 0, \tag{5.11}$$

then without loss of generality we may assume (5.10) holds. We put

$$\bar{X} = X, \qquad \bar{Y} = aY + bZ, \qquad \bar{Z} = bY - aZ,$$

where $a = \cos\theta$, $b = \sin\theta$. Then we have, by using (5.10),

$$\begin{aligned}(5.12)\qquad &(\nabla_{\bar{X}}\,\omega)(\bar{X}) + (\nabla_{\bar{Y}}\,\omega)(\bar{Y})\\ &\quad = a^2[(\nabla_X\,\omega)(X) + (\nabla_Y\,\omega)(Y)]\\ &\qquad + ab[(\nabla_Y\,\omega)(Z) + (\nabla_Z\,\omega)(Y)],\end{aligned}$$

$$\begin{aligned}(5.13)\qquad &(\nabla_{\bar{X}}\,\omega)(\bar{X}) + (\nabla_{\bar{Z}}\,\omega)(\bar{Z})\\ &\quad = b^2[(\nabla_X\,\omega)(X) + (\nabla_Y\,\omega)(Y)]\\ &\qquad - ab\,[(\nabla_Y\,\omega)(Z) + (\nabla_Z\,\omega)(Y)],\end{aligned}$$

$$(5.14)\qquad (\nabla_{\bar{Y}}\,\omega)(\bar{Y}) + (\nabla_{\bar{Z}}\,\omega)(\bar{Z}) = (\nabla_Y\,\omega)(Y) + (\nabla_Z\,\omega)(Z).$$

Thus by choosing θ such that $\tan\theta$ and $\cot\theta$ are not equal to

$$[(\nabla_Y\omega)(Z) + (\nabla_Z\,\omega)(Y)]/[(\nabla_X\,\omega)(X) + (\nabla_Y\,\omega)(Y)],$$

we have

$$(\nabla_{\bar{X}}\,\omega)(\bar{X}) + (\nabla_{\bar{Y}}\,\omega)(\bar{Y}) \neq 0, \qquad (\nabla_{\bar{X}}\,\omega)(\bar{X}) + (\nabla_{\bar{Z}}\,\omega)(\bar{Z}) \neq 0.$$

If $(\nabla_{\bar{Y}}\,\omega)(\bar{Y}) + (\nabla_{\bar{Z}}\,\omega)(\bar{Z}) = 0$, we put

$$X_1 = \cos\phi\,\bar{X} + \sin\phi\,\bar{Y}, \qquad X_2 = \sin\phi\,\bar{X} - \cos\phi\,\bar{Y}, \qquad X_3 = \bar{Z};$$

then by a similar argument as above, we have

$$(\nabla_{X_i}\,\omega)(X_i) + (\nabla_{X_j}\,\omega)(X_j) \neq 0$$

for $i \neq j$, $i, j = 1, 2, 3$. By applying this argument sufficiently many times, we have (5.9) for a suitable orthonormal basis $X_1, \cdots, X_{n-1}$ of $\mathcal{D}_p$. (Q. E. D.)

LEMMA 5.4. *If $X_1, \cdots, X_{n-1}$ give an orthonormal basis of $\mathcal{D}_p$ such that* (5.9) *holds, then $K(\xi, X)$ is independent of the choice of X in $\mathcal{D}_p$ and $R(\xi, X; Y, \xi) = 0$ for orthogonal vectors X, Y in $\mathcal{D}$.*

Proof. From (5.6) we have

$$(\nabla_{X_i}\,\omega)(X_j) = (\nabla_{X_j}\,\omega)(X_i), \qquad i, j = 1, 2, \cdots, n-1.$$

Since every two vectors X, Y in $\mathcal{D}$ are linear combinations of $X_1, \cdots, X_{n-1}$, we have $(\nabla_X\,\omega)(Y) = (\nabla_Y\,\omega)(X)$. Consequently, this lemma follows from Lemma 5.2. (Q. E. D.)

LEMMA 5.5. *If* $\dim M \geqq 5$ *and if* $(\nabla_X\,\omega)(X) + (\nabla_Y\,\omega)(Y) = 0$

for all orthonormal vectors X, Y *in* $\mathcal{D}$, *then* $K(\xi, X)$ *is independent of the choice of* X *in* $\mathcal{D}_p$, $p \in M$, *and* $R(\xi, X; Y, \xi) = 0$ *for orthogonal vectors* X, Y *in* $\mathcal{D}$.

Proof. Let X, Y, Z be orthonormal vectors in $\mathcal{D}$, then we have

$$(\nabla_X \omega)(X) = -(\nabla_Y \omega)(Y) = (\nabla_Z \omega)(Z) = -(\nabla_X \omega)(X).$$

Thus we have $(\nabla_X \omega)(X) = 0$ for any $X \in \mathcal{D}$, from which we may conclude that $(\nabla_X \omega)(Y) = -(\nabla_Y \omega)(X)$ for any X, Y in $\mathcal{D}$. Substituting this into (5.7) we see that $K(\xi, X)$ is independent of the choice of X in $\mathcal{D}_p$. The second part of the lemma follows from (5.5). (Q. E. D.)

Combining the results above, we obtain the following necessary conditions for a symmetric space to admit a quasihypersphere.

THEOREM 5.6. *Let* M *be a (not necessary irreducible) symmetric space of dimension* ≥ 5. *If* M *admits a quasihypersphere* N, *then* M *admits a unit vector* ξ *and a codimension* 2 *subspace* V *in* T_pM *such that (a) the sectional curvatures of* M *satisfy* $K(\xi, X) = K(\xi, Y)$ *for any two unit vectors* X, Y *in* V, (b) $R(X, Y; Z, \xi) = 0$ *for* X, Y, Z *in* V *and* (c) $R(\xi, X; Y, \xi) = 0$ *for orthogonal vectors* X, Y *in* V.

If N is a quasihypersphere in M with ξ as the unit normal vector at p, then there exists a geodesic c through p with ξ as its tangent vector at p. Let B be a maximal flat totally geodesic submanifold of M which contains the geodesic c (and hence p). Then the rank of M is equal to the dimension of B. Thus in particular, if $rk\ M \geq 3$, then the intersection $T_pB \cap \mathcal{D}_p$ is nonempty. For any unit vector X in $T_pB \cap \mathcal{D}_p$ we have $K(\xi, X) = 0$. Consequently, from Theorem 5.6 we have the following.

THEOREM 5.7. *Let* M *be a (not necessary irreducible) symmetric space of dimension* ≥ 5. *If* M *admits a quasihypersphere and if rank of* M *is* ≥ 3, *then the Ricci tensor* S *of* M *satisfies*

$$S(X, X) = K(X, Y) \tag{5.15}$$

for some orthonormal vectors X, Y *tangent to* M.

Now, we give the *proof of Theorem* 5.1. If M is an irreducible symmetric space of dimension ≤ 4, then M is one of the spaces mentioned in Theorem 5.1. So we may assume that the dimension of M is $\geqq 5$. From Theorem 5.7, we see that the rank of M is $\leqq 2$.

If M is of rank 1, then M is one of the spaces mentioned in Theorem 5.1 or M is a quaternion projective space, the Cayley plane or one of their noncompact duals. If M is a quaternion projective space (or its noncompact dual) with real dimension $4n \geqq 8$, then the sectional curvature of any quaternion plane section is 4 times that of totally real plane sections. Since the quaternion section spanned by a unit vector is 4-dimensional, it is clear that there exist no unit vector ξ and a $(4n-2)$-dimensional linear subspace V with $K(\xi, X) = K(\xi, Y)$ for any unit vectors in V. The same argument applies for the Cayley plane and its noncompact dual. Therefore, if M is of rank 1 and M admits a quasihypersphere, M is a sphere, a real projective space, a complex projective space or one of their noncompact duals. Consequently, it suffices to prove that irreducible symmetric spaces of rank 2 admit no quasihyperspheres.

We propose the following *property* (*):

There is a unit vector ξ and a codimension 2 *linear subspace V of the tangent space at some point such that* (a) *the sectional curvature of the plane section $\xi \wedge X$ is independent of the choice of unit vector X in V,* (b) $R(X, Y; Z, \xi) = 0$ *for all X, Y, Z in V, and* (c) $R(\xi, X; Y, \xi) = 0$ *for orthogonal vectors X, Y in V.*

We give the following lemmas.

LEMMA 5.8. *Let $\tilde{B}$ be a complete totally geodesic submanifold of a symmetric space M with $rk\, M = rk\, \tilde{B}$. If M satisfies property* (*), *then $\tilde{B}$ also satisfies the same property.*

Proof. Since $\tilde{B}$ is totally geodesic in M, the curvature tensors R' and R of $\tilde{B}$ and M satisfy $R'(X, Y; Z, W) = R(X, Y; Z, W)$ for all X, Y, Z, W tangent to $\tilde{B}$. In particular, the sectional curvatures of $\tilde{B}$ and M for every plane section in $T\tilde{B}$ are equal. Without loss of generality we may assume that p is a point in $\tilde{B}$ and B is a maximal flat totally geodesic submanifold in M with ξ in $T_p B$. Since B and M have the same rank there is an isometry of M which carries B into $\tilde{B}$ and which fixes p. Thus the intersecion

$T_p\tilde{B} \cap V$ has codimension $\leqq 2$ in T_pB. Let $\tilde{V}$ be a codimension 2 subspace of $T_p\tilde{B} \cap V$. Then ξ and $\tilde{V}$ satisfy (a) of property (*). (b) and (c) of property (*) hold too for ξ and V since $R = R'$ for vectors tangent to $\tilde{B}$. (Q. E. D.)

Now, if M is an irreducible symmetric space of rank 2, then M is one of the following spaces or one of their noncompact duals:

$$AI(3),\ AII(3),\ G^C(2, p)(p \geqq 3),\ G^R(2, p)\ (p \geqq 3),$$

$$G^H(2, p)(p \geqq 2),\ DIII(5),\ EIII,\ EIV,\ GI,\ G_2,\ SU(3),\ Sp(2).$$

From Table VIII of [6] we see that $G^C(2, p)$ $(p \geqq 3)$, $G^R(2, p)$ $(p \geqq 3)$, $G^H(2, p)$ $(p \geqq 3)$, $DIII(5)$ and $EIII$ all contain $G^R(2, 3)$ as totally geodesic submanifold. Moreover, since $G^H(2, 2)$ contains $G^C(2, 2)$ as a totally geodesic submanifold and $G^C(2, 2)$ is $G^R(2, 4)$, by Theorem 5.6 and Lemma 5.4, we have the following.

LEMMA 5.9. *If* $G^R(2, 3)$ *does not satisfy property* (*), *then* $G^C(2, p)$ $(p \geqq 3)$, $G^R(2, p)$ $(p \geqq 3)$, $G^H(2, p)$ $(p \geqq 2)$, $DIII(5)$ *and* $EIII$ *all contain no quasihyperspheres. The same result holds for the corresponding noncompact duals.*

By similar arguments we have

LEMMA 5.10. (a) *If* $AI(3)$ *does not satisfy property* (*), *then* $AI(3)$, $AII(3)$ *and* EIV *contain no quasihyperspheres, and* (b) *if* GI *does not satisfy property* (*), *then* G_2 *and* GI *contain no quasihyperspheres. Similar results hold for the corresponding noncompact duals.*

Consequently, Theorem 5.1 follows immediately from the following.

LEMMA 5.11. $G^R(2, 3)$, $AI(3)$, GI, $SU(3)$, $Sp(2)$ *and their noncompact duals do not satisfy property* (*).

The detailed proof of Lemma 5.11 would be too long to give here. We would like to give the detailed proof for a typical one, namely $M = G^R(2, 3)$, and give only outlines of the proofs for other cases. The proof reads as follows.

Let G be the group of isometries of M, o a point in M and H its isotropy subgroup at o. We have $G = SO(5)$ and $H = SO(2)$

$\times SO(3)$. Let $\mathfrak{G}$ and $\mathfrak{H}$ be the Lie algebras of G and H, respectively. and $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ the Cartan decomposition at o. Then we have

$$\mathfrak{G} = \{X \in gl(5, \boldsymbol{R}) \mid X + {}^tX = 0\} = so(5),$$

$$\mathfrak{H} = \left\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in so(2), \quad B \in so(3)\right\},$$

$$\mathfrak{M} = \left\{\begin{pmatrix} 0 & Z \\ -{}^tZ & 0 \end{pmatrix} \middle| Z \text{ real } (2 \times 3)\text{-matrix}\right\}.$$

For simplicity, we shall identify the matrix $\begin{pmatrix} 0 & Z \\ -{}^tZ & 0 \end{pmatrix}$ with Z. Let

$$E_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then $E_1, \cdots, E_6$ form an orthonormal basis for $\mathfrak{M} = T_0 M$ and $\mathfrak{A} = \boldsymbol{R}E_1 + \boldsymbol{R}E_4$ is a maximal abelian subspace of $\mathfrak{M}$. In particular, we may assume that ξ has the following form

$$\xi = \cos\theta\, E_1 + \sin\theta\, E_4 \tag{5.16}$$

for some θ. Since V is 4-dimensional and it is perpendicular to ξ, there exist vectors X_1 and X_2 in V of the following forms

$$X_1 = -\cos\psi \sin\theta\, E_1 + \sin\psi\, E_3 + \cos\psi \cos\theta\, E_4, \tag{5.17}$$

$$X_2 = -\cos\eta \sin\theta\, E_1 + \sin\eta\, E_2 + \cos\eta \cos\theta\, E_4, \tag{5.18}$$

for some ψ and η. It is easy to see that

$$K(\xi, X_1) = \mathrm{tr}([\xi, X_1] \cdot {}^t[\xi, X_1]) = \frac{1}{2}\sin^2\psi, \tag{5.19}$$

$$K(\xi, X_2) = \frac{1}{2}\sin^2\eta. \tag{5.20}$$

Thus we obtain

$$\sin^2\psi = \sin^2\eta\,. \tag{5.21}$$

Moreover, from $R(X_1, X_2; X_2, \xi) = 0$, we get

$$\sin 2\eta \cos 2\theta \sin\psi = 0\,. \tag{5.22}$$

Similarly, from $R(X_1, X_2; X_1, \xi) = 0$, we obtain

$$\sin 2\psi \cos 2\theta \sin\eta = 0\,. \tag{5.23}$$

Case 1. $\sin\eta = 0$ or $\sin\psi = 0$. In this case, (5.19), (5.20) and (5.21) show that $K(\xi, X) = 0$ for all X in V. Thus $S(\xi, \xi) = K(\xi, \bar{\omega})$ for some vector $\bar{\omega}$. This is a contradiction.

Case 2. $\sin\eta \sin\psi \neq 0$ and either $\cos\eta = 0$ or $\cos\psi = 0$. In this case, (5.21) shows that $\sin^2\eta = \sin^2\psi = 1$ and $\cos\eta = \cos\psi = 0$. Thus, we may put

$$X_1 = E_3 \text{ and } X_2 = E_2\,, \tag{5.24}$$

from which we obtain

$$K(\xi, X_1) = K(\xi, X_2) = \frac{1}{2}\,. \tag{5.25}$$

On the other hand, there exists a unit vector X_3 in V of the following form

$$X_3 = \cos\delta\, E_5 + \sin\delta\, E_6\,. \tag{5.26}$$

It is easy to verify that

$$K(\xi, X_3) = \frac{1}{2}(\cos^2\theta\cos^2\delta + \sin^2\theta\sin^2\delta). \tag{5.27}$$

Thus, we obtain

$$\cos^2\theta\cos^2\delta + \sin^2\theta\sin^2\delta = 1\,. \tag{5.28}$$

Since we have $\cos^2\theta\cos^2\delta + \sin^2\theta\sin^2\delta = 1 - \cos^2\theta\sin^2\delta - \sin^2\theta.\cos^2\delta$, from (5.28) we conclude $\cos\theta\sin\delta = \sin\theta\cos\delta = 0$. Hence either $\cos\theta = 0$ or $\sin\theta = 0$, and it is thus reduced to the remaining case, case (3).

Case 3. $\cos\theta = 0$ or $\sin\theta = 0$. In this case, we have either $\xi = E_1$ or $\xi = E_4$. Since the two cases are similar to each other, we only consider the first one. Thus we have

$$\xi = E_1, \qquad \cos\theta = 1, \qquad \sin\theta = 0\,. \tag{5.29}$$

In this case, there is a unit vector in V of the following form:

$$X_4 = \cos\gamma\, E_4 + \sin\gamma\, E_6 .$$

It is easy to verify that $K(\xi, E_4) = 0$. Thus we obtain $K(\xi, X) = 0$ for all unit vectors X in V. Consequently, we have $S(\xi, \xi) = K(\xi, \bar{\omega})$ for some unit vector $\bar{\omega}$. This gives a contradiction.

This shows that $G^R(2, 3)$ does not satisfy property (*). The other cases can be proved in the following way. Since GI, $SU(3)$ and $Sp(2)$ all contain the product of two 2-spheres $S^2 \times S^2$ as totally geodesic submanifold [6] and they all have rank 2, we may first assume that $\xi = \cos\theta\, E_1 + \sin\theta\, E_2$ for some θ and unit vectors E_1 and E_2 tangent to the correspondent components. Since $S^2 \times S^2$ must satisfy property (*), we may prove that either ξ is tangent to one of the components of $S^2 \times S^2$ or $\cos\theta = \sin\theta = 1/\sqrt{2}$. In the first case, $K(\xi, X) = 0$ for any unit vector X in V, which is impossible by the curvature condition of irreducible symmetric spaces of dimension >2. In the second case, ξ having a very special form, similar arguments as given in case $G^R(2, 3)$ may prove that GI, $SU(3)$ and $Sp(2)$ do not satisfy property (*). $AI(3)$ can be treated by the same method as given for $G^R(2, 3)$.

6. **$J\xi$-quasiumbilical hypersurfaces.** Let N be a quasiumbilical hypersurface in a Kaehler manifold (M, g, J) and ξ a unit normal vector field of N in M. If $J\xi$ is the distinguished direction, that is, $\omega(X) = g(J\xi, X)$, for X tangent to N, then, as in §3, we call N a *$J\xi$-quasiumbilical hypersurface.*

The main purpose of this section is to prove the following.

THEOREM 6.1. *The principal curvatures of every $J\xi$-quasiumbilical hypersurface N in any irreducible Hermitian symmetric space M of dimension >2 are constants. In particular, we have* (a) *N is a quasihypersphere,* (b) *the mean curvature of N is constant, and* (c) *M is either a complex projective space or its noncompact dual.*

Proof. By the assumption, we have

$$h = \alpha g + \beta\omega \otimes \omega, \qquad \omega(X) = g(J\xi, X). \tag{6.1}$$

Let X, Y be two vector fields tangent to N. We have

$$Y\omega(X) = (\nabla_Y \omega)(X) + \omega(\nabla_Y X).$$

On the other hand, (6.1) implies

$$Y\omega(X) = g(\nabla'_Y J\xi, X) + g(J\xi, \nabla_Y X) = g(A_\xi Y, JX) + \omega(\nabla_Y X).$$

Thus we have

$$(\nabla_Y \omega)(X) = g(A_\xi Y, JX), \tag{6.2}$$

for all X, Y in TN. Let $\mathcal{D} = \{X \in TN \mid \omega(X) = 0\}$ as before. Then $A_\xi X = \alpha X$ for X in $\mathcal{D}$. Thus (6.2) gives

$$(\nabla_X \omega)(Y) = \alpha g(X, JY) \tag{6.3}$$

for X, Y in $\mathcal{D}$. In particular, we have

$$(\nabla_X \omega)(J\xi) = 0. \tag{6.4}$$

Similarly, since $A_\xi(J\xi) = (\alpha + \beta)(J\xi)$, (6.2) also gives

$$(\nabla_{J\xi} \omega)(X) = 0 \tag{6.5}$$

for X in $\mathcal{D}$. Substituting (6.3), (6.4) and (6.5) into equation (3.12) of Codazzi, we find

$$R(X, JX; J\xi, \xi) = K(X, \xi) + K(JX, \xi) = -2\alpha\beta \tag{6.6}$$

for X in $\mathcal{D}$. Similarly, we may also prove that

$$R(X, Y; Y, \xi) = X\alpha \tag{6.7}$$

for orthonormal vectors X, Y in $\mathcal{D}$, and

$$R(X, J\xi; J\xi, \xi) = X(\alpha + \beta) \tag{6.8}$$

for X in $\mathcal{D}$. Since M is Einsteinian, the Ricci tensor S of M thus satisfies.

$$0 = S(X, \xi) = X[(n-1)\alpha + \beta] \tag{6.9}$$

for X in $\mathcal{D}$, where $n = \dim N$. Similarly, by $S(\xi, J\xi) = 0$, we find

$$(J\xi)(\alpha) = 0. \tag{6.10}$$

By using (6.3) and (6.10), equation (3.12) of Codazzi gives

$$R(J\xi, Y; Y, \xi) = 0 \tag{6.11}$$

for any vector field Y in $\mathcal{D}$. By taking differentiation of (6.11) with respect to $J\xi$, we find

$$\begin{aligned} O = {} & R(A_\xi(J\xi), JY; Y, \xi) + R(J\xi, \nabla_{J\xi} Y; Y, \xi) \\ & + R(J\xi, Y; \nabla_{J\xi} Y, \xi) - R(J\xi, Y; Y, A_\xi(J\xi)) \end{aligned}$$

for Y in $\mathcal{D}$. Hence, by (6.1), we obtain

$$(6.12)\quad \begin{aligned}&(\alpha+\beta)\{K(\xi, JY)-K(\xi, Y)\}\\ &\qquad = R(J\xi, \nabla_{J\xi} Y;\ Y, \xi)+R(J\xi, Y;\ \nabla_{J\xi} Y, \xi).\end{aligned}$$

On the other hand, (3.11), (6.2), (6.5) and (6.10) give

$$(6.13)\quad \begin{aligned}R(J\xi, \nabla_{J\xi} Y;\ Y, \xi) &= -\beta(\nabla_{\nabla_{J\xi}Y}\,\omega)(Y) = -\beta g(A_\xi(\nabla_{J\xi} Y), JY)\\ &= -\beta g(\nabla_{J\xi} Y, A_\xi(JY)) = -\beta\alpha g(\nabla_{J\xi} Y, JY)\end{aligned}$$

and

$$(6.14)\quad \begin{aligned}&R(J\xi, Y;\ \nabla_{J\xi} Y, \xi)\\ &\quad = -(Y\alpha)g(J\xi, \nabla_{J\xi} Y)-(Y\beta)\omega(\nabla_{J\xi} Y)\\ &\qquad -\beta(\nabla_Y\,\omega)(\nabla_{J\xi} Y)\\ &\quad = -(Y\alpha)g(J\xi, \nabla'_{J\xi} Y)+(Y\beta)(\nabla_{J\xi}\,\omega)(Y)\\ &\qquad -\beta g(A_\xi Y, J\nabla_{J\xi} Y)\\ &\quad = (Y\alpha)g(J\nabla'_{J\xi}\,\xi, Y)-\alpha\beta g(Y, J\nabla_{J\xi} Y)\\ &\quad = \alpha\beta g(JY, \nabla_{J\xi} Y)\end{aligned}$$

for Y in $\mathcal{D}$. Combining (6.12), (6.13) and (6.14) we obtain

$$(6.15)\qquad (\alpha+\beta)\{K(\xi, JX)-K(\xi, X)\}=0$$

for X in $\mathcal{D}$.

Case 1. $\alpha+\beta=0$ identically. From (6.9) and (6.10), we see that $\alpha=-\beta$ is a constant.

Case 2. $\alpha+\beta\neq 0$ for some point $p\in N$. In this case $\alpha+\beta$ is nowhere zero on a nonempty open subset of N. On this subset, (6.15) implies

$$(6.16)\qquad K(\xi, X)=K(\xi, JX)$$

for X in $\mathcal{D}$. Thus from (6.6) we have

$$(6.17)\qquad K(\xi, X)=-\alpha\beta$$

for all unit vectors X in $\mathcal{D}$. This shows that the sectional curvature $K(\xi, X)$ of the totally real section $\xi\wedge X$ is independent of the choice of X. If M is a Hermitian symmetric space of rank $\geqq 2$, then there exists a totally real section $\xi\wedge X$ such that $K(\xi, X)=0$. Thus from (6.17) we have $\alpha\beta=0$. This shows that M admits either a hypercylinder or a umbilical hypersurface. Both cases are impossible (Lemma 2.9 and Theorem 4.1). Thus, M must be a rank 1 Hermitian symmetric space. Consequently, M is either a complex projective space or its noncompact dual. In both cases, M has constant holomorphic sectional curvature, say c. Thus, M has constant totally real sectional curvature $c/4$. So (6.17) implies $\alpha\beta = -c/4$. Combining this with (6.9) and (6.10), we see that both α and β are constant. This proves the theorem.

From the last part of the proof of Theorem 6.1, we have immediately the following

COROLLARY 6.2. *Let N be a $J\xi$-quasiumbilical hypersurface in a complex-space-form of constant holomorphic sectional curvature c. Then α and β are constants satisfying $\alpha\beta = -c/4$, where α and β are given by* (6.7).

7. **Locally symmetric hypersurfaces.** The main purpose of this section is to prove the following.

THEOREM 7.1. *Every n-dimensional $(n \geq 3)$ locally symmetric quasiumbilical hypersurface N in a symmetric space is a quasihypersphere.*

THEOREM 7.2. *The only irreducible symmetric spaces which admit locally symmetric quasiumbilical hypersurfaces are spheres, real projective spaces and their noncompact duals.*

Proof of Theorem 7.1. Let N be a quasiumbilical hypersurface in a symmetric space M. Then the second fundamental form h takes the form (3.10). So by equation (3.4) of Gauss, we have

$$\begin{aligned} R'(X, Y; Z, W) &\\ = R(X, Y; Z, W) &\\ \qquad + \alpha^2\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} &\\ \qquad + \alpha\beta\{g(X, W)\omega(Y)\omega(Z) + g(Y, Z)\omega(X)\omega(W) &\\ \qquad - g(Y, W)\omega(X)\omega(Z) - g(X, Z)\omega(Y)\omega(W)\}. & \end{aligned} \tag{7.1}$$

In particular for X, Y, Z, W in $\mathcal{D}$, we have

$$\begin{aligned} R'(X, Y; Z, W) &\\ = R(X, Y; Z, W) &\\ \qquad + \alpha^2\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}. & \end{aligned} \tag{7.2}$$

From equation (3.5) of Codazzi and (3.10) we also find

$$R(X, Y; Z, \xi) = (X\alpha)g(Y, Z) - (Y\alpha)g(X, Z) \tag{7.3}$$

for X, Y, Z in $\mathcal{D}$.

Let T be any vector tangent to N. By taking differentiation of (7.2) with respect to T and by using (7.1), (7.2) and (7.3), and by a straightforward computation, we may find

$$\begin{aligned}(\nabla_T R')&(X, Y; Z, W)\\ &= (T\alpha^2)\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}\\ &\quad + \alpha(X\alpha)\{g(Y,Z)g(T,W) - g(Y,W)g(T,Z)\}\\ &\quad + \alpha(Y\alpha)\{g(X,W)g(T,Z) - g(X,Z)g(T,W)\}\\ &\quad + \alpha(Z\alpha)\{g(X,W)g(T,Y) - g(X,T)g(Y,W)\}\\ &\quad + \alpha(W\alpha)\{g(Y,Z)g(T,X) - g(Y,T)g(X,Z)\}\,.\end{aligned} \tag{7.4}$$

Since $\dim N \geqq 3$ and N is locally symmetric, (7,4) implies that $T\alpha^2 = 0$ for all T tangent to N. This shows that N is a quasihypersphere.

Proof of Theorem 7.2. Let M be an irreducible symmetric space and N a locally symmetric quasiumbilical hypersurface in M. If $\dim N \leqq 3$, M is one of the rank 1 symmetric spaces given in the theorem. So we may assume that $\dim N \geqq 4$. In this case, Theorem 7.1 shows that N is a quasihypersphere in M, that is, α is constant. If $\alpha = 0$ identically, then N is a hypercylinder and Theorem 7.2 follows from Theorem 4.1. If α is a nonzero constant, Theorem 5.1 shows that M is a sphere, a real projective space, a complex projective space or one of their noncompact duals. Theorem 7.2 follows immediately from the following.

LEMMA 7.3. *Complex projective spaces and their noncompact duals admit no locally symmetric quasiumbilical hypersurfaces of dimension* $\geqq 3$.

Proof. Let M be either a complex projective space or its noncompact dual. If M admits a locally symmetric quasiumbilical hypersurface N, then N is a quasihypersphere in M (Theorm 7.1). Moreover, from Lemma 2.8, we see that N is not totally geodesic in M on a dense open subset of N. Let ξ be a unit vector normal to N and $\mathcal{D} = \{X \in TN \mid \omega(X) = 0\}$ as before. Then Theorem 5.2 tells us that the sectional curvature of M satisfies $K(\xi, X) = K(\xi, Y)$

for all unit vectors X, Y in $\mathcal{D}$. Since M is a complex-space-form and it is non-flat, the totally real sectional curvatures are equal and they are different from the sectional curvatures of non-totally real plane sections. Thus, by the assumption that $\dim M \geqq 5$, the above mentioned property of $\mathcal{D}$ implies that $\mathcal{D}$ is a holomorphic distribution, that is, $J\mathcal{D} \subset \mathcal{D}$.

On the other hand, let X, Y, Z, T be vector fields in $\mathcal{D}$ and put $W = \bar{\omega}$. Then (7.2) holds. By taking differentiation of (7.2) with respect to T, we may find, by using $\nabla_T R' = 0$ and $R(X, Y; Z, \xi) = 0$, that

$$\begin{aligned} g(Y,Z)(\nabla_T\,\omega)(X) - g(X,Z)(\nabla_T\,\omega)(Y) + g(T,Y)(\nabla_Z\,\omega)(X) \\ - g(X,T)(\nabla_Z\,\omega)(Y) + g(T,Z)[(\nabla_Y\,\omega)(X) \\ - (\nabla_X\,\omega)(Y)] = 0\,. \end{aligned}$$

Choosing $Y = T$, X, Z as orthogonal vectors in $\mathcal{D}$, we find

$$(\nabla_Z\,\omega)(X) = 0\,. \tag{7.5}$$

From this we see that for vectors X, Y in $\mathcal{D}$, we have $(\nabla_X\,\omega)(Y) = 0$. In particular, this shows that $\mathcal{D}$ is integrable and that maximal integral submanifolds are totally geodesic in N (Lemma 3.1). Since $\mathcal{D}$ is a holomorphic distribution, every maximal integral submanifold $\tilde{N}$ in M is a complex submanifold of M. Since M is Kaehlerian, $\tilde{N}$ is minimal in M. On the other hand, since $\tilde{N}$ is totally geodesic in N, (3.10) shows that the second fundamental form σ of $\tilde{N}$ in M is given by $\sigma(X, Y) = \alpha g(X, Y)\xi$ for X, Y tangent to $\tilde{N}$. Therefore, by the minimality of $\tilde{N}$ in M, we have $\alpha = 0$. This contradicts to the assumption. (Q. E. D.)

REMARK 7.1. We may also prove that a quasiumbilical hypersurface N in a (locally) symmetric space M is locally symmetric if and only if α is constant and β, ω satisfy the following first order differential equation:

$$\alpha(X\beta)\omega(Y) = 2\alpha\beta\gamma g(X, Y) + 2\beta^2\gamma\omega(X)\omega(Y) - 2\alpha\beta(\nabla_X\,\omega)(Y)$$

for some function γ on N and for vectors X, Y tangent to N, where α, β, ω are given as before.

8. **Conformally flat hypersurfaces.** It is well-known that an

n-dimensional ($n \geqq 4$) hypersurface N in a sphere, a real projective space and their noncompact duals is conformally flat if and only if it is quasiumbilical (see [2] for instance). So it seems natural to ask the following.

PROBLEM 4. *When a quasiumbilical hypersurface of dimension $\geqq 4$ in an irreducible symmetric space is conformally flat?*

The main purpose of this section is to give the following answer to this problem.

THEOREM 8.1. *Let M be an irreducible symmetric space of dimension $n+1 \geqq 5$. Then a quasiumbilical hypersurface N in M is conformally flat if and only if M is a sphere, a real projective space, or one of their noncompact duals.*

Proof. If M is one of the spaces mentioned above, then every quasiumbilical hypersurface is conformally flat.

Conversely, suppose M is an irreducible symmetric space of dimension $\geqq 5$ which admits a conformally flat quasiumbilical hypersurface N. If $\beta \equiv 0$, then N is totally umbilical and Lemma 2.9 shows that N is a space mentioned in the theorem. If $\alpha \equiv 0$, then N is a hypercylinder and the theorem follows from Theorem 4.1. Thus, in the remaining part of the proof, we may assume that both α and β are nonzero. From the assumption, we see that the curvature tensor R' of N satisfies (7.1) for all vectors X, Y, Z, W tanget to N. Let $E_1, \cdots, E_n$ be an orthonormal basis of T_pN, $p \in N$, with $E_n = \bar{\omega}$. Then the Ricci tensor S' of N satisfies

$$\begin{aligned} S'(Y, Z) &= \sum_{i=1}^{n} R'(E_i, Y; Z, E_i) \\ &= S(Y, Z) - R(\xi, Y; Z, \xi) + (n-1)\alpha^2 g(Y, Z) \\ &\quad + \alpha\beta g(Y, Z) + (n-2)\alpha\beta\omega(Y)\omega(Z), \end{aligned} \tag{8.1}$$

where S denotes the Ricci tensor of M. Thus the scalar curvatures ρ' and ρ of N and M satisfy

$$\begin{aligned} \rho' &= \sum_{i=1}^{n} S'(E_i, E_i) \\ &= \rho - 2S(\xi, \xi) + n(n-1)\alpha^2 + 2n(n-1)\alpha\beta \\ &= \frac{n-1}{n+1}\{\rho + n(n+1)\alpha^2 + 2(n+1)\alpha\beta\}, \end{aligned} \tag{8.2}$$

where the last equality holds since M is Einsteinian. Now, by the assumption that N is conformally flat, the Weyl conformal curvature tensor of N vanishes. Thus by (8.1) and (8.2), we see that the curvature tensor R of M satisfies

$$(8.3)\quad \begin{aligned}(n-2)&R(X, Y; Z, W)\\ &= g(Y,W)R(\xi, X; Z,\xi) - g(X,W)R(\xi, Y; Z,\xi)\\ &\quad + g(X,Z)R(\xi, Y; W,\xi) - g(Y,Z)R(\xi, X; W,\xi)\\ &\quad + \frac{\rho}{n+1}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\},\end{aligned}$$

for X, Y, Z, W tangent to N.

On the other hand, from equation (3.5) of Codazzi, we have

$$(8.4)\quad \begin{aligned}R(&X, Y; Z, \xi)\\ &= (X\alpha)g(Y,Z) + (X\beta)\omega(Y)\omega(Z) + \beta(\nabla_X\omega)(Y)\omega(Z)\\ &\quad + \beta\omega(Y)(\nabla_X\omega)(Z) - (Y\alpha)g(X, Z)\\ &\quad - (Y\beta)\omega(X)\omega(Z)\\ &\quad - \beta(\nabla_Y\omega)(X)\omega(Z) - \beta\omega(X)(\nabla_Y\omega)(Z).\end{aligned}$$

Let X, Y, Z, W and T be vector fields tangent to N. By differentiation of (8.3) with respect to T, we may obtain, after a traightforward computation, that

$$(8.5)\quad \begin{aligned}(n-2)\{&[\alpha g(T, X) + \beta\omega(T)\omega(X)]R(\xi, Y; Z, W)\\ &+ [\alpha g(T,Y)+\beta\omega(T)\omega(Y)]R(X, \xi; Z, W)\\ &+ [\alpha g(T,Z) + \beta\omega(T)\omega(Z)]R(X,Y; \xi, W)\\ &+ [\alpha g(T, W) + \beta\omega(T)\omega(W)]R(X, Y; Z, \xi)\}\\ = -&g(X,W)[R(A_\xi T, Y; Z,\xi) + R(\xi, Y; Z, A_\xi T)]\\ &+ g(Y,W)[R(A_\xi T, X; Z,\xi) + R(\xi, X; Z,A_\xi T)]\\ &- g(Y,Z)[R(A_\xi T, X; W,\xi) + R(\xi, X; W, A_\xi T)]\\ &+ g(X,Z)[R(A_\xi T, Y; W,\xi) + R(\xi, Y; W, A_\xi T)].\end{aligned}$$

In particular, for vectors X, Y, Z tangent to N, if we choose $T = W$ in $\mathcal{D}$ to be orthogonal to X, Y, Z, then, from (8.5), we have

$$(8.6)\quad \begin{aligned}g(&T, T)R(X, Y; Z, \xi)\\ &= g(X, Z)R(T, Y; T, \xi) - g(Y, Z)R(T, X; T, \xi).\end{aligned}$$

If X, Y, Z, W are vectors in $\mathcal{D}$ such that $X = W$, $Y = Z$ and X, Y are orthonormal, then by (8.4) and (8.5) we find

$$\begin{aligned}(n-2)\alpha\{(X\alpha)g(T, X) + (Y\alpha)g(T, Y)\} \\ = 2\alpha(T\alpha) - \alpha(Y\alpha)g(T, Y) - \alpha(X\alpha)g(T, X) \\ - (\alpha+\beta)\beta\omega(T)[(\nabla_X\omega)(X) + (\nabla_Y\omega)(Y)] \\ + 2\beta(\bar{\omega}\alpha)\omega(T).\end{aligned} \tag{8.7}$$

In particular, for a vector T in $\mathcal{D}$ and orthonormal vectors X, Y in $\mathcal{D}$ which are perpendicular to T, (8.7) implies

$$X\alpha = 0 \tag{8.8}$$

for X in $\mathcal{D}$. Substituting (8.8) into (8.7) we get

$$\begin{aligned}2\alpha(T\alpha) + 2\beta(\bar{\omega}\alpha)\omega(T) \\ = (\alpha+\beta)\beta\omega(T)[(\nabla_X\omega)(X) + (\nabla_Y\omega)(Y)]\end{aligned} \tag{8.9}$$

for T tangent to N and orthonormal vectors X, Y in $\mathcal{D}$.

If $\alpha + \beta = 0$, then (8.9) gives

$$T\alpha = (\bar{\omega}\alpha)\omega(T). \tag{8.10}$$

If $\alpha + \beta \neq 0$, then (8.9) shows that $(\nabla_X\omega)(X) + (\nabla_Y\omega)(Y)$ is independent of the choice of orthonormal vectors X, Y in $\mathcal{D}$. Since dim N is greater than 3, we thus have

$$(\nabla_X\omega)(X) = (\nabla_Y\omega)(Y) \tag{8.11}$$

for unit vectors X, Y in $\mathcal{D}$. Put $T = \bar{\omega}$. Then, from (8.9) and (8.10), we have

$$\bar{\omega}\alpha = \beta(\nabla_X\omega)(X) \tag{8.12}$$

for unit X in $\mathcal{D}$. Substituting (8.12) into (8.9), we obtain $T\alpha = (\bar{\omega}\alpha)\omega(T)$. Consequently, we have the following.

LEMMA 8.2. *For any vector tangent to N, we have*

$$T\alpha = (\bar{\omega}\alpha)\omega(T). \tag{8.13}$$

By using (8.4) and Lemma 8.2, we have, for unit T in $\mathcal{D}$,

$$R(T, Y;\ T, \xi) = \beta\omega(Y)(\nabla_T\ \omega)(T) - Y\alpha. \tag{8.14}$$

In particular, we have

$$R(T, \bar{\omega};\ T, \xi) = \beta(\nabla_T\omega)(T) - \bar{\omega}\alpha. \tag{8.15}$$

Combining (8.6) and (8.15) we find

$$R(X, \bar{\omega}; X, \xi) = R(T, \bar{\omega}; T, \xi)$$

for unit vectors X, T in $\mathcal{D}$. Therefore (8.15) implies $(\nabla_X \omega)(X) = (\nabla_T \omega)(T)$. Consequently, by (8.9) and (8.10) we obtain

$$Y\alpha = \beta\omega(Y)(\nabla_T \omega)(T) \tag{8.16}$$

for all Y in TN. Thus, (8.6) and (8.14) imply

$$R(X, Y; Z, \xi) = 0 \tag{8.16}$$

for all vectors X, Y, Z tangent to N. By an argument given as before, we may conclude that M admits a totally geodesic hypersurface. Therefore, by Lemma 2.8, we conclude that M is either a sphere, a real projective space, or one of their noncompact duals. This completes the proof of Theorem 8.1.

9. **Einstein hypersurfaces.**

The main purpose of this section is to prove the following.

THEOREM 9.1. *If N is an Einstein quasiumbilical hypersurface in an irreducible symmetric space of dimension $\geqq 4$, then either N has constant principal curvatures or N is a hypercylinder in M. Moreover, M is either a sphere, a real projective space or one of their noncompact duals.*

Proof. Suppose that N is a quasiumbilical hypersurface in an irreducible symmetric space M of dimension $\geqq 4$. Then we have from (8.1) that

$$\begin{aligned} &R(\xi, Y; Z, \xi) \\ &\quad = \left\{(n-1)\alpha^2 + \alpha\beta + \frac{\rho}{n+1} - \frac{\rho'}{n}\right\} g(Y, Z) \\ &\qquad + (n-2)\alpha\beta\omega(Y)\omega(Z), \end{aligned} \tag{9.1}$$

where ρ and ρ' denote the scalar curvatures of M and N, respectively, and $n = \dim N$. By taking differentiation of (9.1) with respect to a tangent vector of N, say T, we have after a straightforward computation, that

$$\begin{aligned}
(9.2)\qquad & g(Y,Z)\{2\beta\omega(T)(\omega\alpha)-[2(n-2)\alpha+\beta](T\alpha)-\alpha(T\beta)\}\\
&\quad +\omega(Y)\omega(Z)\{2\beta\omega(T)(\bar{\omega}\beta)\\
&\quad -(n-4)\alpha(T\beta)-(n-2)\beta(T\alpha)\}\\
&\quad -(n-4)\alpha\beta\{(\nabla_T\omega)(Y)\omega(Z)+\omega(Y)(\nabla_T\omega)(Z)\}\\
&\quad +2\beta^2\omega(T)\{(\nabla_{\bar{\omega}}\omega)(Y)\omega(Z)+\omega(Y)(\nabla_{\bar{\omega}}\omega)(Z)\}\\
&\quad -\alpha\{(Y\alpha)g(T,Z)+(Z\alpha)g(T,Y)\}\\
&\quad -\beta\omega(T)\{(Y\alpha)\omega(Z)+(Z\alpha)\omega(Y)\}\\
&\quad -(\alpha+\beta)\omega(T)\{(Y\beta)\omega(Z)+(Z\beta)\omega(Y)\}\\
&\quad -\beta(\alpha+\beta)\omega(T)\{(\nabla_Y\omega)(Z)+(\nabla_Z\omega)(Y)\}\\
&\quad -\alpha\beta\{(\nabla_Y\omega)(T)\omega(Z)+\omega(Y)(\nabla_Z\omega)(T)\}\\
&\quad -\beta^2\omega(T)\{(\nabla_Y\omega)(\bar{\omega})\omega(Z)+\omega(Y)(\nabla_Z\omega)(\bar{\omega})\}=0.
\end{aligned}$$

Let $Z=T$ and Y be orthonormal vectors in $\mathcal{D}$, then (9.2) gives

$$(9.3)\qquad \alpha(X\alpha)=0$$

for X in $\mathcal{D}$. For Y and Z in $\mathcal{D}$ and $T=\bar{\omega}$, (9.2) gives

$$\begin{aligned}
(9.4)\qquad & \alpha\beta(\alpha+\beta)\{(\nabla_Y\omega)(Z)+(\nabla_Z\omega)(Y)\}\\
&\quad =\alpha g(Y,Z)\{[\beta-2(n-2)\alpha](\bar{\omega}\alpha)-\alpha(\bar{\omega}\beta)\}.
\end{aligned}$$

In particular, we have

$$(9.5)\qquad [\beta-2(n-2)\alpha](\bar{\omega}\alpha)-\alpha(\bar{\omega}\beta)=2\beta(\alpha+\beta)(\nabla_Y\omega)(Y)$$

for unit Y in $\mathcal{D}$.

If $Y=T$ is a unit vector in $\mathcal{D}$ and Y is perpendicular to Z, (9.2) gives

$$(9.6)\qquad \alpha(Z\alpha)=(3-n)\alpha\beta(\nabla_T\omega)(T)\omega(Z)$$

for Z in TN and any unit vector T in $\mathcal{D}$. From this we find

$$(9.7)\qquad \alpha\beta(\nabla_T\omega)(T)=\alpha\beta(\nabla_X\omega)(X)$$

for unit vectors X, T in $\mathcal{D}$. Moreover, from (9.5) and (9.6) we get

$$(9.8)\qquad \alpha(\bar{\omega}\alpha)=(3-n)\alpha\beta(\nabla_T\omega)(T)$$

and

$$(9.9)\qquad \alpha(\omega\beta)=\beta[2(n^2-5n+5)\alpha-(n-1)\beta](\nabla_T\omega)(T).$$

Let T and Y be in $\mathcal{D}$ and $Z=\bar{\omega}$. Then (9.2) and (9.8) give

$$(9.10)\qquad \alpha\beta\{(n-4)(\nabla_T\omega)(Y)+(\nabla_Y\omega)(T)\}=(n-3)\alpha\beta(\nabla_T\omega)(T)$$

from which we get

$$\alpha\beta(\nabla_T \omega)(Y) = \alpha\beta(\nabla_Y \omega)(T). \tag{9.11}$$

On the other hand, if T and Y are orthogonal, (9.4) and (9.11) imply that

$$\alpha\beta(\alpha + \beta)(\nabla_T \omega)(Y) = \alpha\beta(\alpha + \beta)(\nabla_Y \omega)(T) = 0. \tag{9.12}$$

Substituting this in (9.10) we see that if $\alpha\beta(\alpha + \beta) \neq 0$, then $(\nabla_T \omega)(T) = 0$ for T in $\mathcal{D}$. Consequently, on the open subset $U = \{p \in N \mid \alpha\beta(\alpha + \beta) \neq 0 \text{ at } p\}$, α and β are constants. Therefore, U is either the whole hypersurface N or U is empty. In the first case, the principal curvatures are constants. In the second case, we have

$$\alpha\beta(\alpha + \beta) \equiv 0. \tag{9.13}$$

From (8.2), we have

$$\begin{aligned}(n+1)\rho' &= (n-1)\{\rho + n(n+1)\alpha^2 + 2(n+1)\alpha\beta\} \\ &= (n-1)\rho + (n^2-1)\alpha\{2(\alpha+\beta) + (n-2)\alpha\}.\end{aligned} \tag{9.14}$$

Since both ρ and ρ' are constants, $n\alpha^2 + 2\alpha\beta$ is constant. Thus if $\alpha \neq 0$, one of $\beta = 0$ or $\alpha + \beta = 0$ will imply that α is a nonzero constant. Therefore either $\alpha = 0$ identically or both α and β are constants. In the first case, N is a hypercylinder and thus Theorem 4.1 implies that M is one of the mentioned spaces. In the second case with $\alpha \neq 0$, we have either $\beta = 0$ identically, or $(\nabla_T \omega)(T) = 0$ for all T in $\mathcal{D}$ by (9.8). If $\beta = 0$ identically, N is an extrinsic hypersphere. Lemma 2.11 then implies that M is one of the mentioned spaces. If $\beta \neq 0$ and $(\nabla_T \omega)(T) = 0$ for all T in $\mathcal{D}$, (9.10) implies that $(\nabla_X \omega)(Y) = 0$ for X, Y in $\mathcal{D}$. Lemma 3.1 tells us that $\mathcal{D}$ is integrable and every maximal integral submanifold $\tilde{N}$ is totally geodesic in N. Hence by the constancy of α and total umbilicity of N, we see that $\tilde{N}$ is an extrinsic sphere in M. Since $\tilde{N}$ has codimension 2 in M, Lemma 2.11 shows that M contains a totally geodesic hypersurface of constant curvature. Thus M is either a sphere, a real projective space or one of its noncompact duals. This completes the proof of the theorem.

10. Remarks.

REMARK 10.1. Since our discussions in this paper are local, the local versions of the corresponding results hold. In particular, we have the following local results.

THEOREM 4.1′. *The only irreducible locally symmetric spaces which contain hypercylinders are real-space-forms.*

THEOREM 5.1′. *The only irreducible locally symmetric spaces which contain quasihyperspheres are real-space-forms and complex-space-forms.*

THEOREM 6.1′. *The only irreducible Hermitian locally symmetric spaces which contain $J\xi$-quasiumbilical hypersurfaces are complex-space-forms.*

THEOREM 7.1′. *The only irreducible locally symmetric spaces which contain locally symmetric quasiumbilical hypersurfaces are real-space-forms.*

THEOREM 8.1′. *Let M be an irreducible locally symmetric space of dimension $\geqq 5$. Then a quasiumbilical hypersurface is conformally flat if and only if M is a real-space-form.*

THEOREM 9.1′. *The only irreducible locally symmetric spaces which contain Einstein quasiumbilical hypersurfaces are real-space-forms.*

REMARK 10.2. This work was done during his sabbatical leave of the first author from Michigan State University at Katholieke Universiteit Leuven. The first author would like to express his thanks for his colleagues at K. U. L. for their hospitality.

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